2023

Dynamical Aspects in (4+1)-Body Problems

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Dynamical Aspects in $(4+1)$-Body Problems

by

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A thesis
presented to Wilfrid Laurier University
in fulfilment of the
thesis requirement for the degree of
Master of Science
in
Mathematics

Waterloo, Ontario, Canada, 2023

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Author's Declaration

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis.

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Abstract

The n-body problem models a system of n-point masses that attract each other via some binary interaction. The $(n+1)$-body problem assumes that one of the masses is located at the origin of the coordinate system. For example, an $(n+1)$-body problem is an ideal model for Saturn, seen as the central mass, and one of its outer rings. A relative equilibrium (RE) is a special solution of the $(n+1)$-body problem where the non-central bodies rotate rigidly about the centre of mass. In rotating coordinates, these solutions become equilibria.

In this thesis we study dynamical aspects of planar $(4+1)$-body systems with the outer four of equal point mass interacting via a modified gravitational "J_2" attraction potential of the form:

$$V(r) = -\frac{m_i m_j}{r}\left(1 + \frac{\epsilon}{r^2}\right), \qquad 0 < \epsilon << 1 \tag{1}$$

where r denotes the distance between masses m_i and m_j.

Using the rotational symmetry as well as the symmetry induced by the mass equality, we are able to describe qualitatively the phase-space of square-shaped dynamics. We further analyze the existence of square-shaped RE and establish conditions necessary for their stability. We find that the RE appear as a saddle-node bifurcation. For a unit central mass and equal outer bodies of mass m, we find that these RE are unstable.

Acknowledgements

First and foremost, I wish to extend my deepest gratitude to my supervisor and mentor, Dr. Cristina Stoica. I am extremely thankful for her guidance, wisdom, friendship, kindness and patience. Dr. Stoica has had a profoundly positive impact on my life; her advice, mentorship and continuing support have allowed me to grow tremendously as a person over the past three years. In life, there are a rare few people that we encounter that change our lives forever and even rarer are those that change our lives forever in a positive way. For me, Cristina is one of these people and I am forever grateful for everything that she has done for me.

I wish to thank all those that helped me reach this milestone. I am thankful for all of my professors and instructors for everything they have taught me. I am very grateful for all of the lab coordinators that taught and supervised me and all of the staff that helped me in one way or another throughout my time at WLU. Finally, I wish to express my gratitude to my colleagues and friends for all of their help, kindness and friendship, which has been so valuable to me.

I wish to extend a special thanks to my Thesis Advisory Committee. Additionally, I wish to thank Dr. Ioannis Haranas for his friendship and for teaching me much of the physics used in this thesis.

Last, but not least, I wish to thank my parents for everything that they have done for me and for their support. I also wish to thank my brother, my extended family and my friends. Finally, I wish to express my gratitude for the unwavering companionship of my dog and cat.

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Chapter 1

Introduction

The n-body problem models a system of n point masses with mutual binary interactions. The $(n+1)$-body problem assumes a central mass M located at the origin of the coordinate system with n masses orbiting it. For example, an $(n+1)$-body problem is an ideal model for Saturn and an outer ring.

In the n-body problem, relative equilibria (RE) are special solutions in which the bodies rotate rigidly about their centre of mass. In rotating coordinates, these solutions become equilibria parametrized by total angular momentum. In celestial mechanics the RE configurations are known as *central configurations*. With few exceptions, the number of possible central configurations in classical Newtonian n-body problems remains an open problem (see [Albouy & al. 2012] and references within, also [Smale 1998]).

In the case of $(n+1)$-body problems, if the outer masses are equal then regular n-gons are RE configurations for gravitational attraction. It has been shown that such RE are unstable regardless of the magnitude of the central mass or the angular velocity if $n \leq 6$ [Pendse 1935]. Later work by [Roberts 2000] and [Vanderbei & al. 2007] confirmed that regular n-gon RE with $n \geq 7$ are rotationally stable provided that the central mass is sufficiently large.

In this thesis we study planar $(4+1)$-body systems with equal point masses as shown in Figure 1.1. A mass M is positioned at the centre of a Cartesian system of coordinates around which four point masses m_i are orbiting. We employ a modified gravitational mutual attraction potential of the form:

$$V(r) = -\frac{m_i m_j}{r}\left(1 + \frac{\epsilon}{r^2}\right), \qquad 0 < \epsilon << 1 \tag{1.1}$$

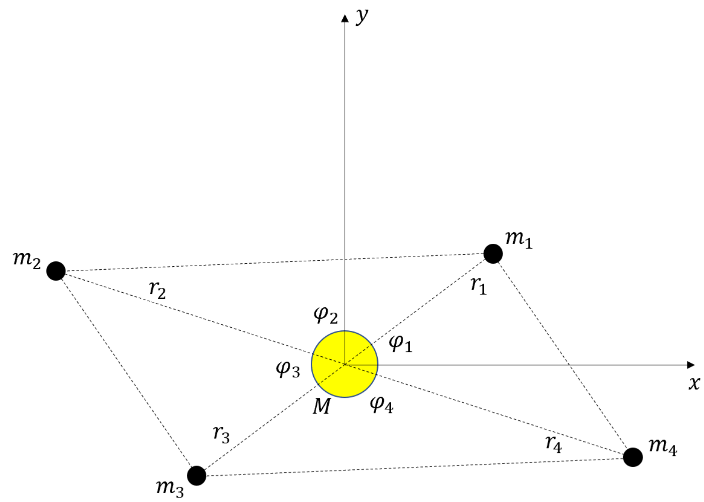


Figure 1.1: Caledonian Problem Configuration

where r denotes the distance between masses m_i and m_j. This augmentation of the Newtonian gravitational law appears in two physically relevant settings:

1. first, as an approximation of the potential manifested between two symmetric ellipsoidal spheroids in a "floating position", that is when their axes of symmetry are parallel to each other, see [Arredondo & al. 2012]. The ϵ-term models the oblateness of the bodies and it is the first term in the expansion in Legendre polynomials of the mutual potential (with ϵ as the "J_2" coefficient; see [Stiefel & al. 1971]);

2. second, as a corrective term emerging from relativity theory, see [Schwarzschild 2008], [Stoica 1997] and [Arredondo & al. 2014] and references within.

For simplicity, we call (1.1) the J_2-potential.

A system of four equal point masses in a parallelogram shaped planar configuration orbiting their centre of mass under gravitational attraction is known as the four-body *Caledonian problem* and was introduced by [Roy & al.2000]. The Caledonian problem was investigated by [Steves & al. 2001], [Széll & al. 2004a], [Széll & al. 2004b] and [Alvarez-Ramirez & al. 2014]. The restriction of the (4 + 1)-body problem to parallelogram-shaped configurations is a Caledonian problem with the inclusion of a non-zero central mass M. The boundedness of the motion of systems in this configuration was studied by [Shoaib 2004] and [Shoaib & al. 2008] and a criterion regarding the "non-escape" of the system was established.

The rhomboidal (4 + 1)-body problem was studied by [Cornelio & al. 2021]. Recently, in the case of homogeneous forces and for $M = 0$, existence of rhomboidal RE was investigated by [Leandro 2019] and by [Lopez & al. 2022] for $M \neq 0$.

The same problem using the classical gravitational attraction between binary pairs of bodies and the Manev potential, i.e. a potential of the form $f(r) = -\frac{1}{r} - \frac{\epsilon}{r^2}$, for the interactions between the central mass and the outer bodies was studied by [Marchesin & al. 2013].

Regardless of the potential between the points, when all non-central point masses are equal, the system admits an invariant dynamical sub-system consisting of motions of square configurations. The latter is a mechanical system with two degrees of freedom with rotational symmetry. Applying reduction, the dynamics reduce to one degree of freedom with angular momentum c and total energy h as parameters. In our work, we fully describe the (homographic) square-shaped solutions in the case of the J_2-potential interaction. We detect the presence of unattainable regions in the configuration space and show that total collisions are possible for non-zero angular momenta. This phenomenon is known in celestial mechanics as the black hole effect (not to be confounded with the relativity black holes) and it is specific to attractive potentials containing terms $-1/r^\alpha, \alpha \geq 2$, see [Delgado & al. 1996], [Pasca & al. 2019], [Stoica 2000] and [Arredondo & al. 2014]. For small angular momenta, we find that there are no RE. As momentum is increased, there exists a threshold value, $c_0 > 0$, marking a saddle-node bifurcation, above which two RE branches appear. Let us denote the distances from the outer masses to the central mass along these RE branches as $r_{1,e}$ and $r_{2,e}$, with $r_{1,e} < r_{2,e}$.

Within an invariant subset of square-shaped configurations, we show that the RE $r_{1,e}$ branch is unstable since it corresponds to a maximum of the effective potential of the square-shaped motions. We further prove that the RE $r_{2,e}$ branch is also unstable. Without loosing generality, we set $M = 1$ and we first consider the linearization along $r_{2,e}$ within the phase-subspace of rhomboidal configurations only. We obtain that for $m > 0.042068394$, the RE $r_{2,e}$ branch is unstable. For $m \in [0, 0.042068394]$ we calculate the linearization along the $r_{2,e}$ branch of the reduced system with 7-degrees of freedom, that is in the full phase-space. Again, using a Taylor expansion in ε of the essential coefficients, we find that the RE $r_{2,e}$ branch is unstable. Thus we conclude:

1. the RE $r_{1,e}$ branch is unstable with respect to perturbations that maintain square-shaped configurations;

2. for $m > 0.042068394$, the RE $r_{2,e}$ branch is stable with respect to perturbations that maintain square-

shaped configurations but unstable with respect to those that break rhomboidal configurations;

3. for $[0, 0.042068394]$ the RE $r_{2,e}$ branch is stable with respect to perturbations that maintain rhomboidal configurations but unstable with respect to perturbations that break these symmetric configurations.

The thesis is organized as follows: we start by setting up the general planar $(4+1)$-body problem within the Lagrangian framework. We use the rotational symmetry to reduce the system to 7 degrees of freedom. Using the Legendre transform, we pass then to the Hamiltonian formulation. Next, we introduce the mass symmetry and write the equations of motion for solutions of parallelogram-shaped configuration. We briefly consider linear motions, listing all possible motions. We then focus on motions of square-shaped configurations. Employing the J_2 potential, we analyze the level sets of the effective potential as function of the angular momentum and energy. Consequently we provide a full qualitative description of the Hills' regions of motions. We calculate the value of the angular momentum which marks a saddle-node bifurcation and the birth of two branches of RE. We deduce that one of these branches is unstable, as it is a maximum of the effective potential. Chapter 3 investigates aspects of RE. First we prove that given the pair-wise equal masses, within the rhomboidal configuration solutions, a RE is a square if and only if *all* masses are equal. We further consider the stability of the square-shaped RE in the case of J_2 potential interactions. As aforementioned, one of the RE branches is unstable. For calculating the stability of the second branch we pass to polar coordinates. In this way, the linearization operator (matrix) at this RE takes a particular simple form. We write the characteristic polynomial and using a Taylor expansion in ε of the essential coefficients, we deduce that if the latter is sufficiently small and $m > 0.042068394$ the (second) RE branch is unstable. Further, for $m \in [0, 0.042068394]$ we calculate the linearization along the (second) RE branch of the reduced 7-degree of freedom system and deduce that for these values, this RE is unstable as well.

The Thesis is augmented by five Appendices. The first Appendix contains the Maple Code used for determining the RE and calculating the existence and stability of the (second) branch of RE. The next Appendices present the theoretical mathematical-physics background employed in our work.

Chapter 2

Model Construction

2.1 General Configuration

In Cartesian coordinates, a general (4 + 1)-body problem can be modelled as shown in the following diagram: In Figure 2.4, the central mass M is centered at the origin. Each point mass P_i has mass m_i for

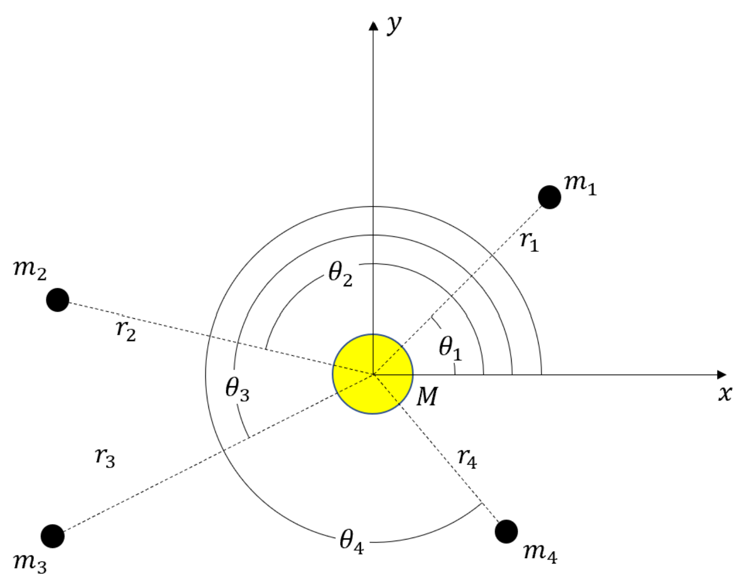


Figure 2.1: General Configuration in Cartesian Coordinates

$i = 1, 2, 3, 4$. The distance from the origin to each point mass is r_i. For $j = 1, 2, 3, 4$, the angle θ_j is the angle formed between the positive x-axis and r_j. All configurations in this thesis are planar.

2.2 Equations of Motion

In polar coordinates, the Lagrangian of a general $(4+1)$-body problem is:

$$L = T - U = \sum_{i=1}^{4} \frac{m_i}{2}\left(\dot{r}_i^2 + r_1^2\dot{\theta}_i^2\right) - U(r_1, r_2, r_3, r_4, \theta_1, \theta_2, \theta_3, \theta_4) \tag{2.1}$$

The potential U is defined as follows where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function on its domain:

$$U(r_1, r_2, r_3, r_4, \theta_1, \theta_2, \theta_3, \theta_4) = \sum_{1\le i\le 4}^{4} M m_i f(r_i) + \sum_{1\le i<j\le 4}^{4} m_i m_j f(d_{ij}) \tag{2.2}$$

In (2.2) d_{ij} is given as:

$$d_{ij} = \sqrt{r_i^2 + r_j^2 - 2r_i r_j \cos(\theta_j - \theta_i)}, \quad 1 \le i < j \le 4 \tag{2.3}$$

We apply the following change of variables:

$$\varphi_1 = \theta_1, \qquad \varphi_2 = \theta_2 - \theta_1, \qquad \varphi_3 = \theta_3 - \theta_1, \qquad \varphi_4 = \theta_4 - \theta_1 \tag{2.4}$$

which yields:

$$\begin{aligned} &L(r_1, r_2, r_3, r_4, \varphi_1, \varphi_2, \varphi_3, \varphi_4) \\ &= T(r_1, r_2, r_3, r_4, \varphi_1, \varphi_2, \varphi_3, \varphi_4) - U(r_1, r_2, r_3, r_4, \varphi_2, \varphi_3, \varphi_4) \\ &= \frac{m_1}{2}(\dot{r}_1^2 + r_1^2\dot{\varphi}_1^2) + \frac{m_2}{2}(\dot{r}_2^2 + r_2^2(\dot{\varphi}_1 + \dot{\varphi}_2)^2) \\ &+ \frac{m_3}{2}(\dot{r}_3^2 + r_3^2(\dot{\varphi}_1 + \dot{\varphi}_3)^2) + \frac{m_4}{2}(\dot{r}_4^2 + r_4^2(\dot{\varphi}_1 + \dot{\varphi}_4)^2) \\ &- U(r_1, r_2, r_3, r_4, \varphi_2, \varphi_3, \varphi_4) \end{aligned} \tag{2.5}$$

Furthermore, applying the change of variables given in (2.4) to (2.3) yields:

$$\begin{aligned} d_{1j} &= \sqrt{r_1^2 + r_j^2 - 2r_1 r_j \cos(\varphi_j)}, \quad j = 2, 3, 4 \\ d_{2j} &= \sqrt{r_2^2 + r_j^2 - 2r_2 r_j \cos(\varphi_j - \varphi_2)}, \quad j = 3, 4 \\ d_{34} &= \sqrt{r_3^2 + r_4^2 - 2r_3 r_4 \cos(\varphi_4 - \varphi_3)} \end{aligned} \tag{2.6}$$

We apply the Legendre transform to obtain the Hamiltonian, $H = K + U$, where:

$$\begin{aligned} K =& \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2m_3} + \frac{p_4^2}{2m_4} + \frac{(p_{\varphi_1} - p_{\varphi_2} - p_{\varphi_3} - p_{\varphi_4})^2}{2m_1 r_1^2} \\ &+ \frac{p_{\varphi_2}^2}{2m_2 r_2^2} + \frac{p_{\varphi_3}^2}{2m_3 r_3^2} + \frac{p_{\varphi_4}^2}{2m_4 r_4^2} \end{aligned} \tag{2.7}$$

Since H is autonomous along any solution, the total energy is conserved, that is $H(t) = h$ where h is a constant. Furthermore, the system is rotationally symmetric which implies that angular momentum is conserved. This is an immediate result of φ_1 being a cyclic coordinate, that is, it does not appear in the Hamiltonian. The equation of motion for p_{φ_1} is:

$$\dot{p}_{\varphi_1} = -\frac{\partial U}{\partial \varphi_1} = 0 \tag{2.8}$$

This implies that along any solution $p_{\varphi_1}(t) = p_{\varphi_1}(t_0) = \hat{c}$ where $\hat{c}$ is a constant. We then substitute $p_{\varphi_1} = \hat{c}$ into the Hamiltonian in order to obtain a reduced Hamiltonian,

$$H_r = K_r + U_r \tag{2.9}$$

where $U_r = U$ and:

$$\begin{aligned} K_r =& \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{p_3^2}{2m_3} + \frac{p_4^2}{2m_4} + \frac{(\hat{c} - p_{\varphi_2} - p_{\varphi_3} - p_{\varphi_4})^2}{2m_1 r_1^2} \\ &+ \frac{p_{\varphi_2}^2}{2m_2 r_2^2} + \frac{p_{\varphi_3}^2}{2m_3 r_3^2} + \frac{p_{\varphi_4}^2}{2m_4 r_4^2} \end{aligned} \tag{2.10}$$

The Hamiltonian (2.9) is a 7-degrees of freedom mechanical system with a configuration space given by $(r_1, r_2, r_3, r_3, \varphi_2, \varphi_3, \varphi_4) \in \mathbb{R}^4 \times \mathcal{S}^1 \times \mathcal{S}^3 \times \mathcal{S}^2$ and $\hat{c} \in \mathbb{R}$ as a parameter.

2.3 Parallelogram-Shaped (Caledonian) Configurations

We consider parallelogram-shaped shaped configurations, as shown in Figure 2.2.

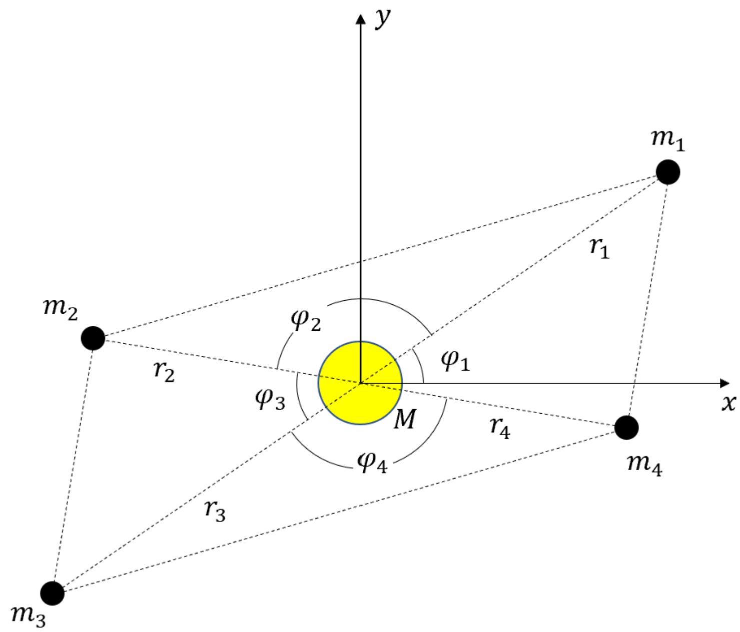


Figure 2.2: Parallelogram-Shaped Configuration

We modify the system in (2.1) by setting $m_1 = m_3$ and $m_2 = m_4$. The phase space subset defined by:

$$\begin{aligned} &r_1 = r_3\,,\ r_2 = r_4\,,\ \varphi_3 = \pi\,,\ \varphi_4 = \pi + \varphi_2 \\ &p_1 = p_3\,,\ p_2 = p_4\,,\ p_{\varphi_2} = p_{\varphi_4} \end{aligned} \tag{2.11}$$

is invariant under the flow for any distance dependent mutual potential $f(r)$. Within this invariant subset, we have that $p_{\varphi_1} = \hat{c} = 2p_{\varphi_2} + 2p_{\varphi_3}$. We then denote:

$$\varphi_2 = \varphi, \quad p_{\varphi_2} = p_\varphi, \quad \hat{c} = c/2 \tag{2.12}$$

The parallelogram-shaped motions are given by the reduced Hamiltonian:

$$H_{\mathrm{red}}(r_1, r_2, \varphi, p_1, p_2, p_\varphi) = \frac{p_1^2}{m_1} + \frac{p_2^2}{m_2} + \frac{(c - p_\varphi)^2}{m_1 r_1^2} + \frac{p_\varphi^2}{m_2 r_2^2} + U(r_1, r_2, \varphi) \tag{2.13}$$

where the potential is:

$$
\begin{aligned}
U(r_1, r_2, \varphi) =& 2Mm_1 f(r_1) + 2Mm_2 f(r_2) + 2m_1 m_2 f\left(\sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos(\varphi)}\right) \\
&+ 2m_1 m_2 f\left(\sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\varphi)}\right) + m_1^2 f(2r_1) + m_2^2 f(2r_2)
\end{aligned}
\tag{2.14}
$$

The equations of motion are:

$$\dot{r}_1 = \frac{\partial H_{\text{red}}}{\partial p_1} = \frac{2p_1}{m_1} \tag{2.15}$$

$$\dot{r}_2 = \frac{\partial H_{\text{red}}}{\partial p_2} = \frac{2p_2}{m_2} \tag{2.16}$$

$$\dot{\varphi} = \frac{\partial H_{\text{red}}}{\partial p_\varphi} = -\frac{2(c - p_\varphi)}{m_1 r_1^2} + \frac{2p_\varphi}{m_2 r_2^2} \tag{2.17}$$

$$\dot{p}_1 = -\frac{\partial H_{\text{red}}}{\partial r_1} = \frac{2(c - p_\varphi)^2}{m_1 r_1^3} - \frac{\partial U(r_1, r_2, \varphi)}{\partial r_1} \tag{2.18}$$

$$\dot{p}_2 = -\frac{\partial H_{\text{red}}}{\partial r_2} = \frac{2p_\varphi^2}{m_2 r_2^3} - \frac{\partial U(r_1, r_2, \varphi)}{\partial r_2} \tag{2.19}$$

$$\dot{p}_\varphi = -\frac{\partial H_{\text{red}}}{\partial \varphi} = -\frac{\partial U(r_1, r_2, \varphi)}{\partial \varphi} \tag{2.20}$$

2.4 Linear Motions

We begin by writing the equations of motion for the pair (φ, p_φ), denoted as (2.17) and (2.20) respectively:

$$\dot{\varphi} = \frac{2p_\varphi}{m_2 r_2^2} - \frac{2(c - p_\varphi)}{m_1 r_1^2} \tag{2.21}$$

$$
\begin{aligned}
\dot{p}_\varphi = 2m_1 m_2 r_1 r_2 \Bigg(& \frac{f'\left(\sqrt{r_1^2 + r_2^2 + r_1 r_2 \cos(\varphi)}\right)}{\sqrt{r_1^2 + r_2^2 + r_1 r_2 \cos(\varphi)}} \\
& - \frac{f'\left(\sqrt{r_1^2 + r_2^2 - r_1 r_2 \cos(\varphi)}\right)}{\sqrt{r_1^2 + r_2^2 - r_1 r_2 \cos(\varphi)}} \Bigg) \sin(\varphi)
\end{aligned}
\tag{2.22}
$$

If we set the angular momentum c to zero, we deduce that linear configurations, that is, motions with $\varphi = 0$ and $p_\varphi = 0$, form an invariant subset, assuming that we include the constraint $r_1 \neq r_2$ (to avoid collisions). Within this subset, the dynamics are given by a system with two degrees of freedom where:

$$H_{\text{red lin}}(r_1, r_2, p_1, p_2) = \frac{p_1^2}{m_1} + \frac{p_2^2}{m_2} + U_{\text{lin}}(r_1, r_2) \tag{2.23}$$

and

$$\begin{aligned} U_{\text{lin}}(r_1, r_2) = 2Mm_1 f(r_1) + 2Mm_2 f(r_2) + 2m_1 m_2 f(r_1 + r_2) \\ + 2m_1 m_2 f(|r_1 - r_2|) + m_1^2 f(2r_1) + m_2^2 f(2r_2) \end{aligned} \tag{2.24}$$

In the case of the J_2-potential, i.e. $f(r) = -\frac{1}{r} - \frac{\epsilon}{r^3} < 0$, we use the energy integral to conclude that $H_{\text{red lin}}(r_1, r_2, p_1, p_2) = \textit{const.} = h$ from which we deduce that $U_{\text{lin}}(r_1, r_2) \le h$. Since the potential is an attractive force, we deduce that for any $h < 0$, the system's motion is bounded, resulting in one of three scenarios (where the bodies form a central configuration at all times):

- The inner masses (the red masses labelled m_{in} in Figure 2.3) collide with the central mass (the yellow mass labelled M in Figure 2.3).
- The inner masses (the red masses labelled m_{in} in Figure 2.3) collide with the outer masses (the blue masses labelled m_{out} in Figure 2.3).
- The system ends in total collision.

In the case where $h \ge 0$, the system's motion will either end in partial or total collision or will be asymptotically bounded and escape to infinity.

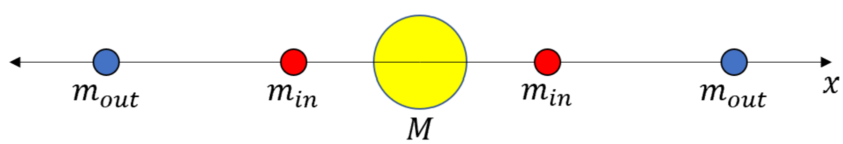


Figure 2.3: Linear Configuration

2.5 $(4+1)$-Square-Shaped Configuration Motions

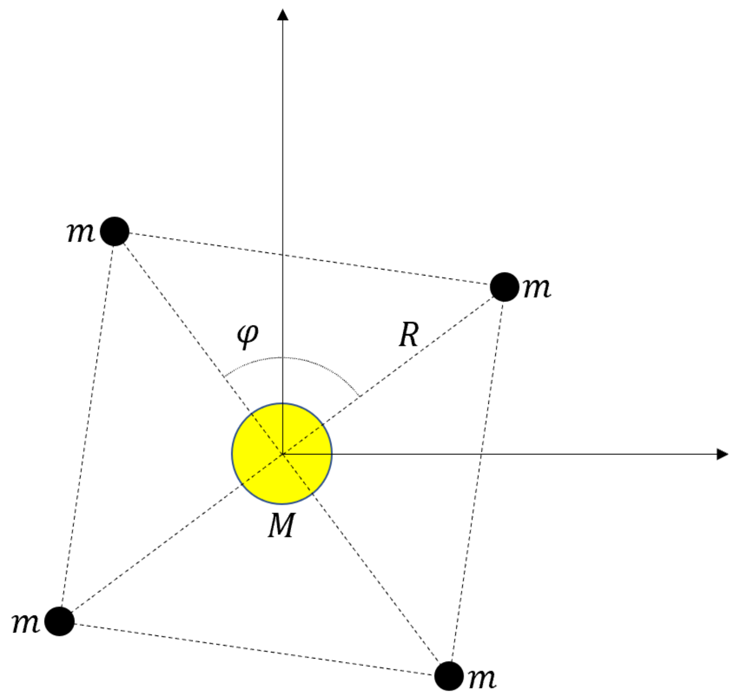


Figure 2.4: Square Configuration

In the case where $m_1 = m_2 = m_3 = m_4 = m$, it can be shown that the subset containing square configurations described by $r_1 = r_2 = r, p_{r_1} = p_{r_2} = p_r, \varphi = \frac{\pi}{2}, p_\varphi = \frac{c}{2}$, is invariant. This subset is a subset of the phase space subset described in (2.11). In this case, the dynamics are given by a system with one degree of freedom where:

$$H_{\text{red sq}}(r, p_r) = \frac{2p_r^2}{m} + \frac{c^2}{2mr^2} + U_{\text{sq}}(r) \tag{2.25}$$

and

$$U_{\text{sq}}(r) = 4Mmf(r) + 4m^2 f(\sqrt{2}r) + 2m^2 f(2r) \tag{2.26}$$

Along any solution $(r(t), p_r(t))$, we have the conservation of energy:

$$H_{\text{red sq}}(r(t), p_r(t)) = \frac{2p_r^2(t)}{m} + \frac{c^2}{2mr^2(t)} + U_{\text{sq}}(r(t)) = h \tag{2.27}$$

Without loss of generality, we consider $c > 0$. Given the binary potential function f, a qualitative description of the square-shaped motions is given by considering the effective potential $U_{\text{eff sq}}$ where:

$$\begin{aligned} U_{\text{eff sq}}(r;c) &= \frac{c^2}{2mr^2} + U_{\text{sq}}(r) \\ &= \frac{c^2}{2mr^2} + 4Mmf(r) + 4m^2 f(\sqrt{2}r) + 2m^2 f(2r) \end{aligned} \tag{2.28}$$

The regions of motion are sets that depend on the total energy h and the angular momentum c as parameters [Arnold 1989]:

$$\{r > 0 | U_{\text{eff sq}}(r;c) \leq h\} \tag{2.29}$$

Physically, these sets describe the radius r from the outer masses to the central mass M when the energy h is fixed. The motions consist of rotations about the central mass M, with the radius r varying in time. We take:

$$f(r) = -\frac{1}{r} - \frac{\epsilon}{r^3} \tag{2.30}$$

and thus:

$$U_{\text{eff sq}}(r;c) = \frac{c^2}{2mr^2} - 4Mm\left(\frac{1}{r} + \frac{\epsilon}{r^3}\right) - 4m^2\left(\frac{1}{\sqrt{2}r} + \frac{\epsilon}{2\sqrt{2}r^3}\right) - 2m^2\left(\frac{1}{2r} + \frac{\epsilon}{8r^3}\right) \tag{2.31}$$

The relative equilibria configurations are found as the roots of $dU_{\text{eff sq}}(r;c) = 0$. A direct calculation using Maple (see Appendix A.1) yields the pair:

$$r_{i,e} = \frac{c^2 \pm \sqrt{c^4 - \varepsilon((51 + 18\sqrt{2})m^6 + (144\sqrt{2} + 60)Mm^5 + 192M^2m^4)}}{2m^2(4M + m(2\sqrt{2} + 1))} \tag{2.32}$$

where $i = 1, 2$, and the roots are real if $c \geq c_0$,

$$c_0^4 = \varepsilon((51 + 18\sqrt{2})m^6 + (144\sqrt{2} + 60)Mm^5 + 192M^2m^4) \tag{2.33}$$

We find that for $c < c_0$, $U_{\text{eff sq}}(r;c)$ admits no critical points since we assume, without loss of generality, that $c > 0$, that is angular momenta are positive. As c increases to the threshold such that $c = c_0$, one critical point appears. As c increases over the threshold such that $c > c_0$, two critical points appear:

$$r_{1,e} < r_{2,e} \tag{2.34}$$

as determined using (2.32). The two critical points are comprised of a maximum and a minimum respectively as shown in Figure 2.5. These critical points are central configurations corresponding to the fixed momentum c. As dynamical solutions, they are relative (rotating) equilibria (RE) with the four external bodies in a uniform rotating motion about the centre M.

Proposition 2.5.1. *Consider the square $(4+1)$-problem with a J_2-augmented potential. Then angular momenta $c = c_0$ marks a saddle-node bifurcation of RE [Guckenheimer & al.]. The RE of smaller side length $r_{1,e}$ is unstable.*

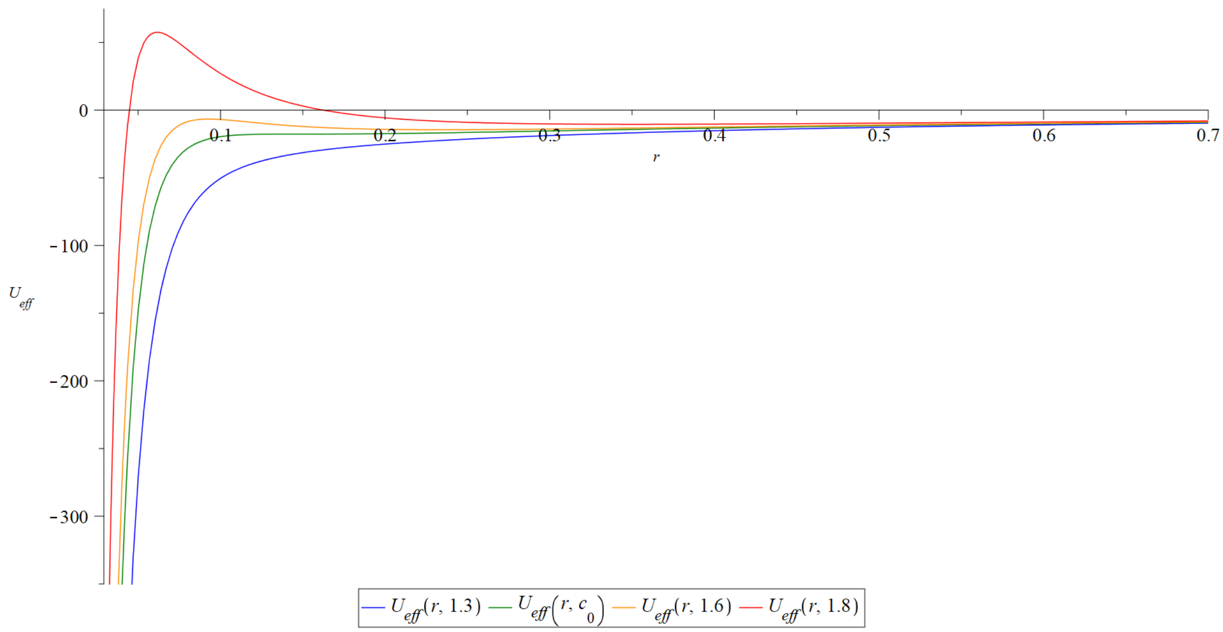


Figure 2.5: $U_{\text{eff}}(r; c)$ for $M = m = 1, \epsilon = 0.01, c_0 = 1.518793902$. Fix total energy $h < 0$. For low momenta $c < c_0$, all motions are bounded and there are no RE (blue curve). If $c = c_0$, all motions are bounded and there is one RE (green curve). As c is increased above c_0, all motions are bounded with two RE present (orange & red curves); forbidden regions are present. For high momenta, unbounded motions are possible.

Remark 2.5.1. *The stability of the $r_{2e}(c)$ branch is determined by considering the corresponding linearization in the larger rhomboidal-shaped configuration's invariant subset and further in the full phase-space.*

Remark 2.5.2. *We note that there are motions that end in collision with non-zero angular momentum. The bodies perform an infinite spin around the central mass and collision is approached asymptotically. This phenomenon is known in celestial mechanics as the black hole effect and it is specific to attractive potentials containing terms $-1/r^\alpha, \alpha \geq 2$; see [Delgado & al. 1996], [Pasca & al. 2019], [Stoica 2000] and [Arredondo & al. 2014].*

From a qualitative standpoint, the regions of motion are as follows:

1. If $c \leq c_0$, then:

(a) for $h < 0$, we have $r \in (0, r_{max}(h,c))$.

(b) for $h \geq 0$, we have $r \in (0, \infty)$.

2. If $c > c_0$ and for c such that $U_{\text{eff}}(r_{1,e}; c) \leq 0$, then:

   (a) for $h \leq U_{\text{eff}}(r_{2,e}, c)$, we have $r \in (0, r_{max}(h,c)]$.

   (b) for $U_{\text{eff}}(r_{2,e}; c) \leq h < U_{\text{eff}}(r_{1,e}; c)$, the radius r is trapped either in $r \in (0, r_{1,max}(h,c)]$ or in a basin $r \in [r_{min}(h,c), r_{2,max}(h,c)]$ depending on the initial position. Radii $(r_{1,max}(h,c), r_{min}(h,c))$ are unattainable.

   (c) for $h = U_{\text{eff}}(r_{1e}; c)$, we have $r \in (0, r_{max}(h,c)]$ with an unstable RE in $r_{1,e}$.

   (d) for $h > U_{\text{eff}}(r_{1,e}; c) < h < 0$, the motion is bounded and $r \in (0, r_{max}(h,c)]$.

   (e) for $h \geq 0$, the motion is unbounded and $r \in (0, \infty)$.

3. If $c > c_0$ and for c such that $U_{\text{eff}}(r_{1,e}; c) > 0$, then:

   (a) for $h < U_{\text{eff}}(r_{2,e}, c)$, we have $r \in (0, r_{max}(h,c)]$.

   (b) for $U_{\text{eff}}(r_{2,e}; c) \leq h < 0$, the radius r is trapped either in $r \in (0, r_{1,max}(h,c)]$ or in a basin $r \in [r_{min}(h,c), r_{2,max}(h,c)]$ depending on the initial position. Radii $[r_{1,max}(h,c), r_{min}(h,c)]$ are unattainable.

   (c) for $0 \leq h < U_{\text{eff}}(r_{1,e}; c)$, the radius r is trapped either in $r \in (0, r_{1,max}(h,c)]$ or is unbounded, that is $r \in [r_{min}(h,c), \infty)$. Radii $[r_{1,max}(h,c), r_{min}(h,c)]$ are unattainable.

   (d) for $h = U_{\text{eff}}(r_{1e}; c)$, we have $r \in (0, r_{max}(h,c)]$ with an unstable RE in $r_{1,e}$.

   (e) for $h > U_{\text{eff}}(r_{1,e}; c)$, the motion is unbounded, that is $r \in (0, \infty)$. No RE are present.

Chapter 3

Relative Equilibria and Stability

3.1 Parallelogram Configurations and RE

The RE of parallelogram-shaped configurations are determined by equating (2.15) to (2.20) to zero. For a fixed angular momentum c, solutions of the configuration equations determine the central configuration $(r_1(c), r_2(c), \varphi(c))$. The associated RE solution is:

$$(r_1, r_2, \varphi, p_1, p_2, p_\varphi) = (r_1(c), r_2(c), \varphi(c), 0, 0, p_\varphi(c)) \tag{3.1}$$

where

$$p_\varphi(c) = \frac{cm_2 r_2^2(c)}{m_1 r_1^2(c) + m_2 r_2^2(c)} \tag{3.2}$$

and $(r_1(c), r_2(c), \varphi(c))$ are given by the central configuration equations:

$$c^2 = \frac{(m_1 r_1^2 + m_2 r_2^2)^2}{2m_1 r_1} \frac{\partial U}{\partial r_1}(r_1, r_2, \varphi) \tag{3.3}$$

$$c^2 = \frac{(m_1 r_1^2 + m_2 r_2^2)^2}{2m_2 r_2} \frac{\partial U}{\partial r_2}(r_1, r_2, \varphi) \tag{3.4}$$

$$\frac{\partial U(r_1, r_2, \varphi)}{\partial \varphi} = 0 \tag{3.5}$$

The configuration $(r_1(c), r_2(c), \varphi(c))$ of the RE is called a *central configuration* and is given by the central configuration equations (3.3)-(3.5). We must solve Equations (3.3) to (3.5) in order to find relative equilibria for parallelogram-shaped configurations. However, we restrict the parallelogram-shaped configurations to rhomboidal or colinear configurations, i.e. $\varphi = \frac{\pi}{2}$ or 0. This is done to ensure that (3.5) holds true for any values of the variables r_1, r_2 and parameters m_1, m_2. Therefore, we must solve (3.6) and (3.7) numerically for (r_{1e}, r_{2e}, c) for each specified c value.

$$c^2 = \frac{(m_1 r_1^2 + m_2 r_2^2)^2}{2m_1 r_1} \Bigg(2Mm_1 f'(r_1) + 2m_1^2 f'(2r_1) + 2m_1 m_2 f'\left(\sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos(\varphi)}\right) \left(\frac{2r_1 + 2r_2 \cos(\varphi)}{\sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos(\varphi)}} \right) \Bigg) \tag{3.6}$$

$$c^2 = \frac{(m_1 r_1^2 + m_2 r_2^2)^2}{2m_2 r_2} \Bigg(2Mm_2 f'(r_2) + 2m_2^2 f'(2r_2) + 2m_1 m_2 f'\left(\sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos(\varphi)}\right) \left(\frac{2r_2 + 2r_1 \cos(\varphi)}{\sqrt{r_1^2 + r_2^2 + 2r_1 r_2 \cos(\varphi)}} \right) \Bigg) \tag{3.7}$$

For each c value, an equilibrium solution of the form (r_{1e}, r_{2e}, c) needs to be determined using numerical methods. We completed some work on this problem, however, it is beyond the scope of this thesis. Instead we study square configurations and provide the necessary and sufficient conditions for the existence of square RE.

Proposition 3.1.1. *Within the rhomboidal configuration solutions, a RE is a square (i.e. $r_1 = r_2$ and $\varphi = \pi/2$) if and only if $m_1 = m_2$.*

Proof: If $m_1 = m_2$ then it is immediate that there is a square-shaped RE. For the reciprocal, let us denote $r_1 = r_2 = r$. Given that the RE is a square, we have that $\varphi = \pi/2$, and so $\cos\varphi = 0$. Also, we have

$$\left.\frac{\partial U}{\partial r_1}\right|_{r_1=r_2=r} = \left.\frac{\partial U}{\partial r_2}\right|_{r_1=r_2=r} \tag{3.8}$$

and so, since $c \neq 0$, from the equations of motion,

$$c^2 = \frac{(m_1 r_1^2 + m_2 r_2^2)^2}{2m_1 r_1} \frac{\partial U}{\partial r_1} = \frac{(m_1 r_1^2 + m_2 r_2^2)^2}{2m_2 r_2} \frac{\partial U}{\partial r_2} \implies m_1 = m_2 \; \square$$

3.2 Square Configurations and RE

In the case of parallelogram-shaped configuration, we introduce the polar coordinates:

$$r_1 = R\cos(\psi), \quad r_2 = R\sin(\psi) \qquad R > 0, \;\; \psi \in \left(0, \frac{\pi}{2}\right) \tag{3.9}$$

and

$$p_{r_1} = p_R \cos(\psi) - \frac{p_\varphi}{R}\sin(\psi), \quad p_{r_2} = p_R \sin(\psi) - \frac{p_\varphi}{R}\cos(\psi) \tag{3.10}$$

Square configurations are only possible if $m_1 = m_2$ which implies that $\psi = \pi/4$ and $\varphi = \pi/2$. Applying the change of coordinates to (2.13) and setting $m_1 = m_2 = m$, we obtain:

$$H(R,\psi,\varphi,p_R,p_\psi,p_\varphi) = \frac{1}{m}\left(p_R^2 + \frac{p_\psi^2}{R^2}\right) + \frac{(c-p_\varphi)^2}{mR^2\cos^2(\psi)} + \frac{p_\varphi^2}{mR^2\sin^2(\psi)} + U(R,\psi,\varphi) \tag{3.11}$$

with

$$\begin{aligned} U(R,\psi,\varphi) &= 2Mmf(R\cos(\psi)) + 2Mmf(R\sin(\psi)) \\ &\quad + 2m^2 f\left(R\sqrt{1+\sin(2\psi)\cos(\varphi)}\right) + 2m^2 f\left(R\sqrt{1-\sin(2\psi)\cos(\varphi)}\right) \\ &\quad + m^2 f(2R\cos(\psi)) + m^2 f(2R\sin(\psi)) \end{aligned} \tag{3.12}$$

Rescaling $H \to (1/2)H$, we obtain:

$$H(R,\psi,\varphi,p_R,p_\psi,p_\varphi) = \frac{1}{2m}\left(p_R^2 + \frac{p_\psi^2}{R^2}\right) + \frac{(c-p_\varphi)^2}{2mR^2\cos^2(\psi)} + \frac{p_\varphi^2}{2mR^2\sin^2(\psi)} + \hat{U}(R,\psi,\varphi) \tag{3.13}$$

with

$$\begin{aligned} \hat{U}(R,\psi,\varphi) &= Mmf(R\cos(\psi)) + Mmf(R\sin(\psi)) \\ &\quad + m^2 f\left(R\sqrt{1+\sin(2\psi)\cos(\varphi)}\right) + m^2 f\left(R\sqrt{1-\sin(2\psi)\cos(\varphi)}\right) \\ &\quad + \left(\frac{m^2}{2}\right) f\left(2R\cos(\psi)\right) + \left(\frac{m^2}{2}\right) f\left(2R\sin(\psi)\right) \end{aligned} \tag{3.14}$$

3.3 Stability of Square RE

The stability of a rotationally symmetric Hamiltonian system, modulo rotations, is determined by computing the eigenvalues of the linearization matrix L of the associated reduced Hamiltonian (3.13). The eigenvalues of L at a RE of a Hamiltonian system come in quadruples $\lambda_{1,2,3,4} = \pm\alpha \pm i\beta$. Spectral stability is obtained if the real part of all of the eigenvalues is zero. A complete discussion of the theory used here is provided in Appendix C.

Here we determine the stability of square-shaped configuration RE within the parallelogram-shaped invariant motions.

The linearization at a RE, $(R_0, \pi/4, \pi/2, 0, 0, c/2)$ is:

$$\begin{bmatrix} 0 & 0 & 0 & \frac{1}{m} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{mR_0^2} & 0 \\ 0 & -\frac{4c}{mR_0^2} & 0 & 0 & 0 & \frac{4}{mR_0^2} \\ \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & \sigma & 0 & 0 & 0 & \frac{4c}{mR_0^2} \\ 0 & 0 & \tau & 0 & 0 & 0 \end{bmatrix} \tag{3.15}$$

where

$$\rho = -\frac{3c^2}{mR_0^4} - \frac{\partial^2 U}{\partial R^2}\bigg|_{RE} = -\frac{3c^2}{mR_0^4} - Mmf''\left(R_0\frac{\sqrt{2}}{2}\right) - 2m^2 f''(R_0) - 2m^2 f''(R_0\sqrt{2})$$

$$\sigma = -\frac{4c^2}{mR_0^2} - \frac{\partial^2 U}{\partial \psi^2}\bigg|_{RE} = -\frac{4c^2}{mR_0^2} + \sqrt{2}R_0 Mmf'\left(R_0\frac{\sqrt{2}}{2}\right) - R_0^2 Mmf''\left(R_0\frac{\sqrt{2}}{2}\right) + \sqrt{2}R_0 m^2 f'(\sqrt{2}R_0) - 2R_0^2 m^2 f''(\sqrt{2}R_0)$$

$$\tau = -\frac{\partial^2 U}{\partial \varphi^2}\bigg|_{RE} = \frac{-R_0^2 m^2 f''(R_0) + R_0 m^2 f'(R_0)}{2}$$

For the potential function f:

$$f(r) = -\frac{1}{r} - \frac{\varepsilon}{r^3} \tag{3.16}$$

we obtain the following:

$$\rho = \frac{4\sqrt{2}m^2\left(m\left(\frac{R_0^2}{4} + \frac{3\varepsilon}{4}\right) + M(R_0^2 + 12\varepsilon)\right) + m^3(4R_0^2 + 24\varepsilon) - 3c^2 R_0}{mR_0^5}$$

$$\sigma = \frac{24\sqrt{2}m^2\left(m\left(\frac{R_0^2}{4} + \frac{5\varepsilon}{8}\right) + M(R_0^2 + 10\varepsilon)\right) - 16c^2 R_0}{4mR_0^3}$$

$$\tau = \frac{3m^2(R_0^2 + 5\varepsilon)}{2R_0^3}$$

The characteristic polynomial is:

$$P(\lambda) = \frac{(\sqrt{2}R_0^5 m^2\lambda^2 - \alpha)(2\sqrt{2}R_0^{10}m^2\lambda^4 + \beta\lambda^2 + \gamma)}{4m^4 R_0^{15}} \tag{3.17}$$

where

$$\alpha = 4\sqrt{2}R_0^2m^3 + 8MR_0^2m^2 + 24\sqrt{2}\varepsilon m^3 + 2R_0^2m^3 + 96M\varepsilon m^2 - 3\sqrt{2}R_0c^2 + 6\varepsilon m^3$$

$$\beta = -12\sqrt{2}R_0^7m^3 - 24MR_0^7m^2 - 60\sqrt{2}R_0^5\varepsilon m^3 - 6R_0^7m^3 - 240MR_0^5\varepsilon m^2 + 8\sqrt{2}R_0^6c^2 - 15R_0^5\varepsilon m^3$$

$$\gamma = 144MR_0^4m^3 + 36R_0^4m^4 + 2160MR_0^2\varepsilon m^3 + 270R_0^2\varepsilon m^4 + 7200M\varepsilon^2m^3 + 450\varepsilon^2m^4$$

The $(\sqrt{2}R_0^5m^2\lambda^2 - \alpha)$ term of the characteristic polynomial (3.17) is associated with perturbations that preserve square configurations. We find (using Maple in Appendix A.3) that for all permitted values of c, α is negative and thus $(\sqrt{2}R_0^5m^2\lambda^2 - \alpha) = 0$ always has two imaginary solutions. This is in agreement with the results of Section 2.5, where $r_{2,e}$ is shown to be stable with respect to perturbations that preserve square configurations. We substitute y for λ^2 and obtain:

$$P(y) = \frac{(\sqrt{2}R_0^5m^2y - \alpha)(2\sqrt{2}R_0^{10}m^2y^2 + \beta y + \gamma)}{4m^4R_0^{15}} \tag{3.18}$$

We isolate the following piece of (3.18) and obtain:

$$Q(y) = (2\sqrt{2}R_0^{10}m^2y^2 + \beta y + \gamma) \tag{3.19}$$

The perturbations associated with $Q(y)$ are those that preserve parallelogram-shaped configurations. We analyze $Q(y) = [Ay^2 + By + C]$ where:

$$A = 2\sqrt{2}m^2R_0^{10} \tag{3.20}$$

$$B = -12\sqrt{2}R_0^7m^3 - 24MR_0^7m^2 - 60\sqrt{2}R_0^5\varepsilon m^3 - 6R_0^7m^3 - 240MR_0^5\varepsilon m^2 + 8\sqrt{2}R_0^6c^2 - 15R_0^5\varepsilon m^3 \tag{3.21}$$

$$C = 144MR_0^4m^3 + 36R_0^4m^4 + 2160MR_0^2\varepsilon m^3 + 270R_0^2\varepsilon m^4 + 7200M\varepsilon^2m^3 + 450\varepsilon^2m^4 \tag{3.22}$$

For stability (within parallelogram-shaped motions), we require the roots $y_{1,2}$ of $Ay^2 + By + C = 0$ to be real and negative; for this to be true the following three conditions must be satisfied:

- $B^2 - 4AC \geq 0$
- $S = -B/A < 0$
- $P = C/A > 0$

As shown in Section 2.5, when the two branches of RE exist, the smaller RE, denoted $r_{1,e}$ in Section 2.5, corresponding here to $R_{1,e}$, is an unstable RE. The branch of larger RE, denoted $r_{2,e}$ in Section 2.5 and is given by:

$$r_{2,e} = \frac{c^2 + \sqrt{c^4 - \varepsilon((51+18\sqrt{2})m^6 + (144\sqrt{2}+60)Mm^5 + 192M^2m^4)}}{2m^2(4M + m(2\sqrt{2}+1))} \tag{3.23}$$

The RE is multiplied by $\sqrt{2}$ to account for the change to polar coordinates. Without loss of generality, we set $M = 1$ and obtain the larger RE, which we denote as $R_{2,e}$:

$$R_{2,e} = \sqrt{2}\left(\frac{c^2 + \sqrt{c^4 - \varepsilon((51+18\sqrt{2})m^6 + (144\sqrt{2}+60)Mm^5 + 192M^2m^4)}}{2m^2(4M + m(2\sqrt{2}+1))}\right) \tag{3.24}$$

We consider the discriminant of $Q(y)$, denoted as Δ and determined using Maple as shown in Appendix A.2. We factor out R_0^{10} from the discriminant since $R_0 > 0$ and denote this as Δ. We then expand Δ in ε and substitute $R_{2,e}$ for R_0 and obtain:

$$\Delta = T_0(m,c) + T_1(m,c)\varepsilon + \mathcal{O}(\varepsilon^2) \tag{3.25}$$

with

$$T_0(m,c) = -\frac{16c^8(68\sqrt{2}m^2 + 272\sqrt{2}m - 9m^2 - 8m - 16)}{m^4(2\sqrt{2}m + 4 + m)^4} \tag{3.26}$$

$$T_1(m,c) = \frac{72c^4}{2\sqrt{2}m + 4 + m)^4}\Big[(22\sqrt{2}m^2 - 176\sqrt{2}m - 15m^2 + 20m + 64)(7m^2 - 8m - 16)\Big] \tag{3.27}$$

For ε small enough, the sign of Δ is the same as the sign of $T_0(m,c)$. We denote:

$$g(m) = 68\sqrt{2}m^2 + 272\sqrt{2}m - 9m^2 - 8m - 16, \qquad m \geq 0 \tag{3.28}$$

such that

$$T_0(m,c) = -\frac{16c^8 g(m)}{m^4(2\sqrt{2}m + 4 + m)^4} \tag{3.29}$$

We observe that $g(m) > 0$ for $m > (-4(34\sqrt{2}-49)/(68\sqrt{2}-9)) \approx 0.042068394$ in which case $T_0(m,c) < 0$ which implies that $\Delta < 0$. Thus the roots of $Q(y)$ are complex with non-zero real parts, which implies that the eigenvalues of $P(\lambda)$ have non-zero real parts and thus the equilibrium $R_{2,e}$ is unstable when $m > (-4(34\sqrt{2}-49)/(68\sqrt{2}-9)) \simeq 0.042068394$.

For $m \in [0, (-4(34\sqrt{2}-49)/(68\sqrt{2}-9)) \approx 0.042068394$ we calculate the linearization at the RE $R_{2,e}$ of the mechanical system in the full phase-space with 7-degrees of freedom given by the Hamiltonian (2.9) (see Appendices A.4 and A.5). The linearization is a 14×14 matrix and the corresponding characteristic polynomial is a polynomial of degree 14 of the form:

$$P(\lambda) = a_0 + a_2\lambda^2 + a_4\lambda^4 + \ldots + a_{14}\lambda^{14} \tag{3.30}$$

with the coefficients $a_j = a_j(m, c, \varepsilon)$, $j = 0, 1, 2, \ldots, 14$. We denote $y = \lambda^2$ and obtain:

$$P(y) = a_0 + a_2 y + \ldots + a_{12}y^6 + a_{14}y^7 \tag{3.31}$$

The requirement for the stability of $R_{2,e}$ is that all roots of $P(y)$ be real and negative. In particular, this implies that the product of the roots must be negative. Using algebra, we determine the product of the roots:

$$\text{Prod:} = y_1 y_2 \ldots y_7 = (-1)^7 \frac{a_0}{a_{14}} \tag{3.32}$$

The calculations performed in Maple (see Appendix A.5) show that $a_{14} = 1$. Further, expanding $a_0(m, c, \varepsilon)$ in terms of ε

$$a_0(m, c, \varepsilon) = A_0(m, c) + A_1(m, c)\varepsilon + \mathcal{O}\left(\varepsilon^2\right) \tag{3.33}$$

we observe that the term of order zero is always negative (see Appendix A.5) and therefore the product of the roots is positive. Thus, we obtain

Proposition 3.3.1. *In the $(4+1)$-body problem with J_2 potential, if ε is sufficiently small, all the square RE are unstable.*

Chapter 4

Conclusion

In this thesis we have considered $(4+1)$-body problems with Newtonian potential and a J_2 correction. Using Lagrangian and Hamiltonian mechanics, we constructed a model of the standard $(4+1)$-body problem and reduced the model to rhomboidal-shaped configurations and to square-shaped configurations. We used angular momentum to reduce the problem by its rotational symmetry.

We proved that the RE of square-shaped configurations exist if and only if the outer masses are equal. We then determined the minimum necessary angular momentum, c_0, to allow for RE to exist. Furthermore, we obtained equations that allow for the location of RE, $r_{1,e}, r_{2,e}$, where $r_{1,e} < r_{2,e}$, to be determined explicitly.

We provided a theoretical argument that $r_{1,e}$ is unstable. We also presented a numerical based argument that $r_{2,e}$ is also unstable for $m > (-4(34\sqrt{2} - 49)/(68\sqrt{2} - 9))$. We then proposed that the $(4+1)$-body problem with J_2 potential has unstable square RE if $m \in [0, (-4(34\sqrt{2} - 49)/(68\sqrt{2} - 9)))$

There are two main possibilities that exist for further work. The first and foremost possibility involves examining square-shaped RE using a different potential function, however depending on the potential function, analytic solutions may not be possible. Another possibility is to study other RE configurations, such as rhomboidal or colinear RE configurations. From preliminary work done on this problem, it appears that numerical methods are needed to determine the location of any rhomboidal or colinear RE. It may also be possible to employ numerical methods to determine RE in parallelogram-shaped configurations, that is, configurations where $\varphi \in (0, \pi/2)$.

Appendix A

Maple Code for Stability Analysis of RE

A.1 Maple Code for Determining the RE

Here we present the Maple code used to determine RE.

> *#Roots calculated using r*

>

> *restart* :

>

> $f(x) := -\frac{1}{x} - \frac{\epsilon}{x^3};$

$$f := x \mapsto -\frac{1}{x} - \frac{\epsilon}{x^3} \quad \textbf{(1)}$$

> $U(r) := 4 \cdot M \cdot m \cdot f(r) + 4 \cdot m^2 \cdot f(r \cdot \mathrm{sqrt}(2)) + 2 \cdot m^2 \cdot f(2 \cdot r);$

$$U := r \mapsto 4 \cdot M \cdot m \cdot f(r) + 4 \cdot m^2 \cdot f\left(r \cdot \sqrt{2}\right) + 2 \cdot m^2 \cdot f(2 \cdot r) \quad \textbf{(2)}$$

> $U_{eff}(r) := \left(\frac{c^2}{2 \cdot m \cdot r^2}\right) + U(r)$

$$U_{eff} := r \mapsto \frac{c^2}{2 \cdot m \cdot r^2} + U(r) \quad \textbf{(3)}$$

> $U_{eff}(r);$

$$\frac{c^2}{2\, m\, r^2} + 4\, M\, m \left(-\frac{1}{r} - \frac{\epsilon}{r^3}\right) + 4\, m^2 \left(-\frac{\sqrt{2}}{2\, r} - \frac{\epsilon \sqrt{2}}{4\, r^3}\right) + 2\, m^2 \left(-\frac{1}{2\, r} - \frac{\epsilon}{8\, r^3}\right) \quad \textbf{(4)}$$

> $dU_{eff}(r) := diff\left(U_{eff}(r), r\right);$

$$dU_{eff} := r \rightarrow \frac{\mathrm{d}}{\mathrm{d}r} U_{eff}(r) \quad \textbf{(5)}$$

> $r\_roots := solve\left(dU_{eff}(r) = 0, r\right)$

$$r\_roots := \frac{c^2 + \sqrt{-51\, \epsilon\, m^6 - 144\, M \sqrt{2}\, \epsilon\, m^5 - 18 \sqrt{2}\, \epsilon\, m^6 - 192\, M^2\, \epsilon\, m^4 - 60\, M\, \epsilon\, m^5 + c^4}}{2\, m^2 \left(2\sqrt{2}\, m + 4\, M + m\right)}, \quad \textbf{(6)}$$
$$-\frac{-c^2 + \sqrt{-51\, \epsilon\, m^6 - 144\, M \sqrt{2}\, \epsilon\, m^5 - 18 \sqrt{2}\, \epsilon\, m^6 - 192\, M^2\, \epsilon\, m^4 - 60\, M\, \epsilon\, m^5 + c^4}}{2\, m^2 \left(2\sqrt{2}\, m + 4\, M + m\right)}$$

>

A.2 Maple Code for the Stability of RE $R_{2,e}$

Here, on the following page we present Maple code and results for the analysis of the second RE in the square-shaped configuration case.

> *restart* :

> *with*(*Student*[*Calculus1*]) : *with*(*LinearAlgebra*) : *with*(*Student*[*VectorCalculus*]) :

>

> **local** gamma :

>

> $f(x) := -\frac{1}{x} - \frac{\epsilon}{x^3}$:

> ##

>

> *#Hessian and Linearization:*

> $H(R, \psi, \varphi, p_R, p_\psi, p_\varphi) := \frac{1}{2\cdot m}\cdot\left(p_R^{\,2} + \frac{p_\psi^{\,2}}{R^2}\right) + \frac{(c - p_\varphi)^2}{2\cdot R^2\cdot m\cdot\cos^2(\psi)} + \frac{p_\varphi^{\,2}}{2\cdot R^2\cdot m\cdot\sin^2(\psi)}$:

>

> $U(R, \psi, \varphi) := M\cdot m\cdot f(R\cdot\cos(\psi)) + M\cdot m\cdot f(R\cdot\sin(\psi)) + m^2\cdot f(R\cdot \mathrm{sqrt}(1 + \sin(2\cdot\psi)\cdot\cos(\varphi))) + m^2\cdot f(R\cdot\mathrm{sqrt}(1 - \sin(2\cdot\psi)\cdot\cos(\varphi))) + \left(\frac{m^2}{2}\right)\cdot f(2\cdot R\cdot\cos(\psi)) + \left(\frac{m^2}{2}\right)\cdot f(2\cdot R\cdot\sin(\psi))$:

>

> $H_{full}(R, \psi, \varphi, p_R, p_\psi, p_\varphi) := H(R, \psi, \varphi, p_R, p_\psi, p_\varphi) + U(R, \psi, \varphi)$:

>

> $Hess := Hessian\left(H_{full}(R, \psi, \varphi, p_R, p_\psi, p_\varphi), [R, \psi, \varphi, p_R, p_\psi, p_\varphi] = \left[R_0, \frac{\mathrm{Pi}}{4}, \frac{\mathrm{Pi}}{2}, 0, 0, \frac{c}{2}\right]\right)$:

> $J := Matrix([[0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 1], [-1, 0, 0, 0, 0, 0], [0, -1, 0, 0, 0, 0], [0, 0, -1, 0, 0, 0]])$:

> $L := simplify(J \bullet Hess)$;

$$L := \left[\left[0, 0, 0, \frac{1}{m}, 0, 0\right],\right. \tag{1}$$

$$\left[0, 0, 0, 0, \frac{1}{R_0^{\,2}\, m}, 0\right],$$

$$\left[0, -\frac{4\,c}{R_0^{\,2}\, m}, 0, 0, 0, \frac{4}{R_0^{\,2}\, m}\right],$$

$$\left[\frac{4\left(\frac{\left(R_0^2+3\epsilon\right)m}{4}+M\left(R_0^2+12\epsilon\right)\right)m^2\sqrt{2}+4m^3\left(R_0^2+6\epsilon\right)-3c^2R_0}{R_0^5 m}, 0, 0, 0, 0, 0\right],$$

$$\left[0, \frac{2\left(3\left(\frac{\left(R_0^2+\frac{5\epsilon}{2}\right)m}{4}+M\left(R_0^2+10\epsilon\right)\right)m^2\sqrt{2}-2c^2R_0\right)}{R_0^3 m}, 0, 0, 0, \frac{4c}{R_0^2 m}\right],$$

$$\left[0, 0, \frac{3m^2\left(R_0^2+5\epsilon\right)}{2R_0^3}, 0, 0, 0\right]\Bigg]$$

\>

\> $CP := factor(CharacteristicPolynomial(L, \lambda));$

$$CP := \frac{1}{4R_0^{15}m^4}\Big(\big(-R_0^5\lambda^2m^2\sqrt{2}+4\sqrt{2}R_0^2m^3+8MR_0^2m^2+24\sqrt{2}\epsilon m^3+2R_0^2m^3+96M\epsilon m^2-3R_0c^2\sqrt{2}+6\epsilon m^3\big)\big(-2R_0^{10}\lambda^4m^2\sqrt{2}+12R_0^7\lambda^2m^3\sqrt{2}+24MR_0^7\lambda^2m^2+60R_0^5\epsilon\lambda^2m^3\sqrt{2}+6R_0^7\lambda^2m^3+240MR_0^5\epsilon\lambda^2m^2-8R_0^6c^2\lambda^2\sqrt{2}+15R_0^5\epsilon\lambda^2m^3-144MR_0^4m^3-36R_0^4m^4-2160MR_0^2\epsilon m^3-270R_0^2\epsilon m^4-7200M\epsilon^2m^3-450\epsilon^2m^4\big)\Big) \tag{2}$$

\> #Form of Characteristic Polynomial: $\frac{\left(\sqrt{2}R_0^5m^2\lambda^2-\alpha\right)\left(2\sqrt{2}R_0^{10}m^2\lambda^4+\beta\lambda^2+\gamma\right)}{4R_0^{15}m^4}$

\>

\> $\alpha := \left(4\sqrt{2}R_0^2m^3+8MR_0^2m^2+24\sqrt{2}\epsilon m^3+2R_0^2m^3+96M\epsilon m^2-3R_0c^2\sqrt{2}+6\epsilon m^3\right):$

\> $\beta := \left(-12\sqrt{2}R_0^7m^3-24MR_0^7m^2-60\sqrt{2}R_0^5\epsilon m^3-6R_0^7m^3-240MR_0^5\epsilon m^2+8\sqrt{2}R_0^6c^2-15R_0^5\epsilon m^3\right):$

\> $\gamma := 144MR_0^4m^3+36R_0^4m^4+2160MR_0^2\epsilon m^3+270R_0^2\epsilon m^4+7200M\epsilon^2m^3+450\epsilon^2m^4:$

\>

> $CP := \frac{\left(\sqrt{2}\,R_0^{5} m^2 \lambda^2 - \alpha\right)\left(2\sqrt{2}\,R_0^{10} m^2 \lambda^4 + \beta\lambda^2 + \gamma\right)}{4 R_0^{15}\, m^4}$:

>

> *#We now consider the term* $\left(2\sqrt{2}\,R_0^{10} m^2\lambda^4 + \beta\lambda^2 + \gamma\right)$, *rewriting it as* $Q(y) := \left(2\sqrt{2}\,R_0^{10} m^2 y^2 + \beta y + \gamma\right) = \left(A y^2 + B y + C\right)$:

>

> $A := 2\,R_0^{10}\, m^2 \sqrt{2}$:

> $B := -12\sqrt{2}\,R_0^{7} m^3 - 24\,M R_0^{7} m^2 - 60\sqrt{2}\,R_0^{5}\,\epsilon\, m^3 - 6\,R_0^{7} m^3 - 240\,M R_0^{5}\,\epsilon\, m^2 + 8\sqrt{2}\,R_0^{6} c^2 - 15\,R_0^{5}\,\epsilon\, m^3$:

> $C := 144\,M R_0^{4} m^3 + 36\,R_0^{4} m^4 + 2160\,M R_0^{2}\,\epsilon\, m^3 + 270\,R_0^{2}\,\epsilon\, m^4 + 7200\,M \epsilon^2 m^3 + 450\,\epsilon^2 m^4$:

>

> *#Calculating* Δ, *S and P:*

>

> *#Since* $R_0 > 0$, *we factor* R_0^{10} *from* Δ.

> $\Delta_{factor} := factor\left(B^2 - 4\cdot A\cdot C\right)$:

> $\Delta := simplify\Big(-576\,M\sqrt{2}\,R_0^{4} m^5 - 144\sqrt{2}\,R_0^{4} m^6 + 576\,M^2 R_0^{4} m^4 - 8640\,M\sqrt{2}\,R_0^{2}\,\epsilon\, m^5 + 288\,M R_0^{4} m^5 - 1080\sqrt{2}\,R_0^{2}\,\epsilon\, m^6 + 324\,R_0^{4} m^6 + 11520\,M^2 R_0^{2}\,\epsilon\, m^4 - 384\,M\sqrt{2}\,R_0^{3} c^2 m^2 - 28800\,M\sqrt{2}\,\epsilon^2 m^5 + 3600\,M R_0^{2}\,\epsilon\, m^5 - 96\sqrt{2}\,R_0^{3} c^2 m^3 - 1800\sqrt{2}\,\epsilon^2 m^6 + 3060\,R_0^{2}\,\epsilon\, m^6 + 57600\,M^2 \epsilon^2 m^4 - 3840\,M\sqrt{2}\,R_0 c^2\,\epsilon\, m^2 + 7200\,M \epsilon^2 m^5 - 240\sqrt{2}\,R_0 c^2\,\epsilon\, m^3 - 384\,R_0^{3} c^2 m^3 + 7425\,\epsilon^2 m^6 - 1920\,R_0 c^2\,\epsilon\, m^3 + 128\,R_0^{2} c^4\Big)$:

>

> *#Since A>0, we consider the sum as -B.*

> $S := simplify(-B)$:

>

> *#Since A>0, we consider the product as C.*

> $P := simplify(C)$:

>

> *#WLOG we set the central mass M to 1.*

> $M := 1$:

>

> *#*RE_2 *from Roots file. This is the larger RE.*

> $R_0 := \frac{c^2 + \sqrt{-51\,\epsilon\, m^6 - 144\,M\sqrt{2}\,\epsilon\, m^5 - 18\sqrt{2}\,\epsilon\, m^6 - 192\,M^2\,\epsilon\, m^4 - 60\,M\,\epsilon\, m^5 + c^4}}{2\,m^2\left(2\sqrt{2}\,m + 4\,M + m\right)} \cdot \sqrt{2}$;

$$R_0 := \frac{\left(c^2 + \sqrt{-51\,\epsilon\, m^6 - 144\sqrt{2}\,\epsilon\, m^5 - 18\sqrt{2}\,\epsilon\, m^6 - 192\,\epsilon\, m^4 - 60\,\epsilon\, m^5 + c^4}\right)\sqrt{2}}{2\, m^2 \left(2\sqrt{2}\, m + 4 + m\right)} \tag{3}$$

> # Second Order Taylor Series Expansions about ϵ=0 for Δ and S

> $Taylor\_Series\_\Delta := simplify(taylor(\Delta, \epsilon = 0, 2))$ assuming $c \geq 0$

$$Taylor\_Series\_\Delta := \frac{16\, c^8 \left(-68\, m\,(m+4)\sqrt{2} + 9\, m^2 + 8\, m + 16\right)}{m^4 \left(2\sqrt{2}\, m + 4 + m\right)^4} + 72\, \frac{c^4 \left(22\, m^2 \sqrt{2} - 176\sqrt{2}\, m - 15\, m^2 + 20\, m + 64\right)\left(7\, m^2 - 8\, m - 16\right)}{\left(2\sqrt{2}\, m + 4 + m\right)^4}\, \epsilon + \mathrm{O}(\epsilon^2) \tag{4}$$

>

> $Taylor\_Series\_S := simplify(taylor(S, \epsilon = 0, 2))$ assuming $c^2 \geq 0$

$$Taylor\_Series\_S := -\frac{16\sqrt{2}\, c^{14}}{\left(2\sqrt{2}\, m + 4 + m\right)^6 m^{12}} + 96\, \frac{c^{10} \left(12\, m\,(m+8) + \left(17\, m^2 + 20\, m + 64\right)\sqrt{2}\right)}{m^8 \left(2\sqrt{2}\, m + 4 + m\right)^6}\, \epsilon + \mathrm{O}(\epsilon^2) \tag{5}$$

>

> $Taylor\_Series\_P := simplify(taylor(P, \epsilon = 0, 2))$ assuming $c^2 \geq 0$

$$Taylor\_Series\_P := \frac{144\, c^8\,(m+4)}{m^5 \left(2\sqrt{2}\, m + 4 + m\right)^4} - 432\, \frac{c^4 \left(\left(m^3 + 12\, m^2 + 32\, m\right)\sqrt{2} + \frac{23\, m^3}{4} - 12\, m^2 + 44\, m + 96\right)}{m \left(2\sqrt{2}\, m + 4 + m\right)^4}\, \epsilon + \mathrm{O}(\epsilon^2) \tag{6}$$

>

> $solve\left(68\, m^2 \sqrt{2} + 272\sqrt{2}\, m - 9\, m^2 - 8\, m - 16\right)$

$$-\frac{4\left(34\sqrt{2} - 49\right)}{68\sqrt{2} - 9},\ -\frac{4\left(34\sqrt{2} + 47\right)}{68\sqrt{2} - 9} \tag{7}$$

> $evalf\left(-\frac{4\left(34\sqrt{2} - 49\right)}{68\sqrt{2} - 9}\right)$

$$0.04206839400 \tag{8}$$

>

A.3 Maple Code for the Sign of α

Here, on the following page we present the Maple code that was used to prove that $\alpha < 0$.

> *restart* :

>

> $M := 1;$

$$M := 1 \quad (1)$$

>

> $CP := \frac{\left(\sqrt{2}\, R_0^{\,5} m^2 \lambda^2 - \alpha\right)\left(2\sqrt{2}\, R_0^{\,10} m^2 \lambda^4 + \beta \lambda^2 + \gamma\right)}{4 R_0^{\,15}\, m^4};$

$$CP := \frac{\left(\sqrt{2}\, R_0^{\,5}\, m^2\, \lambda^2 - \alpha\right)\left(2\sqrt{2}\, R_0^{\,10}\, m^2\, \lambda^4 + \beta\, \lambda^2 + \gamma\right)}{4\, R_0^{\,15}\, m^4} \quad (2)$$

>

> $\alpha := \left(4\sqrt{2}\, R_0^{\,2}\, m^3 + 8\, M R_0^{\,2}\, m^2 + 24\sqrt{2}\, \epsilon\, m^3 + 2\, R_0^{\,2}\, m^3 + 96\, M \epsilon\, m^2 - 3\, R_0\, c^2 \sqrt{2} + 6\, \epsilon\, m^3\right);$

$$\alpha := 4\sqrt{2}\, R_0^{\,2}\, m^3 + 8\, R_0^{\,2}\, m^2 + 24\sqrt{2}\, \epsilon\, m^3 + 2\, R_0^{\,2}\, m^3 + 96\, \epsilon\, m^2 - 3\, R_0\, c^2 \sqrt{2} + 6\, \epsilon\, m^3 \quad (3)$$

>

> $R_0 := \frac{c^2 + \sqrt{-51\, \epsilon\, m^6 - 144\, M \sqrt{2}\, \epsilon\, m^5 - 18\sqrt{2}\, \epsilon\, m^6 - 192\, M^2\, \epsilon\, m^4 - 60\, M \epsilon\, m^5 + c^4}}{2\, m^2 \left(2\sqrt{2}\, m + 4\, M + m\right)} \cdot \sqrt{2};$

$$R_0 := \frac{\left(c^2 + \sqrt{-51\, \epsilon\, m^6 - 144\sqrt{2}\, \epsilon\, m^5 - 18\sqrt{2}\, \epsilon\, m^6 - 192\, \epsilon\, m^4 - 60\, \epsilon\, m^5 + c^4}\right)\sqrt{2}}{2\, m^2 \left(2\sqrt{2}\, m + 4 + m\right)} \quad (4)$$

>

> $\alpha_{simplified} := simplify(\alpha);$

$$\alpha_{simplified} := \frac{1}{m^2 \left(2\sqrt{2}\, m + 4 + m\right)} \Big(-\sqrt{-18\, \epsilon\, m^5\, (m+8)\sqrt{2} + \left(-51\, m^6 - 60\, m^5 - 192\, m^4\right)\epsilon + c^4}\; c^2 + 18\, \epsilon\, m^5\, (m+8)\sqrt{2} + 51\, \epsilon\, m^6 + 60\, \epsilon\, m^5 + 192\, \epsilon\, m^4 - c^4 \Big) \quad (5)$$

>

> # $c^4 \geq 51\, \epsilon\, m^6 + 144\sqrt{2}\, \epsilon\, m^5 + 18\sqrt{2}\, \epsilon\, m^6 + 192\, \epsilon\, m^4 + 60\, \epsilon\, m^5$, *therefore, we consider the terms* $18\, \epsilon\, m^5\, (m+8)\sqrt{2} + 51\, \epsilon\, m^6 + 60\, \epsilon\, m^5 + 192\, \epsilon\, m^4 - c^4$ *in the preceding equation.*

>

> $simplify\left(\left(18\, \epsilon\, m^5\, (m+8)\sqrt{2} + 51\, \epsilon\, m^6 + 60\, \epsilon\, m^5 + 192\, \epsilon\, m^4\right) - \left(51\, \epsilon\, m^6 + 144\sqrt{2}\, \epsilon\, m^5 + 18\sqrt{2}\, \epsilon\, m^6 + 192\, \epsilon\, m^4 + 60\, \epsilon\, m^5\right)\right);$

$$0 \quad (6)$$

>

> #*Thus the terms:* $\left(18\, \epsilon\, m^5\, (m+8)\sqrt{2} + 51\, \epsilon\, m^6 + 60\, \epsilon\, m^5 + 192\, \epsilon\, m^4 - c^4\right) \leq 0.$

>

> #*Therefore, since the first square root will either be greater than or eqaul to zero, we see that* $\alpha_{simplifed} < 0$ *and therefore*, $\alpha < 0$, *for all permitted values of c.*

>

A.4 Maple Code for Determining the Second RE in the Full Phase Space

Here, on the following page we present the Maple code that was used to determine the RE in the full phase space.

> *restart* :

> *with*(*Student*[*Calculus1*]) : *with*(*LinearAlgebra*) : *with*(*Student*[*VectorCalculus*]) :

>

> #WLOG M=1

>

> $f(x) := -\frac{1}{x} - \frac{\epsilon}{x^3}$:

>

> #Let p_{φ_i} be represented as q_i

>

> $K(r_1, r_2, r_3, r_4, \varphi_2, \varphi_3, \varphi_4, p_1, p_2, p_3, p_4, q_2, q_3, q_4, c, m, \epsilon) := \frac{1}{2 \cdot m} \cdot (p_1^2 + p_2^2 + p_3^2 + p_4^2) + \frac{(c - q_2 - q_3 - q_4)^2}{2 \cdot m \cdot r_1^2} + \frac{q_2^2}{2 \cdot m \cdot r_2^2} + \frac{q_3^2}{2 \cdot m \cdot r_3^2} + \frac{q_4^2}{2 \cdot m \cdot r_4^2}$:

>

> $U(r_1, r_2, r_3, r_4, \varphi_2, \varphi_3, \varphi_4, m, \epsilon) := m \cdot (f(r_1) + f(r_2) + f(r_3) + f(r_4)) + m^2 \cdot (f(\mathrm{sqrt}(r_1^2 + r_2^2 - 2 \cdot r_1 \cdot r_2 \cdot \cos(\varphi_2))) + f(\mathrm{sqrt}(r_1^2 + r_3^2 - 2 \cdot r_1 \cdot r_3 \cdot \cos(\varphi_3))) + f(\mathrm{sqrt}(r_1^2 + r_4^2 - 2 \cdot r_1 \cdot r_4 \cdot \cos(\varphi_4))) + f(\mathrm{sqrt}(r_2^2 + r_3^2 - 2 \cdot r_2 \cdot r_3 \cdot \cos(\varphi_3 - \varphi_2))) + f(\mathrm{sqrt}(r_2^2 + r_4^2 - 2 \cdot r_2 \cdot r_4 \cdot \cos(\varphi_4 - \varphi_2))) + f(\mathrm{sqrt}(r_3^2 + r_4^2 - 2 \cdot r_3 \cdot r_4 \cdot \cos(\varphi_4 - \varphi_3))))$:

>

> $H_{full}(r_1, r_2, r_3, r_4, \varphi_2, \varphi_3, \varphi_4, p_1, p_2, p_3, p_4, q_2, q_3, q_4, c, m, \epsilon) := (K(r_1, r_2, r_3, r_4, \varphi_2, \varphi_3, \varphi_4, p_1, p_2, p_3, p_4, q_2, q_3, q_4, c, m, \epsilon) + U(r_1, r_2, r_3, r_4, \varphi_2, \varphi_3, \varphi_4, m, \epsilon))$:

>

> $R_{eq} := -\mathit{diff}(H_{full}(r_1, r_2, r_3, r_4, \varphi_2, \varphi_3, \varphi_4, p_1, p_2, p_3, p_4, q_2, q_3, q_4, c, m, \epsilon), r_1)$:

>

> $R_0 := \mathit{simplify}\left(\mathit{subs}\left(r_1 = r, r_2 = r, r_3 = r, r_4 = r, \varphi_2 = \frac{\pi}{2}, \varphi_3 = \pi, \varphi_4 = \frac{3 \cdot \pi}{2}, p_1 = 0, p_2 = 0, p_3 = 0, p_4 = 0, q_2 = 0, q_3 = 0, q_4 = 0, R_{eq}\right)\right)$ assuming $r \geq 0$;

$$R_0 := \frac{-8\left(r^2 + \frac{3\epsilon}{2}\right) m^3 \sqrt{2} + (-4 r^2 - 3\epsilon) m^3 + (-16 r^2 - 48\epsilon) m^2 + 16 c^2 r}{16 m r^4} \quad \textbf{(1)}$$

>

> $\mathit{solve}(R_0 = 0, r)$;

$$\frac{1}{2 m^2 (8\sqrt{2} + 7m - 4)} \Big(8\sqrt{2}\, c^2 - 4 c^2 \quad \textbf{(2)}$$

$$+ \big(612\,\epsilon\, m^5 + 42\sqrt{2}\,\epsilon\, m^6 - 1728\,\epsilon\, m^4 - 1056\sqrt{2}\,\epsilon\, m^5 - 315\,\epsilon\, m^6 + 144 c^4 + 768\sqrt{2}\,\epsilon\, m^4$$

$$- 64\sqrt{2}\,c^4\Big)^{1/2}\Big), \frac{1}{2\,m^2\,(8\sqrt{2}+7\,m-4)}\Big(8\sqrt{2}\,c^2 - 4\,c^2$$

$$- \big(612\,\epsilon\,m^5 + 42\sqrt{2}\,\epsilon\,m^6 - 1728\,\epsilon\,m^4 - 1056\sqrt{2}\,\epsilon\,m^5 - 315\,\epsilon\,m^6 + 144\,c^4 + 768\sqrt{2}\,\epsilon\,m^4$$

$$- 64\sqrt{2}\,c^4\big)^{1/2}\Big)$$

>

> *#Thus we use the RE,* $\frac{1}{2\,m^2\,(8\sqrt{2}+7\,m-4)}\Big(8\sqrt{2}\,c^2 - 4\,c^2$

$$+ \big(612\,\epsilon\,m^5 + 42\sqrt{2}\,\epsilon\,m^6 - 1728\,\epsilon\,m^4 - 1056\sqrt{2}\,\epsilon\,m^5 - 315\,\epsilon\,m^6 + 144\,c^4 + 768\sqrt{2}\,\epsilon\,m^4$$

$$- 64\sqrt{2}\,c^4\big)^{1/2}\Big)$$

>

A.5 Maple Code for the Stability of the Second RE in the Full Phase Space

Here, on the following page we present the Maple code that was used to prove that the larger RE is unstable.

> *restart* :

> *with*(*Student*[*Calculus1*]) : *with*(*LinearAlgebra*) : *with*(*Student*[*VectorCalculus*]) : *with*(*plots*) :

>

> *#WLOG M=1*

>

> $f(x) := -\frac{1}{x} - \frac{\epsilon}{x^3}$:

>

> *#Let* p_{φ_i} *be represented as* q_i

>

> $K(r_1, r_2, r_3, r_4, \varphi_2, \varphi_3, \varphi_4, p_1, p_2, p_3, p_4, q_2, q_3, q_4, c, m, \epsilon) := \frac{1}{2 \cdot m} \cdot (p_1^2 + p_2^2 + p_3^2 + p_4^2) + \frac{(c - q_2 - q_3 - q_4)^2}{2 \cdot m \cdot r_1^2} + \frac{q_2^2}{2 \cdot m \cdot r_2^2} + \frac{q_3^2}{2 \cdot m \cdot r_3^2} + \frac{q_4^2}{2 \cdot m \cdot r_4^2}$:

>

> $U(r_1, r_2, r_3, r_4, \varphi_2, \varphi_3, \varphi_4, m, \epsilon) := m \cdot (f(r_1) + f(r_2) + f(r_3) + f(r_4)) + m^2 \cdot (f(\mathrm{sqrt}(r_1^2 + r_2^2 - 2 \cdot r_1 \cdot r_2 \cdot \cos(\varphi_2))) + f(\mathrm{sqrt}(r_1^2 + r_3^2 - 2 \cdot r_1 \cdot r_3 \cdot \cos(\varphi_3))) + f(\mathrm{sqrt}(r_1^2 + r_4^2 - 2 \cdot r_1 \cdot r_4 \cdot \cos(\varphi_4))) + f(\mathrm{sqrt}(r_2^2 + r_3^2 - 2 \cdot r_2 \cdot r_3 \cdot \cos(\varphi_3 - \varphi_2))) + f(\mathrm{sqrt}(r_2^2 + r_4^2 - 2 \cdot r_2 \cdot r_4 \cdot \cos(\varphi_4 - \varphi_2))) + f(\mathrm{sqrt}(r_3^2 + r_4^2 - 2 \cdot r_3 \cdot r_4 \cdot \cos(\varphi_4 - \varphi_3))))$:

>

> $H_{full}(r_1, r_2, r_3, r_4, \varphi_2, \varphi_3, \varphi_4, p_1, p_2, p_3, p_4, q_2, q_3, q_4, c, m, \epsilon) := (K(r_1, r_2, r_3, r_4, \varphi_2, \varphi_3, \varphi_4, p_1, p_2, p_3, p_4, q_2, q_3, q_4, c, m, \epsilon) + U(r_1, r_2, r_3, r_4, \varphi_2, \varphi_3, \varphi_4, m, \epsilon))$:

>

> $Hess := Hessian\left(H_{full}(r_1, r_2, r_3, r_4, \varphi_2, \varphi_3, \varphi_4, p_1, p_2, p_3, p_4, q_2, q_3, q_4, c, m, \epsilon), [r_1, r_2, r_3, r_4, \varphi_2, \varphi_3, \varphi_4, p_1, p_2, p_3, p_4, q_2, q_3, q_4] = \left[R_0, R_0, R_0, R_0, \frac{\pi}{2}, \pi, \frac{3 \cdot \pi}{2}, 0, 0, 0, 0, 0, 0, 0\right]\right)$:

>

> $J :=$ *Matrix*([[0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0], [0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1], [−1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, −1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, −1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, −1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, −1, 0, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, −1, 0, 0, 0, 0, 0, 0, 0, 0], [0, 0, 0, 0, 0, 0, −1, 0, 0, 0, 0, 0, 0, 0]]) :

>

> $L := simplify(J \cdot Hess)$ assuming $c \geq 0, R_0 \geq 0$:

>

> $CP := CharacteristicPolynomial(L, \lambda)$:

>

> $CP_{sub} := subs\left(R_0 = \frac{1}{2\,m^2\,(8\sqrt{2}+7\,m-4)}\left(8\sqrt{2}\,c^2 - 4\,c^2 + \left(612\,\epsilon\,m^5 + 42\sqrt{2}\,\epsilon\,m^6 - 1728\,\epsilon\,m^4 - 1056\sqrt{2}\,\epsilon\,m^5 - 315\,\epsilon\,m^6 + 144\,c^4 + 768\sqrt{2}\,\epsilon\,m^4 - 64\sqrt{2}\,c^4\right)^{1/2}\right), CP\right):$

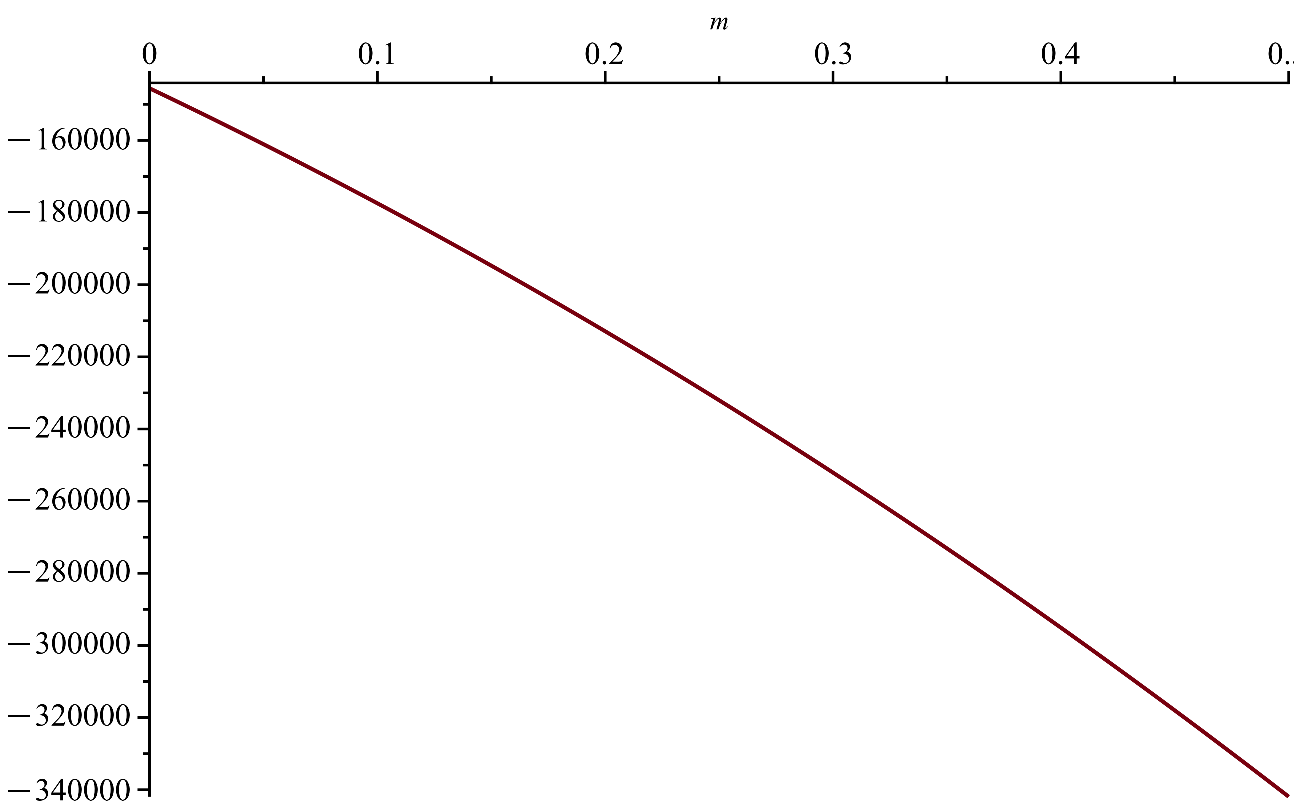
>

> $T := simplify\left(taylor\left(coeff\left(CP_{sub}, \lambda, 0\right), \epsilon = 0, 1\right)\right)$ assuming $c \geq 0$;

$$T := \frac{1}{4503599627370496\,c^{42}\,(2\sqrt{2}-1)^{21}}\left(3\,m^{45}\left(19\sqrt{2}\,m^4 - 72\sqrt{2}\,m^3 + 6\,m^4 - 176\sqrt{2}\,m^2 + 328\,m^3 - 7680\sqrt{2}\,m - 2496\,m^2 - 9728\sqrt{2} - 11264\,m - 6144\right)(8\sqrt{2}+7\,m-4)^{21}\right) + O(\epsilon) \quad \textbf{(1)}$$

>

> $plot\left(\left(19\sqrt{2}\,m^4 - 72\sqrt{2}\,m^3 + 6\,m^4 - 176\sqrt{2}\,m^2 + 328\,m^3 - 7680\sqrt{2}\,m - 2496\,m^2 - 9728\sqrt{2} - 11264\,m - 6144\right)\cdot\left(8\sqrt{2}+7\,m-4\right), m = 0..0.5\right)$



>

> *#Thus $T<0$ when m is in [0, 0.042068394].*

>

Appendix B

Equilibria and Stability for Autonomous ODEs

This appendix is a brief overview of the underlying theory behind the equilibria and stability of autonomous ODEs. The relevant references here are *Relative Equilibria of Isosceles Triatomic Molecules in Classical Approximation* by Damaris McKinley ([McKinley 2014]), *Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry, and Engineering* by Steven Strogatz and *Geometric Mechanics and Symmetry: From Finite to Infinite Dimensions* by Holm et. al. ([Holm & al. 2009]).

We begin by defining equilibria points of non-linear autonomous systems:

Definition B.0.1. *Given a non-linear autonomous system defined as follows:*

$$\dot{x}(t) = f(x(t)),\ x \in D \subset \mathbb{R}^n,\ D \text{ is open} \tag{B.1}$$

where $f : D \to \mathbb{R}^n$ *is* $\mathcal{C}^1(D)$*. A point* $x_e \in D$ *is called an equilibrium point of* $\dot{x}(t)$ *if it is a solution of* $f(x(t)) = 0$*. A solution* $x(t) = x_e(t)\ \forall t$ *is valid if it holds for* (B.1).

Next we define the stability of an equilibrium point:

Definition B.0.2. *An equilibrium point* x_e *is Lyapunov stable if and only if for any* $\epsilon > 0$ *there exists a* $\delta > 0$ *such that if* $|x(t_0) - x_e| < \delta$*, then* $|x(t) - x_e| < \epsilon$ *for all* $t > t_0$

Definition B.0.3. *Furthermore, an equilibrium point* x_e *is asymptotically stable if and only if it is Lya-*

punov stable and if there exists $\delta > 0$ such that if $|x(t_0) - x_e| < \delta$ then:

$$\lim_{t\to\infty} |x(t) - x_e| = 0 \tag{B.2}$$

Now, without loss of generality, we assume that $x_e = 0$, which can be always be achieved with a linear change of coordinates.

Definition B.0.4. *Let $F(x)$ be a continuous function on a domain D containing the origin and assume that $F(0) = 0$. Under these assumptions:*

1. *$F(x)$ is positive definite on D if $F(x) > 0$, $\forall x \in D \setminus \{0\}$*
2. *$F(x)$ is positive semidefinite on D if $F(x) \geq 0$, $\forall x \in D \setminus \{0\}$*
3. *$F(x)$ is negative definite on D if $F(x) < 0$, $\forall x \in D \setminus \{0\}$*
4. *$F(x)$ is negative semidefinite on D if $F(x) \leq 0$, $\forall x \in D \setminus \{0\}$*

Now, we define a Lyapunov function:

Definition B.0.5. *Let $x_e = 0$ be the equilibrium point for (B.1) and let $\hat{D}$ be an open neighbourhood surrounding the origin and $F : \hat{D} rightarrow \mathbb{R}$ be a function. Then $F(x)$ is a Lyapunov function if:*

1. *$F(x)$ is continuously differentiable on $\hat{D}$*
2. *$F(0) = 0$*
3. *$F(x) > 0$ for all $x \in \hat{D} \setminus \{0\}$*
4. *$\dot{F}(x(t)) \leq 0$ along any solution $x(t)$ with $x(t_0) \in \hat{D}$*

The following example is given in order to illustrate Lyapunov functions.

Example B.0.1. *Consider a system given as:*

$$\begin{cases} \dot{x} = -x + y - xy^2 \\ \dot{y} = -x - y \end{cases} \tag{B.3}$$

Then given a Lyapunov function F of the form $F(x, y) = x^2 + ky^2$ where $k \in \mathbb{R}$, the value of k that allows F to be a Lyapunov function must be determined.

It is easy to see that the point $(0, 0)$ *is an equilibrium point. Then the easiest condition to verify is Condition 2, which requires that* $F(0, 0) = 0$ *which it clearly does regardless of the value of* k*. Next, it is easy to see that Condition 3 can be met so long as* k *is non-negative or* $x^2 > ky^2$ *for all* $(x, y) \in D \setminus \{0, 0\}$*. Now taking the first derivative, the following is obtained:*

$$\begin{aligned} \frac{d}{dt}F(x(t), y(t)) &= \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} \\ &= (2x)(-x + y - xy^2) + (2ky)(-x - y) \\ &= -2x^2 + 2xy - 2x^2y^2 - 2kxy - 2ky^2 \\ &= -2x^2 + 2xy(1 - k) - 2x^2y^2 - 2ky^2 \end{aligned} \tag{B.4}$$

Now, Condition 3 can be satisfied by setting $k = 1$*, resulting in:*

$$\dot{F}(x, y) = -2x^2 - 2x^2y^2 - 2y^2 \tag{B.5}$$

The above equation is always negative for any $(x, y) \neq (0, 0)$*, thus satisfying Condition 4. From here it can be easily seen that* F *is continuously differentiable, satisfying Condition 1. Therefore, since all conditions are satisfied,* F *is indeed a Lyapunov function.*

Now we give the First and Second Lyapunov Stability Theorems which give important results on the stability of equilibria.

Theorem B.0.1 (Lyapunov's First Stability Theorem)**.** *Suppose that a Lyapunov function* $F(x)$ *can be defined on a neighbourhood of the origin and* $x_e = 0$*. Then the origin is Lyapunov stable.*

Theorem B.0.2 (Lyapunov's Second Stability Theorem)**.** *Suppose that* $x_e = 0$ *is an equilibrium point for* (B.1) *and* $F(x)$ *is a Lyapunov function that is continuous on* D*, which contains the equilibrium point. If* $\dot{F}(x) < 0$ *for all* $x \in D \setminus \{0\}$*, then* $x_e = 0$ *is asymptotically stable.*

If the Lyapunov test fails, then the stability of the equilibria cannot be determined by this method, rather the behavior of the linear system at $x = x_e$ must be studied. We now define linearization at an equilibrium point of an autonomous ODE system:

Definition B.0.6. *The linear system given as* $\dot{x} = Ax$*, with* $A = Df(x_e)$ *is called the linerization of* (B.1) *at the equilibrium point* x_e*.*

This gives rise to the following conditions regarding the stability of an equilibrium point x_e.

Definition B.0.7. *In the following list, it shall be assumed that x_e is a hyperbolic equilibrium of* (B.1). *x_e is called a hyperbolic equilibrium if none of the eigenvalues of A have a real part of zero. Then there are various possibilities regarding the stability of a hyperbolic equilibrium x_e as shown in the following list.*

1. *x_e is unstable if at least one of the eigenvalues of A has a positive real part.*

2. *x_e is asymptotically stable if all of the eigenvalues of A have strictly negative real parts.*

3. *x_e is spectrally stable if all of the eigenvalues of A have real parts that are less than or equal to zero.*

It is possible that a non-hyperbolic equilibrium may be spectrally stable.

Now we can establish the criteria to determine what type of equilibrium point we are dealing with.

Definition B.0.8. *Given the eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ of A, the following list outlines the stability of the equilibrium point x_e.*

1. *If for all i where $1 < i < n$, $Re(\lambda_i) < 0$, then x_e is a sink.*

2. *If for all i where $1 < i < n$, $Re(\lambda_i) > 0$, then x_e is a source.*

We now continue our previous example and aim to classify the equilibria points and determine their stability.

Example B.0.2. *Consider the following system of differential equations:*

$$\begin{cases} \dot{x} = xy + 3x - y - 3 \\ \dot{y} = xy - 5y - x + 5 \end{cases} \tag{B.6}$$

The system can be easily rewritten to make the equilibria obvious:

$$\begin{cases} \dot{x} = (x-1)(y+3) \\ \dot{y} = (x-5)(y-1) \end{cases} \tag{B.7}$$

Therefore the two equilibria are $(1, 1)$ and $(5, -3)$. The next step is to compute the Jacobian as follows:

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} y+3 & x-1 \\ y-1 & x-5 \end{pmatrix} = A \tag{B.8}$$

The matrix A is the linearization of the system.

Example B.0.3. *Recall in the previous example, a linearization matrix A was found for a system with equilibria points $(1,1)$ and $(5,-3)$. The matrix A was found to be:*

$$A = \begin{pmatrix} y+3 & x-1 \\ y-1 & x-5 \end{pmatrix} \tag{B.9}$$

Then evaluating A at the equilibrium $(1,1)$ yields:

$$A_{(1,1)} = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix} \iff \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \iff \begin{cases} \dot{x} = 4x \\ \dot{y} = -4y \end{cases} \tag{B.10}$$

Evaluating the eigenvalues of matrix $A_{(1,1)}$ yields $\lambda_1 = 4$ and $\lambda_2 = -4$ meaning that the hyperbolic equilibrium point $(1,1)$ is an unstable saddle ($\lambda_1 > 0$ and $\lambda_2 < 0$).

Furthermore, evaluating A at the equilibrium $(5,-3)$ yields:

$$A_{(5,-3)} = \begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix} \iff \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 4 \\ -4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \iff \begin{cases} \dot{x} = 4y \\ \dot{y} = -4x \end{cases} \tag{B.11}$$

Evaluating the eigenvalues of matrix $A_{(5,-3)}$ yields $\lambda_1 = 4i$ and $\lambda_2 = -4i$, a non-hyperbolic equilibrium, meaning that the equilibrium point $(5,-3)$ is a spectrally stable centre.

The next appendix provides an outline of the mathematical and physical theory underlying the work done in this thesis, specifically the mathematical formulation of conservative mechanical N-point mass systems.

Appendix C

Newtonian Mechanics

This appendix is a brief overview of the underlying Newtonian mechanics employed in this paper. The relevant references here are *Relative Equilibria of Isosceles Triatomic Molecules in Classical Approximation* by Damaris McKinley ([McKinley 2014]), *Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry, and Engineering* by Steven Strogatz ([Strogatz 2008]) and *Geometric Mechanics and Symmetry: From Finite to Infinite Dimensions* by Holm et. al. ([Holm & al. 2009]). Part 1 outlines Newtonian N-body systems and their associated dynamical quantities such as linear momentum, angular momentum, etc. This section also examines binary interactions in N-body systems, central force problems and various examples. Part 2 introduces and outlines the Lagrangian formulation of N-body problems and the Euler-Lagrange equations. Part 3 discusses the Legendre transform and Hamiltonian mechanics, outlining how to obtain the Hamiltonian of a system via the Legendre Transform. The section then discusses Hamiltonian mechanics, outlining various definitions, theorems, proofs and examples. The penultimate section covers cyclic and polar coordinates which are used in this paper to reduce the Hamiltonian. This leads to the final section, which outlines the theory behind finding relative equilibria which is a major part of the thesis. To summarize, the appendix is organized as follows:

1. Part 1: Newtonian N-Body Systems

2. Part 2: Lagrangian Mechanics and the Formulating Lagrangians for Newtonian N-Body Systems

3. Part 3: The Legendre Transform and Hamiltonian Mechanics for Newtonian N-Body Systems

4. Part 4: Cyclic Coordinates and Reduced Hamiltonians

5. Part 5: Equilibria and Relative Equilibria of Newtonian N-Body Systems

C.1 Part 1: Newtonian N-Body Systems

In this section, a system of N bodies is considered with each body having a mass of m_i with mutual (binary) interactions. A fixed frame of reference is denoted by $\mathbb{R}^d$, where d is the spatial dimension, $d = 1$, 2 or 3. Each of the N bodies is represented by a single point mass.

Definition C.1.1 (Point Mass)**.** *A point mass is defined as an idealised zero-dimensional object that is completely described by its mass and spatial position. It is assumed to have a constant mass and its position varies as a function of time.*

This leads to the configuration and configuration space of a system which are defined as follows:

Definition C.1.2 (Configuration and Configuration Space)**.** *The configuration of a system comprised of N point masses that interact with one another is given by the multi-vector $\boldsymbol{q} = (\boldsymbol{q_1}, \boldsymbol{q_2}, ..., \boldsymbol{q_N}) \in \mathbb{R}^{dN}$, where $\boldsymbol{q_i}$ denotes the position vector of each of the point masses. The configuration space is defined as the set of all possible configurations where in the absence of constraints, it is either $\mathbb{R}^{dN}$ if no collisions are possible or an open subset of $\mathbb{R}^{dN}$. As a function of time, the velocity of each point mass is given by:*

$$\frac{d\boldsymbol{q_i}}{dt} = \dot{\boldsymbol{q}}_{\boldsymbol{i}}(t) \tag{C.1}$$

Furthermore, as a function of time, the acceleration of each point mass is given by:

$$\frac{d^2\boldsymbol{q_i}}{dt^2} = \ddot{\boldsymbol{q}}_{\boldsymbol{i}}(t) \tag{C.2}$$

Example C.1.1. *The preceding definition can be explained with an example. If there is a single particle in open space, then its configuration space is $\mathbb{R}^3$, since the particle can occupy any point in space. Similarly, a point mass confined to a straight line would have its configuration space as $\mathbb{R}$.*

If, for example, there was a system with N non-interacting particles in space, the largest possible configuration space is $\mathbb{R}^{3N}$. However, the actual configuration space would be a proper subset of $\mathbb{R}^{3N}$ since two or more particles cannot simultaneously occupy the same point in space.

Now that we have defined point masses, we can discuss perhaps the most famous law of physics, Newton's Second Law.

Definition C.1.3 (Newton's Second Law)**.** *Newton's Second Law for the system is given as:*

$$m_i\ddot{\boldsymbol{q}}_{\boldsymbol{i}} = \boldsymbol{F_i} \text{ for } i = 1, 2, ..., N \tag{C.3}$$

In the above, $\boldsymbol{F_i}$ is the total force on point mass i.

A crucial aspect of Newtonian Mechanics is the idea of inertia and inertial frames of reference, which we define below:

Definition C.1.4 (Inertial Frame)**.** *A frame of reference in which Newton's Second Law applies is called an inertial frame.*

An inertial frame of reference is assumed throughout this thesis. Furthermore, it is assumed that each $\boldsymbol{F_i}$ is a smooth function of configurations only, that is $\boldsymbol{F_i} = \boldsymbol{F_i}(\boldsymbol{q_1}, \boldsymbol{q_2}, ..., \boldsymbol{q_N})$. Thus, the system (C.3) is a second order autonomous ODE system with $\mathbb{R}^{dN}$ equations and $\mathbb{R}^{dN}$ unknowns.

The following dynamical quantities play central roles in the dynamics of the preceding system. In many situations, these quantities are conserved, that is, they are constant along any trajectory given below, of the system (C.3), so long as it satisfies equation (C.3).

$$\boldsymbol{q}(t) = (\boldsymbol{q_1}(t), \boldsymbol{q_2}(t), ..., \boldsymbol{q_N}(t)) \tag{C.4}$$

Here we provide definitions of important dynamical quantites for Newtonian N-body systems, such as various momenta, the centre of mass and kinetic energy.

Definition C.1.5 (Dynamical Quantities)**.** *The linear momentum, or impulse of a single point mass with position $\boldsymbol{q_i}$ is $\boldsymbol{p_i} = m_i \dot{\boldsymbol{q}}_{\boldsymbol{i}}$.*

The total linear momentum of a system of N point masses is the sum of the linear momenta of each of the point masses. This is given as:

$$\boldsymbol{p} = \sum_{i=1}^{N} \boldsymbol{p_i} = \sum_{i=1}^{N} m_i \dot{\boldsymbol{q}}_{\boldsymbol{i}} \tag{C.5}$$

The centre of mass, $c(t)$ is a unique location in space such that:

$$c(t) = \frac{\sum_{i=1}^{N} m_i q_i}{\sum_{i=1}^{N} m_i} \tag{C.6}$$

The angular momentum, about the origin of coordinates $\boldsymbol{q} = 0$, of a single point mass with position $\boldsymbol{q_i} \in \mathbb{R}^3$ is:

$$\boldsymbol{\pi_i} = \boldsymbol{q_i} \times m_i \dot{\boldsymbol{q}}_{\boldsymbol{i}} = \boldsymbol{q_i} \times \boldsymbol{p_i} \tag{C.7}$$

In the preceding equation $\times$ denotes the three-dimensional cross product. Now the total angular momentum of a spatial system ($d = 3$) of N point masses taken about the origin of coordinates where $\boldsymbol{q} = \boldsymbol{0}$ is the sum of the angular momenta of each point mass, given as:

$$\boldsymbol{\pi} = \sum \pi_i = \sum \boldsymbol{q_i} \times m_i \dot{\boldsymbol{q}}_{\boldsymbol{i}} = \sum \boldsymbol{q_i} \times \boldsymbol{p_i} \tag{C.8}$$

Geometrically, this is the sum of the N oriented areas given by the cross-products of pairs of vectors $\boldsymbol{q_i}$ and $\boldsymbol{p_i}$. For a planar system ($d = 2$), the total angular momentum is the scalar defined by:

$$\pi = \sum m_i(q_i^1 \dot{q}_i^2 - q_i^2 \dot{q}_i^1) = \sum q_i^1 p_i^2 - q_i^2 p_i^1 \tag{C.9}$$

In the preceding equation $\boldsymbol{q_i} = (q_i^1, q_i^2)$ and $\boldsymbol{p_i} = (p_i^1, p_i^2)$. It should be noted that if the vectors $\boldsymbol{q_i}$ are embedded in $\mathbb{R}^3$ as $(q_i^1, q_i^2, 0)$, then π as defined above is the third component of the vector $\boldsymbol{\pi}$ as defined in (C.8). It should also be noted that angular momentum is undefined for systems defined on a line ($d = 1$).

The total kinetic energy of the system of N point masses is given as:

$$K = \frac{1}{2}\sum_{i=1}^{N} m_i ||\dot{\boldsymbol{q}}_{\boldsymbol{i}}||^2 \tag{C.10}$$

Here $||\dot{\boldsymbol{q}}_{\boldsymbol{i}}||^2 = \dot{\boldsymbol{q}}_{\boldsymbol{i}} \cdot \dot{\boldsymbol{q}}_{\boldsymbol{i}} = \sum_j (\dot{q}_i^j)^2$ is the squared Euclidean norm of $\dot{\boldsymbol{q}}_{\boldsymbol{i}}$.

The following remark is of great importance in this thesis; it outlines how to change coordinates from Cartesian coordinates into polar coordinates.

Remark C.1.1. *One can convert the total angular momentum and kinetic energy into polar coordinates by making a change of coordinates as shown below:*

$$\boldsymbol{q_i} = (r_i \cos\theta_i, r_i \sin\theta_i) \text{ for } i = 1, 2, ..., N \tag{C.11}$$

This then yields:

$$\begin{aligned}
\pi &= \sum m_i (r_i \cos\theta_i, r_i \sin\theta_i) \times (\dot{r}_i \cos\theta_i - r_i\dot{\theta}_i \sin\theta_i, \dot{r}_i \sin\theta_i + r_i\dot{\theta}_i \cos\theta_i) \\
&= \sum m_i r_i^2 \dot{\theta}_i \\
K &= \frac{1}{2}\sum m_i ||(\dot{r}_i \cos\theta_i - r_i\dot{\theta}_i \sin\theta_i, \dot{r}_i \sin\theta_i + r_i\dot{\theta}_i \cos\theta_i)||^2 \\
&= \frac{1}{2}\sum m_i(\dot{r}_i^2 + r_i^2\dot{\theta}_i^2)
\end{aligned} \tag{C.12}$$

Under specific conditions, linear momentum, angular momentum and energy are conserved.

The following definition shows how a body's position described in polar coordiantes can be differentiated to obatin its velocity in polar coordinates.

Definition C.1.6. *In two dimensional space, the velocity in xy-components is given as:*

$$\begin{pmatrix} \dot{x}_i \\ \dot{y}_i \end{pmatrix} = \dot{r}_i \begin{pmatrix} \cos\theta_i \\ \sin\theta_i \end{pmatrix} + r\dot{\theta}_i \begin{pmatrix} -\sin\theta_i \\ \cos\theta_i \end{pmatrix} \tag{C.13}$$

where in the preceding equation r_i is the radius from the origin to the point i and θ is the counter-clockwise angle formed between the positive x-axis and the radius for point i.

The following theorems and definitions provide important results regarding momenta, torque and binary particle interactions in systems of N point masses.

Theorem C.1.1 (Conservation of Linear Momentum)**.** *If $\sum \boldsymbol{F_i} = 0$, then the total linear momentum is conserved.*

The following definition is required before proceeding.

Definition C.1.7. *The torque on a point mass with position $\boldsymbol{q} \in \mathbb{R}^3$ and force $\boldsymbol{F}$ is $\boldsymbol{q} \times \boldsymbol{F}$. The total torque of a system of N point masses is $\sum \boldsymbol{q_i} \times \boldsymbol{F_i}$.*

Theorem C.1.2 (Conservation of Angular Momentum)**.** *The Conservation of Angular Momentum Theorem states that if, for a system, $\sum \boldsymbol{q_i} \times \boldsymbol{F_i} = 0$, then its total angular momentum is conserved.*

Definition C.1.8 (Binary Particle Interaction)**.** *The force being exerted on a single mass point by another mass point within the system will now be considered.*

In many systems, the only forces on the point masses are forces of binary interaction $\boldsymbol{F_{ij}}$, each parallel to the inter-particle position vector $\boldsymbol{d_{ij}} = \boldsymbol{q_i} - \boldsymbol{q_j}$, such that $\boldsymbol{F_{ij}} = -\boldsymbol{F_{ij}}$ always holds and the total force on each point mass i is $\boldsymbol{F_i} = \sum_j \boldsymbol{F_{ij}}$. This system is called a closed system. Moreover, such forces are called internal and any other forces are called external.

Proposition C.1.1. *In any closed system, the total force and total torque are both zero.*

Corollary C.1.1. *The total angular momentum of a closed system is conserved.*

We now present an extensive example of a Newtonian 2-body system that illustrates many of the preceeding theorems and definitions.

Example C.1.2. *(Newtonian 2-Body Systems) Consider a system formed by two mass points m_1 and m_2, with mutual interaction, so that the motion is described by:*

$$m_1\ddot{\boldsymbol{q}}_{\mathbf{1}} = -\boldsymbol{F}(\boldsymbol{q}_{\mathbf{2}} - \boldsymbol{q}_{\mathbf{1}}) \tag{C.14}$$

$$m_2\ddot{\boldsymbol{q}}_{\mathbf{2}} = \boldsymbol{F}(\boldsymbol{q}_{\mathbf{2}} - \boldsymbol{q}_{\mathbf{1}}) \tag{C.15}$$

In the preceding equations $F : D \subset \mathbb{R}^d \to \mathbb{R}^d$ is a force which depends upon the relative vector $(\boldsymbol{q}_{\mathbf{2}} - \boldsymbol{q}_{\mathbf{1}})$ only. Then, since the sum of the forces is zero, the linear momentum is conserved, yielding the following:

$$m_1\dot{\boldsymbol{q}}_{\mathbf{1}}(t) + m_2\dot{\boldsymbol{q}}_{\mathbf{2}}(t) = constant = \boldsymbol{c}_{\mathbf{1}} \tag{C.16}$$

In the preceding equation, the constant $\boldsymbol{c}_{\mathbf{1}}$ is determined by the initial conditions. (The relation above can be also immediately observed from the equations of motion).

The preceding equation leads to:

$$\boldsymbol{c(t)} = \frac{m_1\boldsymbol{q}_{\mathbf{1}}(t) + m_2\boldsymbol{q}_{\mathbf{2}}(t)}{m_1 + m_2} = \frac{\boldsymbol{c}_{\mathbf{1}}}{m_1 + m_2}t + \boldsymbol{c}_{\mathbf{2}} \tag{C.17}$$

In the preceding equation, $\boldsymbol{c}_{\mathbf{2}}$ is a constant determined by the initial conditions. Now, let $\boldsymbol{r} = \boldsymbol{q}_{\mathbf{2}} - \boldsymbol{q}_{\mathbf{1}}$ be the relative vector of m_1 and m_2 and consider the change of coordinates $(\boldsymbol{q}_{\mathbf{1}}, \boldsymbol{q}_{\mathbf{2}}) \to (\boldsymbol{r}, \boldsymbol{c})$ given below as:

$$\boldsymbol{r} = \boldsymbol{q}_{\mathbf{2}} - \boldsymbol{q}_{\mathbf{1}} \tag{C.18}$$

$$\boldsymbol{c} = \frac{m_1\boldsymbol{q}_{\mathbf{1}} + m_2\boldsymbol{q}_{\mathbf{2}}}{m_1 + m_2} \tag{C.19}$$

Then $\boldsymbol{r}$ is given by the following equation:

$$\begin{aligned} \boldsymbol{r}(t) = \boldsymbol{q}_{\mathbf{2}}(t) - \boldsymbol{q}_{\mathbf{1}}(t) &= \frac{m_1 + m_2}{m_2}\boldsymbol{c}(t) - \frac{m_1}{m_2}\boldsymbol{q}_{\mathbf{1}}(t) - \boldsymbol{q}_{\mathbf{1}}(t) \\ &= \frac{m_1 + m_2}{m_2}\boldsymbol{c}(t) - \frac{m_1 + m_2}{m_2}\boldsymbol{q}_{\mathbf{1}}(t) \end{aligned} \tag{C.20}$$

Then the following is deduced using the preceding equation and the first equation in this example, yielding:

$$\ddot{\boldsymbol{r}}\boldsymbol{(t)} = \frac{m_1 + m_2}{m_1 m_2}\boldsymbol{F}(\boldsymbol{r}) \tag{C.21}$$

Let $\tilde{\boldsymbol{r}}(t)$ be a solution of the preceding equation, then the motion $\boldsymbol{r}(t)$ is given by:

$$\boldsymbol{r}(t) = \tilde{\boldsymbol{r}}(t) + \frac{m_1 + m_2}{m_2}\boldsymbol{c}(t) \tag{C.22}$$

In the preceding equation $\boldsymbol{r}(t)$ is the sum of the relative vector dynamics $\tilde{\boldsymbol{r}}(t)$ and dynamics of centre of mass vector $\boldsymbol{c}(t)$. Since the dynamics of the centre of mass is known (see (C.6)) from the initial conditions (see (C.17)), it is sufficient to solve only (C.21).

Thus, without loss of generality, in the case of Newtonian 2-body systems, it can be assumed that $\boldsymbol{c_1} = \boldsymbol{c_2} = \boldsymbol{0}$ and then study the dynamics of the relative vector given by (C.21). It is customary to write this system as follows:

$$M\ddot{\boldsymbol{r}} = \boldsymbol{f}(\boldsymbol{r}) \tag{C.23}$$

In the preceding, M is the relative mass and given as:

$$M = \frac{m_1 m_2}{m_1 + m_2} \tag{C.24}$$

Unlike linear and angular momenta, the kinetic energy of closed system is not necessarily conserved. However, the total energy is conserved if no forces are present. We now expand upon the idea of Newtonian N-body systems by defining Newtonian N-body potential systems.

Definition C.1.9 (Newtonian N-Body Potential Systems)**.** *A Newtonian N-Body Potential System is a mechanical system formed by N point masses where the force on each point mass is the gradient of a function. Specifically,*

$$m_i \ddot{\boldsymbol{q}}_{\boldsymbol{i}} = -\frac{\partial V}{\partial \boldsymbol{q}_{\boldsymbol{i}}}, \text{ for } i = 1, ..., N \tag{C.25}$$

In the preceding equation $V : D \subset \mathbb{R}^{dN} \to \mathbb{R}$ is a smooth function called the potential energy.

Examples of Newtonian N-body potential systems are central force problems, which are defined as follows:

Definition C.1.10 (Central Force Problem)**.** *A central force problem is a mechanized system where:*

$$F(\boldsymbol{q}) = F(|q|)\frac{\boldsymbol{q}}{|q|} \tag{C.26}$$

This system is a Newtonian potential system with potential given by:

$$V(\boldsymbol{q}) = -U(|q|) \tag{C.27}$$

In the potential, $U(|q|)$ is an anti-derivative of F.

The following example is a very famous central force problem, the Kepler Problem.

Example C.1.3 (Central Force Problem: The Kepler Problem)**.** *The motion of a planet of mass m in the gravitational field of a sun of mass M, with $M >> m$ and G is the gravitational constant, is modeled by:*

$$m\ddot{\boldsymbol{q}} = -\frac{GmM}{||\boldsymbol{q}||^3}\boldsymbol{q} = -\frac{GmM}{||\boldsymbol{q}||^2}\hat{\boldsymbol{q}} \tag{C.28}$$

In the above equation:

$$\hat{\boldsymbol{q}} = \frac{\boldsymbol{q}}{||\boldsymbol{q}||} \tag{C.29}$$

An additional example of a Newtonian N-body potential system is given in the following example.

Example C.1.4. *(The Classical N-Body Problem) Consider the motion of N point masses under their mutual Newtonian gravitational forces (i.e., inverse-square law). This system is a Newtonian potential system with:*

$$V(\boldsymbol{q}) = \sum_{1 \leq i < j \leq N} \frac{-Gm_i m_j}{||\boldsymbol{q_i} - \boldsymbol{q_j}||} \tag{C.30}$$

The general problem of solving this system or determining its characteristics is called the Newtonian N-body problem.

The following example is a continuation of the previous Newtonian 2-body system example; this example examines potential energy.

Example C.1.5. *(Newtonian 2-Body Systems cont.) In the first part of Example* (C.1.2) *given previously, if one considers that $\boldsymbol{F}$ depends only upon the distance between m_1 and m_2, that is:*

$$\boldsymbol{F} = F(|\boldsymbol{q_2} - \boldsymbol{q_1}|)\frac{\boldsymbol{q_2} - \boldsymbol{q_1}}{|\boldsymbol{q_2} - \boldsymbol{q_1}|} = F(r)\frac{\boldsymbol{r}}{r} \tag{C.31}$$

then one finds that:

$$V(\boldsymbol{r}) = -U(r) = \int (-F(\rho))d\rho = -\int F(\rho)d\rho \tag{C.32}$$

In the above equation, $\int F(\rho)d\rho$ is an anti-derivative of F. Therefore, (C.23) *can be equivalently written as:*

$$M\ddot{\boldsymbol{r}} = -\frac{\partial V}{\partial \boldsymbol{r}} \tag{C.33}$$

We now consider the total energy of a system and state the Conservation of Energy Theorem with proof.

Definition C.1.11. *The total energy of a Newtonian N-body potential system with potential energy V is:*

$$E = K + V \tag{C.34}$$

In the preceding equation, K is the kinetic energy.

This allows one to arrive at the following conservation law.

Theorem C.1.3 (Conservation of Energy Theorem)**.** *In any Newtonian N-body potential system, the total energy is conserved.*

Proof C.1.1.

$$\begin{aligned}
\frac{dE}{dt} = \frac{d}{dt}(K+V) = \frac{d}{dt}\left(\frac{1}{2}\sum_{i=1}^{N} m_i||\dot{\boldsymbol{q}}_{\boldsymbol{i}}||^2 + \sum_{i=1}^{N} V(\boldsymbol{q}_{\boldsymbol{i}})\right) \\
= \frac{1}{2}\left(\sum_{i=1}^{N} m_i \dot{\boldsymbol{q}}_{\boldsymbol{i}} \cdot \ddot{\boldsymbol{q}}_{\boldsymbol{i}} + \sum_{i=1}^{N} m_i \ddot{\boldsymbol{q}}_{\boldsymbol{i}} \cdot \dot{\boldsymbol{q}}_{\boldsymbol{i}}\right) + \sum_{i=1}^{N} \frac{\partial V}{\partial \boldsymbol{q}_{\boldsymbol{i}}} \cdot \frac{d\boldsymbol{q}_{\boldsymbol{i}}}{dt} \\
= \sum_{i=1}^{N} m_i \dot{\boldsymbol{q}}_{\boldsymbol{i}} \cdot \ddot{\boldsymbol{q}}_{\boldsymbol{i}} + \sum_{i=1}^{N} \frac{\partial V}{\partial \boldsymbol{q}_{\boldsymbol{i}}} \cdot \frac{d\boldsymbol{q}_{\boldsymbol{i}}}{dt} \\
= \sum_{i=1}^{N} \dot{\boldsymbol{q}}_{\boldsymbol{i}} \cdot \left(m_i \ddot{\boldsymbol{q}}_{\boldsymbol{i}} + \frac{\partial V}{\partial \boldsymbol{q}_{\boldsymbol{i}}}\right) = 0
\end{aligned} \tag{C.35}$$

Remark C.1.2. *For this reason, Newtonian N-body potential systems are also referred to as conservative.*

We now define the conditions for a function to be translationally and rotationally invariant. Furthermore, we prove that the potential energy of the Classical N-Body Problem is translationally invariant.

Definition C.1.12. *A function $V : \mathbb{R}^{dN} \to \mathbb{R}$ is translationally invariant if:*

$$V(\boldsymbol{q_1} + \boldsymbol{t}, ..., \boldsymbol{q_N} + \boldsymbol{t}) = V(\boldsymbol{q_1}, ..., \boldsymbol{q_N}) \tag{C.36}$$

for any $\boldsymbol{q} = (\boldsymbol{q_1}, ..., \boldsymbol{q_N})$ and any $\boldsymbol{t} \in \mathbb{R}^d$.

A function $V : \mathbb{R}^{dN} \to \mathbb{R}$ is rotationally invariant if:

$$V(R\boldsymbol{q_1}, ..., R\boldsymbol{q_N}) = V(\boldsymbol{q_1}, ..., \boldsymbol{q_N}) \tag{C.37}$$

for any $\boldsymbol{q} = (\boldsymbol{q_1}, ..., \boldsymbol{q_N})$ and any $d \times d$ rotational matrix R (it is assumed that $d = 2$ or $d = 3$).

Example C.1.6. *It can be shown that the following potential is translationally invariant:*

$$V(\boldsymbol{q}) = \sum_{i,j=1}^{N} \frac{-Gm_i m_j}{||\boldsymbol{q_i} - \boldsymbol{q_j}||} \tag{C.38}$$

Let $\boldsymbol{t}$ *be a scalar quantity, then:*

$$V(\boldsymbol{q} + \boldsymbol{t}) = \sum_{i,j=1}^{N} \frac{-Gm_i m_j}{||\boldsymbol{q_i} + \boldsymbol{t} - \boldsymbol{q_j} - \boldsymbol{t}||} = \sum_{i,j=1}^{N} \frac{-Gm_i m_j}{||\boldsymbol{q_i} - \boldsymbol{q_j}||} = V(\boldsymbol{q}) \tag{C.39}$$

Note: It can also be shown that V *is rotationally invariant as well.*

C.2 Part 2: Lagrangian Formulation

In this section we introduce the Lagrangian and the Euler-Lagrange equations and their application to N-body systems. Additionally, we provide related definitions, examples, theorems and proofs. We begin this section by defining the Lagrangian of a Newtonian N-body potential system.

Definition C.2.1. *The Lagrangian of a Newtonian* N*-body potential system in* $\mathbb{R}^d$ *with Cartesian coordinates is a smooth function* $L : D \times \mathbb{R}^{dN} \to \mathbb{R}$, $D \subset \mathbb{R}^{dN}$ *open, of the form:*

$$L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\dot{\boldsymbol{q}}^T M \dot{\boldsymbol{q}} - V(\boldsymbol{q}) \tag{C.40}$$

where $\boldsymbol{q} = (\boldsymbol{q_1}, \boldsymbol{q_2}, \ldots, \boldsymbol{q_N})$.

In the above, the mass matrix M *is defined by:*

$$M = \begin{pmatrix} m_1 & & & & & & \\ & m_1 & & & & \mathit{0} & \\ & & m_1 & & & & \\ & & & \ddots & & & \\ & & & & m_N & & \\ & \mathit{0} & & & & m_N & \\ & & & & & & m_N \end{pmatrix} \tag{C.41}$$

In the above matrix, $m_i > 0$, *for all* $i = 1, 2, ..., N$.

Remark C.2.1. *The Lagrangian consists of* $L = K - V$, *where* K *is the kinetic energy and* V *is the potential energy.*

From the Lagrangian we now consider the equations of motion.

Theorem C.2.1. *The equations of motion of a Newtonian N-body potential system:*

$$m_i \ddot{\boldsymbol{q}}_{\boldsymbol{i}} = -\frac{\partial V}{\partial \boldsymbol{q}_{\boldsymbol{i}}},\ i = 1, 2, ..., N \tag{C.42}$$

are equivalent to the Euler-Lagrange Equations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \boldsymbol{q}_{\boldsymbol{i}}}\right) - \frac{\partial L}{\partial \boldsymbol{q}_{\boldsymbol{i}}} = 0,\ i = 1, 2, ..., N \tag{C.43}$$

for the Lagrangian L defined by (C.40).

Note that here, $\dot{\boldsymbol{q}}$ is considered as an independent variable, so that $L : \mathbb{R}^{2dN} \to \mathbb{R}$ and (C.40) has partial derivatives with respect to both $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$. Yet, when evaluating (C.40) and its derivatives, one must substitute $\dot{\boldsymbol{q}}(t) = \frac{d}{dt}\boldsymbol{q}(t)$

Proof C.2.1.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial L}{\partial \boldsymbol{q}} = \frac{d}{dt}(M\dot{\boldsymbol{q}}) + \frac{\partial V}{\partial \boldsymbol{q}} = M\ddot{\boldsymbol{q}} + \frac{\partial V}{\partial \boldsymbol{q}} = 0 \tag{C.44}$$

We now provide a definition for a time-independent Lagrangian system and state with proof a theorem regarding the coordinate-independence of the Euler-Lagrange equations.

Definition C.2.2. *A time-independent Lagrangian system on a configuration space $\mathbb{R}^{dN}$ is the system of ODEs in* (C.43)*, i.e. the Euler-Lagrange equations, for some function $L : D \times \mathbb{R}^{dN} \to \mathbb{R}, D \subset \mathbb{R}^{dN}$ open, $L = L(\boldsymbol{q}, \dot{\boldsymbol{q}})$, called the Lagrangian.*

Theorem C.2.2. *The Euler-Lagrange equations are coordinate-independent.*

Proof C.2.2. *Consider a change of coordinates $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{r})$, where $\boldsymbol{q}(\boldsymbol{r})$ is a smooth map with a smooth inverse. Then:*

$$\dot{\boldsymbol{q}} = \frac{d\boldsymbol{q}}{dt} = \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{r}} \cdot \frac{d\boldsymbol{r}}{dt} = \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{r}} \cdot \dot{\boldsymbol{r}}, \quad \textit{so} \quad \frac{\partial \dot{\boldsymbol{q}}}{\partial \boldsymbol{r}} = \frac{\partial^2 \boldsymbol{q}}{\partial \boldsymbol{r}^2} \cdot \dot{\boldsymbol{r}} \quad \textit{and} \quad \frac{\partial \dot{\boldsymbol{q}}}{\partial \dot{\boldsymbol{r}}} = \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{r}} \tag{C.45}$$

If it is also assumed that $\boldsymbol{q}(t)$ is smooth, then the equality of mixed partials gives:

$$\frac{d}{dt}\left(\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{r}}\right) = \frac{\partial \dot{\boldsymbol{q}}}{\partial \boldsymbol{r}}, \quad \textit{so} \quad \frac{d}{dt}\left(\frac{\partial \dot{\boldsymbol{q}}}{\partial \dot{\boldsymbol{r}}}\right) = \frac{d}{dt}\left(\frac{\partial \boldsymbol{q}}{\partial \boldsymbol{r}}\right) = \frac{\partial \dot{\boldsymbol{q}}}{\partial \boldsymbol{r}} \tag{C.46}$$

If the Euler-Lagrange equations hold for $(\boldsymbol{q}, \dot{\boldsymbol{q}})$, then:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{r}}}\right) - \frac{\partial L}{\partial \boldsymbol{r}} &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot \frac{\partial \dot{\boldsymbol{q}}}{\partial \dot{\boldsymbol{r}}}\right) - \left(\frac{\partial L}{\partial \boldsymbol{q}} \cdot \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{r}} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot \frac{\partial \dot{\boldsymbol{q}}}{\partial \boldsymbol{r}}\right) \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) \cdot \frac{\partial \dot{\boldsymbol{q}}}{\partial \dot{\boldsymbol{r}}} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot \frac{d}{dt}\left(\frac{\partial \dot{\boldsymbol{q}}}{\partial \dot{\boldsymbol{r}}}\right) - \frac{\partial L}{\partial \boldsymbol{q}} \cdot \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{r}} - \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot \frac{\partial \dot{\boldsymbol{q}}}{\partial \boldsymbol{r}} \\
&= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) \cdot \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{r}} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot \frac{\partial \dot{\boldsymbol{q}}}{\partial \boldsymbol{r}} - \frac{\partial L}{\partial \boldsymbol{q}} \cdot \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{r}} - \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot \frac{\partial \dot{\boldsymbol{q}}}{\partial \boldsymbol{r}} \\
&= \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial L}{\partial \boldsymbol{q}}\right) \cdot \frac{\partial \boldsymbol{q}}{\partial \boldsymbol{r}} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot \frac{\partial \dot{\boldsymbol{q}}}{\partial \boldsymbol{r}} - \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \cdot \frac{\partial \dot{\boldsymbol{q}}}{\partial \boldsymbol{r}} \\
&= \left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) - \frac{\partial L}{\partial \boldsymbol{q}}\right) = 0
\end{aligned} \tag{C.47}$$

By using the preceding theorem, one can perform a change of coordinate by expressing $L = K - V$ in the new coordinates in which the (new) Euler-Lagrange equations can be computed.

Definition C.2.3. *A Lagrangian system on a configuration space $\mathbb{R}^{dN}$ is the system of ODEs in the Euler-Lagrange equations, given below, for some function $L : \mathbb{R}^{2dN} \to \mathbb{R}$, called the Lagrangian.*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right) - \left(\frac{\partial L}{\partial \boldsymbol{q}}\right) = 0 \tag{C.48}$$

Here, an example is provided to illustrate the Lagrangian formulation and the Euler-Lagrange equations.

Example C.2.1 (Newtonian 2-Body Potential Systems)**.** *Recall from the previous Newtonian 2-body potential system example, the equation of motion* (C.33) *of the relative vector between two mass points with binary interaction. The Lagrangian for this problem is:*

$$L(\boldsymbol{r}, \dot{\boldsymbol{r}}) = \frac{M}{2}\dot{\boldsymbol{r}}^2 - V(\boldsymbol{r}) \tag{C.49}$$

The preceding Lagrangian can be expressed in components as:

$$L(r_x, r_y, \dot{r}_x, \dot{r}_y) = \frac{M}{2}(\dot{r}_x^2 + \dot{r}_y^2) - V\left(\sqrt{r_x^2 + r_y^2}\right) \tag{C.50}$$

The Euler-Lagrange equations are given by:

$$M\ddot{r}_x = -\frac{\partial V}{\partial r_x}, \qquad M\ddot{r}_y = -\frac{\partial V}{\partial r_y} \tag{C.51}$$

In this problem it is more convenient to use polar coordinates. Passing L to polar coordinates, the following is obtained:

$$L(r, \theta, \dot{r}, \dot{\theta}) = \frac{M}{2}\left(\dot{r}^2 + (r\dot{\theta})^2\right) - V(r, \theta) \tag{C.52}$$

with the Euler-Lagrange equations given by:

$$\frac{d}{dt}\frac{\partial}{\partial \dot{r}}\left(\frac{M}{2}\left(\dot{r}^2+(r\dot{\theta})^2\right)-V(r)\right)=\frac{\partial}{\partial r}\left(\frac{M}{2}\left(\dot{r}^2+(r\dot{\theta})^2\right)-V(r)\right) \tag{C.53}$$

$$\frac{d}{dt}\frac{\partial}{\partial \dot{\theta}}\left(\frac{M}{2}\left(\dot{r}^2+(r\dot{\theta})^2\right)-V(r)\right)=\frac{\partial}{\partial \theta}\left(\frac{M}{2}\left(\dot{r}^2+(r\dot{\theta})^2\right)-V(r)\right) \tag{C.54}$$

Calculating the above, the Euler-Lagrange equations become:

$$M\ddot{r}=Mr\dot{\theta}^2-V'(r) \tag{C.55}$$

$$\frac{d}{dt}Mr^2\dot{\theta}=0 \tag{C.56}$$

We have

$$Mr^2(t)\dot{\theta}(t)=const. \tag{C.57}$$

which can be used to reduce the system.

To conclude this section we briefly address the energy function for a Lagrangian and prove that this energy function is conserved in any time-independent Lagrangian system.

Definition C.2.4. *The energy function for a Lagrangian $L(\boldsymbol{q},\dot{\boldsymbol{q}})$ is:*

$$E=\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\cdot\dot{\boldsymbol{q}}-L \tag{C.58}$$

In particular, for Lagrangians of the form (C.40), the energy is given by:

$$E=\sum m_i||\dot{\boldsymbol{q}}_{\boldsymbol{i}}||^2-L=\frac{1}{2}\sum m_i||\dot{\boldsymbol{q}}_{\boldsymbol{i}}||^2+V=K+V \tag{C.59}$$

Theorem C.2.3. *In any time-independent Lagrangian system, the energy function is conserved.*

Proof C.2.3.

$$\begin{aligned}\frac{dE}{dt}=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\dot{\boldsymbol{q}}-L(\boldsymbol{q},\dot{\boldsymbol{q}})\right)&=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\dot{\boldsymbol{q}}\right)-\frac{dL(\boldsymbol{q},\dot{\boldsymbol{q}})}{dt}\\&=\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)\cdot\dot{\boldsymbol{q}}+\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\right)\cdot\ddot{\boldsymbol{q}}-\frac{\partial L}{\partial \boldsymbol{q}}\cdot\dot{\boldsymbol{q}}-\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\cdot\ddot{\boldsymbol{q}}\\&=\frac{\partial L}{\partial \boldsymbol{q}}\cdot\dot{\boldsymbol{q}}+\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\cdot\ddot{\boldsymbol{q}}-\frac{\partial L}{\partial \boldsymbol{q}}\cdot\dot{\boldsymbol{q}}-\frac{\partial L}{\partial \dot{\boldsymbol{q}}}\cdot\ddot{\boldsymbol{q}}=0\end{aligned} \tag{C.60}$$

The following section discusses the Hamiltonian formulation of the dynamics for mechanical systems, which serves as an alternative to the Lagrangian formulation discussed in this section.

C.3 Part 3: The Legendre Transform and Hamiltonian Mechanics

In this section we discuss the Lengendre transform which is used to derive the Hamiltonian of a system from the Lagrangian of a system. Furthermore, we introduce and discuss Hamiltonian mechanics and how the Hamiltonian can be applied to N-body systems. We begin by defining the Legendre transform.

Definition C.3.1. *The Legendre transform for a Lagrangian $L(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is the change of variables $(\boldsymbol{q}, \dot{\boldsymbol{q}}) \mapsto (\boldsymbol{q}, \boldsymbol{p})$ given by:*

$$\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}} \tag{C.61}$$

The new variables $\boldsymbol{p}$ are called the momenta.

Remark C.3.1. *If $L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\boldsymbol{q}^T M \boldsymbol{q} - V(\boldsymbol{q}) = \sum \frac{1}{2} m_i ||\dot{\boldsymbol{q}}_{\boldsymbol{i}}||^2 - V(\boldsymbol{q})$ then:*

$$\boldsymbol{p_i} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{i}}} = m_i \dot{\boldsymbol{q}}_{\boldsymbol{i}} \tag{C.62}$$

We now discuss hyperregular Lagrangians and how they can be transformed, via a Legendre transform, into a Hamiltonian system, which we briefly introduce.

Definition C.3.2. *A Lagrangian L is hyperregular if the Legendre transform for L is a diffeomorphism, that is, a differentiable map with a differentiable inverse.*

Remark C.3.2. *If $L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \sum \frac{1}{2} m_i ||\dot{\boldsymbol{q}}_{\boldsymbol{i}}||^2 - V(\boldsymbol{q})$ then the Legendre transform defined by $\boldsymbol{p_i} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{i}}} = m_i \dot{\boldsymbol{q}}_{\boldsymbol{i}}$ is hyperregular, since it is differentiable and has a differentiable inverse given by $\dot{\boldsymbol{q}}_{\boldsymbol{i}} = \frac{1}{m_i} \boldsymbol{p_i}$.*

Theorem C.3.1. *If $L : \mathbb{R}^{2dN} \to \mathbb{R}$ is any hyperregular Lagrangian, the the Euler-Lagrange equations:*

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{i}}}\right) - \frac{\partial L}{\partial \boldsymbol{q_i}} = 0, \qquad i = 1, ..., N \tag{C.63}$$

are equivalent to Hamilton's equations of motion,

$$\dot{\boldsymbol{q}} = \frac{\partial H}{\partial \boldsymbol{p}}, \qquad \dot{\boldsymbol{p}} = -\frac{\partial H}{\partial \boldsymbol{q}}, \tag{C.64}$$

for the Hamiltonian:

$$H(\boldsymbol{q}, \boldsymbol{p}) = \boldsymbol{p} \cdot \dot{\boldsymbol{q}}(\boldsymbol{q}, \boldsymbol{p}) - L(\boldsymbol{q}, \dot{\boldsymbol{q}}(\boldsymbol{q}, \boldsymbol{p})) \tag{C.65}$$

where $\dot{\boldsymbol{q}}(\boldsymbol{q}, \boldsymbol{p})$ is the second component of the inverse Legendre transform.

Proof C.3.1. *The Euler-Lagrange equations are second order ODEs in the variables* $\boldsymbol{q} = (\boldsymbol{q_1}, ..., \boldsymbol{q_N})$*, but they can also be interpreted as part of an equivalent first order system of ODEs in the variables* $(\boldsymbol{q}, \dot{\boldsymbol{q}})$ *with the extra equations* $\frac{d}{dt}\boldsymbol{q} = \dot{\boldsymbol{q}}$*. Applying the Legendre transform gives the equivalent system:*

$$\frac{d}{dt}\boldsymbol{q_i} = \dot{\boldsymbol{q}}_{\boldsymbol{i}}(\boldsymbol{q}, \boldsymbol{p}), \qquad \frac{d}{dt}\boldsymbol{p_i} - \frac{\partial L}{\partial \boldsymbol{q_i}} = 0 \tag{C.66}$$

The partial derivatives of H *can then be calculated as follows:*

$$\frac{\partial H}{\partial \boldsymbol{q_i}} = \boldsymbol{p_i} \cdot \frac{\partial \dot{\boldsymbol{q}}_{\boldsymbol{i}}(\boldsymbol{q_i}, \boldsymbol{p_i})}{\partial \boldsymbol{q_i}} - \frac{\partial L}{\partial \boldsymbol{q_i}} - \frac{\partial L}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{i}}} \cdot \frac{\partial \dot{\boldsymbol{q}}_{\boldsymbol{i}}(\boldsymbol{q_i}, \boldsymbol{p_i})}{\partial \boldsymbol{q_i}} = \frac{\partial L}{\partial \boldsymbol{q_i}} \tag{C.67}$$

$$\frac{\partial H}{\partial \boldsymbol{p_i}} = \dot{\boldsymbol{q}}_{\boldsymbol{i}}(\boldsymbol{q_i}, \boldsymbol{p_i}) + \boldsymbol{p_i} \cdot \frac{\partial \dot{\boldsymbol{q}}_{\boldsymbol{i}}(\boldsymbol{q_i}, \boldsymbol{p_i})}{\partial \boldsymbol{q_i}} - \frac{\partial L}{\partial \dot{\boldsymbol{q}}_{\boldsymbol{i}}} \cdot \frac{\partial \dot{\boldsymbol{q}}_{\boldsymbol{i}}(\boldsymbol{q_i}, \boldsymbol{p_i})}{\partial \boldsymbol{p_i}} = \dot{\boldsymbol{q}}_{\boldsymbol{i}}(\boldsymbol{q_i}, \boldsymbol{p_i}) \tag{C.68}$$

Consequently, the Euler-Lagrange equations are equivalent to:

$$\frac{d}{dt}\boldsymbol{q_i} = \frac{\partial H}{\partial \boldsymbol{p_i}}, \qquad \frac{d}{dt}\boldsymbol{p_i} = -\frac{\partial H}{\partial \boldsymbol{q_i}} \tag{C.69}$$

Remark C.3.3. *The Hamiltonian in the preceding theorem is the energy function* E *expressed as a function of* $\boldsymbol{p}$ *and* $\boldsymbol{q}$*. Note that this change of variables is valid because the Legendre transform has a differentiable inverse, that is,* L *is hyperregular.*

Remark C.3.4. *If* $L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \sum \frac{1}{2} m_i ||\dot{\boldsymbol{q}}_{\boldsymbol{i}}||^2 - V(\boldsymbol{q})$ *then* $\boldsymbol{p_i} = m_i \dot{\boldsymbol{q}}_{\boldsymbol{i}}$ *and*

$$H(\boldsymbol{q}, \boldsymbol{p}) = \boldsymbol{p} \cdot \dot{\boldsymbol{q}}(\boldsymbol{q}, \boldsymbol{p}) - L(\boldsymbol{q}, \dot{\boldsymbol{q}}(\boldsymbol{q}, \boldsymbol{p})) = \sum \frac{m_i}{2} ||\dot{\boldsymbol{q}}_{\boldsymbol{i}}||^2 + V = \sum \frac{1}{2m_i} ||\boldsymbol{p_i}||^2 + V \tag{C.70}$$

To summarize, if $L = K - V$ *then* $H = K + V$

Remark C.3.5. *Since the Euler-Lagrange equations are coordinate independent, then for any Hamiltonian deduced via hyperregular Legendre transform, Hamilton's equations are coordinate independent as well.*

Here an example of a simple harmonic oscillator is used to illustrate how a Legendre transform can be used to obtain the Hamiltonian for the system from the system's Lagrangian.

Example C.3.1 (Simple Harmonic Oscillator Example)**.** *Consider a particle of mass* m *moving in one-dimensional space under the influence of a spring with spring constant* k*. The position of the particle is given by* q *instead of* x*. The Lagrangian of this system is:*

$$L(q, \dot{q}) = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} k q^2 \tag{C.71}$$

Since there is only one degree of freedom ($N = 1$), the corresponding phase space is two dimensional ($2N$). Now parameterizing by (q, p),

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} \tag{C.72}$$

Inverting yields:

$$\dot{q} = \frac{p}{m} \tag{C.73}$$

Then the Hamiltonian is:

$$H(q,p) = \dot{q}p - L = \frac{p^2}{m} - \left(\frac{p^2}{2m} - \frac{kq^2}{2}\right) = \frac{p^2}{2m} + \frac{kq^2}{2} \tag{C.74}$$

Now that the Hamiltonian has been obtained from the Lagrangian via a Legendre transform, the equations of motion can be determined:

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}, \qquad \dot{p} = -\frac{\partial H}{\partial q} = -kq \tag{C.75}$$

Here, a simple example of transforming a Lagrangian into a Hamiltonian is given.

Example C.3.2. *A system of two point masses with Lagrangian:*

$$L = \frac{(m_1\dot{q}_1^2 + m_2\dot{q}_2^2)}{2} - V(q_1, q_2) \tag{C.76}$$

has Hamiltonian:

$$H = \frac{1}{2m_1}p_1^2 + \frac{1}{2m_2}p_2^2 + V(q_1, q_2) \tag{C.77}$$

We now define integrals of motion:

Definition C.3.3. *Given $\dot{x} = f(x), x \in D \subset \mathbb{R}^n$, a function $F : D \to \mathbb{R}$ is an integral of motion if $F(x(t)) = const.$, $\forall x(t)$ along a solution. Practically, it is sufficient to show $\frac{d}{dt}F(x(t)) = 0$, $\forall x(t)$ along a solution.*

Now that we have introduced Hamiltonian mechanics, we state and prove the following conservation theorem for Hamiltonians.

Theorem C.3.2. *In any Hamiltonian system with a Hamiltonian $H(\boldsymbol{q}, \boldsymbol{p})$, the Hamiltonian is conserved.*

Proof C.3.2.

$$\dot{H} = \sum_{i=1}^{N}\left(\frac{\partial H}{\partial \boldsymbol{q_i}} \cdot \dot{\boldsymbol{q}}_{\boldsymbol{i}} + \frac{\partial H}{\partial \boldsymbol{p_i}} \cdot \dot{\boldsymbol{p}}_{\boldsymbol{i}}\right) = \sum_{i=1}^{N}\left(\frac{\partial H}{\partial \boldsymbol{q_i}} \cdot \frac{\partial H}{\partial \boldsymbol{p_i}} - \frac{\partial H}{\partial \boldsymbol{p_i}} \cdot \frac{\partial H}{\partial \boldsymbol{q_i}}\right) = 0 \tag{C.78}$$

Now, the Legendre transform applied to the Lagrangian of the Newtonian N-body potential system (C.40) leads to:

$$\boldsymbol{p_i} = m_i \dot{\boldsymbol{q}}_{\boldsymbol{i}} \tag{C.79}$$

and hence:

$$H(\boldsymbol{q},\boldsymbol{p}) = \boldsymbol{p}\cdot\dot{\boldsymbol{q}}(\boldsymbol{q},\boldsymbol{p}) - L(\boldsymbol{q},\dot{\boldsymbol{q}}(\boldsymbol{q},\boldsymbol{p})) = \sum \frac{m_i}{2}||\dot{\boldsymbol{q}}_{\boldsymbol{i}}||^2 + V(\boldsymbol{q}) = \sum \frac{1}{2m_i}||\boldsymbol{p_i}||^2 + V \tag{C.80}$$

Thus the Hamiltonian of a Newtonian N-body potential system takes the form:

$$H(\boldsymbol{q},\boldsymbol{p}) = \frac{1}{2}\boldsymbol{p}^T M^{-1}\boldsymbol{p} + V(\boldsymbol{q}) \tag{C.81}$$

where M^{-1} is the inverse of the mass matrix (C.41).

Remark C.3.6. *The Hamiltonian in the preceding theorem is the energy function E expressed as a function of $\boldsymbol{q}$ and $\boldsymbol{p}$.*

Here we continue with our previous example of a Newtonian 2-body potential system, applying the Legendre transform to obtain the Hamiltonian and obtain the associated equations of motion.

Example C.3.3 (Newtonian 2-Body Potential Systems in Polar Coordinates)**.** *Recall from the previous example, the Lagrangian (C.52) giving the relative motion for a Newtonian 2-body potential system in polar coordinates. By applying the Legendre transform, the corresponding Hamiltonian is obtained:*

$$H(r,\theta,p_r,p_\theta) = \frac{1}{2M}\left(p_r^2 + \frac{p_\theta^2}{r^2}\right) + V(r) \tag{C.82}$$

for which the associated equations of motion are:

$$\dot{r} = \frac{p_r}{M} \qquad \dot{p}_r = \frac{p_\theta^2}{Mr^3} - V'(r) \tag{C.83}$$

$$\dot{\theta} = \frac{p_\theta}{mr^2} \qquad \dot{p}_\theta = 0 \tag{C.84}$$

C.4 Part 4: Cyclic Coordinates

In this section we discuss cyclic coordinates which are important as they allow for a reduced Hamiltonian to be obtained in this thesis. We begin the section by defining a cyclic coordinate.

Definition C.4.1. *Consider a smooth Lagrangian* $L : D \times R^N \to \mathbb{R},\ D \subset R^N$ *open:*

$$L = L(q_1, q_2, ..., q_N, \dot{q}_1, \dot{q}_2, ..., \dot{q}_N) \tag{C.85}$$

A coordinate q_i *for some* $i = 1, ..., N$ *is called cyclic if it does not appear explicitly in the Lagrangian expression.*

Now, without loss of generality, let q_1 be a cyclic coordinate. Then the Lagrangian takes the form:

$$L = L(q_2, q_3, ..., q_N, \dot{q}_1, \dot{q}_2, ..., \dot{q}_N) \tag{C.86}$$

Using the Legendre Transform, the corresponding Hamiltonian is:

$$H = H(q_2, q_3, ..., q_N, p_1, p_2, ..., p_N) \tag{C.87}$$

Then the following is obtained, where $i = 2, ..., N$:

$$\dot{q}_1 = \frac{\partial H}{\partial p_1} \tag{C.88}$$

$$\dot{p}_1 = -\frac{\partial H}{\partial q_1} \tag{C.89}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{C.90}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{C.91}$$

Given some initial conditions $(q_1(t_0), q_2(t_0), q_3(t_0), ..., q_N(t_0), p_1(t_0), p_2(t_0), ..., p_N(t_0))$, the equation for $\dot{p}_1$ leads to the momentum conservation:

$$p_1(t) = constant = p_1(t_0) = c \tag{C.92}$$

Further, provided the system given by $H(q_2, q_3, ..., q_N, p_1, p_2, ..., p_N)|_{p_1=c}$ is solved and explicit expressions for $q_i(t; c)$ and $p_i(t; c)$, $i = 2, 3, ..., N$ are found, then one can determine:

$$q_1(t) = q_1(t_0) + \int_{t_0}^{t} \frac{\partial H}{\partial p_1}(q_2(\tau), ..., q_N(\tau), p_2(\tau), ..., p_N(\tau); c)d\tau \tag{C.93}$$

So the presence of a cyclic coordinate implies that the corresponding momentum is constant along the motion (the constant being fixed by the initial conditions) and the number of degrees of freedom drops by one. This leads to the reduced Hamiltonian which is defined as follows:

Definition C.4.2. *The Reduced Hamiltonian is:*

$$H_{red}(q_2, q_3, ..., q_N, p_2, ..., p_N; c) = H(q_2, q_3, ..., q_N, p_1, p_2, ..., p_N)|_{p_1=c} \tag{C.94}$$

In H_{red}, the constant c may be treated as a parameter, which is allowed to take all the values permitted by the initial conditions.

In the particular case of Newtonian N-body potential systems with a Hamiltonian of the form:

$$H = H(q_2, q_3, ..., q_N, p_1, p_2, ..., p_N) = \sum_{i=1}^{N} \frac{1}{2m_i} p_i^2 + V(q_2, q_3, ..., q_N) \tag{C.95}$$

the following is obtained:

$$H_{red}(q_2, q_3, ..., q_N, p_2, ..., p_N; c) = \frac{1}{2m_1} c^2 + \sum_{i=2}^{N} \frac{1}{2m_i} p_i^2 + V(q_2, q_3, ..., q_N) \tag{C.96}$$

where $c = p_1(t) = const. = p_1(t_0)$

We now continue to expand our example of a Newtonian 2-body potential system.

Example C.4.1 (Newtonian 2-Body Potential Systems cont.)**.** *Recall from previous examples that the Lagrangian is* (C.52) *with corresponding Hamiltonian* (C.82)*. The Lagrangian and Hamiltonian are stated again below:*

$$L(r, \theta, \dot{r}, \dot{\theta}) = \frac{M}{2} \left(\dot{r}^2 + (r\dot{\theta})^2 \right) - V(r) \tag{C.97}$$

$$H(r, \theta, p_r, p_\theta) = \frac{1}{2M} \left(p_r^2 + \frac{p_\theta^2}{r^2} \right) + V(r) \tag{C.98}$$

It can be observed that θ is cyclic in L as this coordinate is missing from the expression. Consequently, the corresponding momentum p_θ is a conserved quantity. The is immediate from the Hamiltonian equations of motion:

$$\dot{r} = \frac{p_r}{M} \qquad \dot{p}_r = \frac{p_\theta^2}{Mr^3} - V'(r) \tag{C.99}$$

$$\dot{\theta} = \frac{p_\theta}{mr^2} \qquad \dot{p}_\theta = 0 \tag{C.100}$$

From the last equation of (C.100)*, it can be determined that $p_\theta(t) = const. = p(t_0) = c$. The reduced*

Hamiltonian is:

$$H_{red}(r, p_r; c) = \frac{1}{2M}p_r^2 + \frac{c^2}{2Mr^2} + V(r) \tag{C.101}$$

Definition C.4.3. *The effective potential is defined as:*

$$V_{\text{eff}} = \frac{c^2}{2Mr^2} + V(r) \tag{C.102}$$

C.5 Part 5: Equilibria and Relative Equilibria

In this section we discuss the equilibria and relative equilibra of a Hamiltonian and then discuss methods to determine the stability of these equilibria. We begin by defining an equilibrium point of a Hamiltonian system.

Definition C.5.1. *Given a Hamiltonian:*

$$H = H(\boldsymbol{q}, \boldsymbol{p}) = H(q_1, q_2, ..., q_N, p_1, p_2, ..., p_N) \tag{C.103}$$

a Hamiltonian equilibrium is an equilibrium $(\boldsymbol{q_e}, \boldsymbol{p_e})$ *point of the Hamiltonian system of equations associated to* H*.*

The methods for studying the stability of each equilibrium point, as outlined in Part A of the Appendix, are used for studying the stability of a Hamiltonian equilibrium. However, some specific features of the analysis occur due to the structure of the Hamiltonian equations. Let $(\boldsymbol{q_e}, \boldsymbol{p_e})$ be an equilibrium for (C.103).

To apply the Lyapunov Method, the Lyapunov function is defined as:

$$F(\boldsymbol{q}, \boldsymbol{p}) = H(\boldsymbol{q}, \boldsymbol{p}) - H(\boldsymbol{q_e}, \boldsymbol{p_e}) \tag{C.104}$$

Then,

1. $F(\boldsymbol{q_e}, \boldsymbol{p_e}) = H(\boldsymbol{q_e}, \boldsymbol{p_e}) - H(\boldsymbol{q_e}, \boldsymbol{p_e}) = 0$

2. For any solution $(\boldsymbol{q}(t), \boldsymbol{p}(t))$,

$$\frac{d}{dt}F(\boldsymbol{q}(t), \boldsymbol{p}(t)) = \frac{d}{dt}H(\boldsymbol{q}(t), \boldsymbol{p}(t)) = 0 \tag{C.105}$$

   since the Hamiltonian is conserved along solutions.

So the second and fourth conditions of Lyapunov's First Stability Theorem are satisfied for any Hamiltonian system. The third condition is satisfied if $(\boldsymbol{q_e}, \boldsymbol{p_e})$ is a local minimum point of the Hamiltonian. The first condition is satisfied if F is continuously differentiable on its domain.

We now discuss Lyapunov's Stability Theorem for Hamiltonian systems and discuss the stability of equilibria.

Theorem C.5.1 (Lyapunov's Stability Theorem for Hamiltonian Systems)**.** *An equilibrium of a Hamiltonian system is stable if it is a local minimum point of the Hamiltonian.*

Recall that a Newtonian N-body potential system, with Hamiltonian $H(\boldsymbol{q}, \boldsymbol{p})$ defined by (C.81) has the equations of motion:

$$\dot{\boldsymbol{q}} = M^{-1}\boldsymbol{p}, \quad \dot{\boldsymbol{p}} = -\frac{\partial V}{\partial \boldsymbol{q}} \tag{C.106}$$

It follows that equilibria are always of the form $(\boldsymbol{q_e}, \boldsymbol{p_e}) = (\boldsymbol{q_{cr}}, \boldsymbol{0})$, where $\boldsymbol{q_{cr}}$ is a critical point of the potential $V(\boldsymbol{q})$. Further,

$$H(\boldsymbol{q_e}, \boldsymbol{p_e}) = H(\boldsymbol{q_{cr}}, \boldsymbol{0}) = \frac{\boldsymbol{0}^T M^{-1}\boldsymbol{0}}{2} + V(\boldsymbol{q_{cr}}) = V(\boldsymbol{q_{cr}}) \tag{C.107}$$

Since the diagonal of the matrix M has strictly positive entries, the matrix M^{-1} has strictly positive entries as well and so for any vector $\boldsymbol{p}$, $\boldsymbol{p}M^{-1}\boldsymbol{p} \geq 0$. Hence:

$$H(\boldsymbol{q}, \boldsymbol{p}) = \frac{\boldsymbol{p}M^{-1}\boldsymbol{p}}{2} + V(\boldsymbol{q}) \geq V(\boldsymbol{q}) \text{ for all } (\boldsymbol{q}, \boldsymbol{p}) \tag{C.108}$$

If $\boldsymbol{q_{cr}}$ is a local minimum for $V(\boldsymbol{q})$, then using (C.107) and (C.108), the following is obtained:

$$H(\boldsymbol{q}, \boldsymbol{p}) - H(\boldsymbol{q_e}, \boldsymbol{p_e}) \geq V(\boldsymbol{q}) - V(\boldsymbol{q_{cr}}) \geq 0 \text{ for all } (\boldsymbol{q}, \boldsymbol{p}) \tag{C.109}$$

and so $(\boldsymbol{q_e}, \boldsymbol{p_e}) = (\boldsymbol{q_{cr}}, \boldsymbol{0})$ is a local minimum for the Hamiltonian.

Proposition C.5.1 (Lyapunov's Stability Theorem for Newtonian N-Body Potential Systems)**.** *If $\boldsymbol{q_{cr}}$ is a local minimum point of the potential then the equilibrium $(\boldsymbol{q_{cr}}, \boldsymbol{0})$ of a Newtonian N-body potential system is stable.*

Corollary C.5.1. *If the Hessian $D^2V(\boldsymbol{q_e})$ is positive definite, then the equilibrium $(\boldsymbol{q_{cr}}, \boldsymbol{0})$ is stable.*

If the Lyapunov criterion fails, then one must resort to spectral analysis. It can be shown (Meyer et. al. (1992)) that the eigenvalues of the linearization matrix of a Hamiltonian system always form conjugate quadruplets, that is, they take the form:

$$\lambda_{1,2,3,4} = \pm\alpha \pm i\beta \tag{C.110}$$

This implies that asymptotical stability (where $\mathrm{Re}(\lambda) < 0$ for all λ) is not possible in any Hamiltonian system, but only spectral stability. The latter is ensured by:

$$\mathrm{Re}(\lambda) = 0 \text{ for all } \lambda \tag{C.111}$$

In the presence of cyclic coordinates, the dynamics are reduced by one degree of freedom and the reduced Hamiltonian depends on a parameter.

Definition C.5.2. *An equilibrium point of the reduced Hamiltonian H_{red} is a relative equilibrium of the unreduced Hamiltonian H.*

Note that equilibria of H_{red} depend parametrically on c. The methods for studying Hamiltonian equilibria may now be applied to the case of relative equilibria with the distinction that the existence and stability of relative equilibria depend upon c.

Remark C.5.1. *An equilibrium of H_{red} with $c = 0$ is an equilibrium of H.*

Appendix D

Bifurcations of Differential Equations with One External Parameter

D.1 First-order systems of differential equations

In this Appendix we briefly describe the typical bifurcations that appear in first-order systems of differential equations in the presence of an external parameter.

Definition D.1.1. *A first-order system of differential equations is defined as:*

$$\dot{x} = f(x) \tag{D.1}$$

where $x(t)$ is a real-valued function of time t and f is a smooth real-valued function.

Definition D.1.2. *An equilibrium of* (D.1) *is a solution $x(t)$ that is constant in time, that is $x(t)$ =const.= $x(t_0)$.*

If parameters are present, equilibria can be created and destroyed, or their stability can change. These qualitative changes are known as *bifurcations* and the parameter values where they occur are called *bifurcation points*.

D.1.1 Saddle-Node Bifurcation

Saddle-node bifurcations consist of a mechanism by which two equilibria are created and destroyed as a parameter is varied.

Consider the equation $\dot{x} = r + x^2$ with $r \in \mathbb{R}$ is the parameter. Based upon the value of r the system will display two, one or zero fixed points (see Figure D.1). When $r < 0$, there are two fixed points, the left point is stable and the right point is unstable as shown in Figure D.1 (a). When $r = 0$, there is one semi-stable fixed point as shown in Figure D.1 (b). Finally, when $r > 0$, there are zero fixed points as shown in Figure D.1 (c).

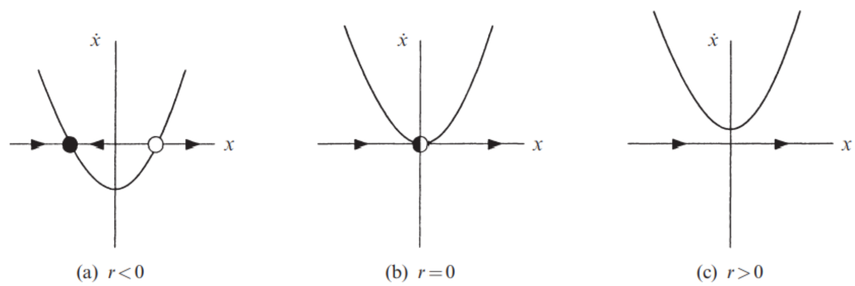


Figure D.1: Plot of $\dot{x} = r + x^2$ with Varied r

As $r \to 0$ from below, the parabola moves up and the two fixed points move towards each other and eventually become a single half-stable fixed point when $r = 0$. The single fixed point vanishes as soon as $r > 0$ and thus $r = 0$ is considered a bifurcation point. We can now consider the curve $r = -x^2$, i.e., $\dot{x} = 0$ which shows fixed points for different r values. Unstable fixed points are denoted using a dashed line and stable fixed points are denoted using a solid line as shown in Figure D.2, which is called a bifurcation diagram.

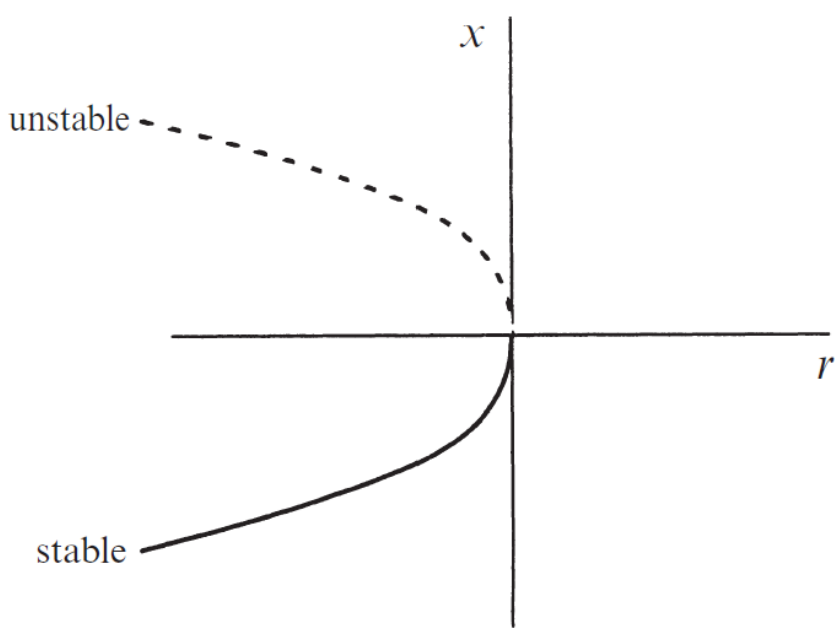


Figure D.2: Bifurcation Diagram of $\dot{x} = r + x^2$

D.1.2 Transcritical Bifurcation

A *transcritical bifurcation* is the mechanism by which the stability of an equilibrium may change. A typical transcritical bifurcation occurs in the equation:

$$\dot{x} = rx - x^2 \tag{D.2}$$

Equilibria exist for all values of the parameter r. Figure D.3 shows the vector field of the system as r is varied.

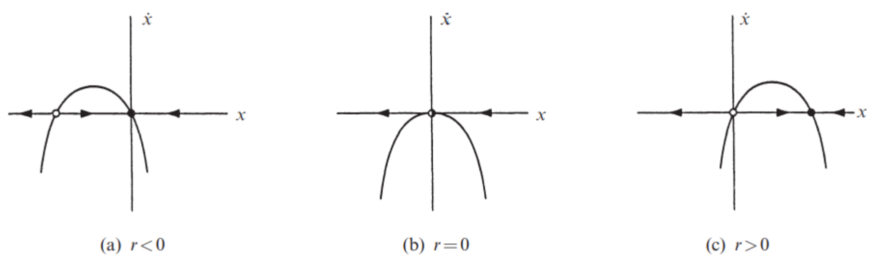


Figure D.3: Plot of $\dot{x} = rx - x^2$ with Varied r

As shown in Figure D.3, the equilibrium $x^* = 0$ is stable when $r < 0$, half-stable when $r = 0$ and unstable when $r > 0$. As r is varied, the stability changes, but $x^* = 0$ remains a fixed point for all values of r. Figure D.4 is the bifurcation diagram of this transcritical bifurcation.

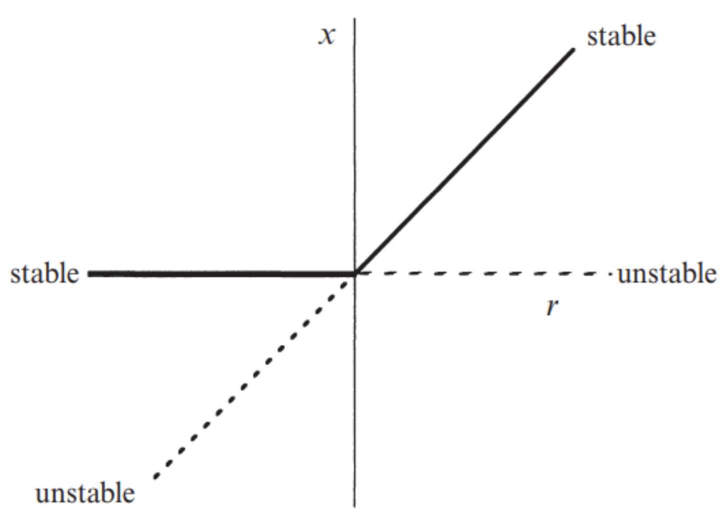


Figure D.4: Bifurcation Diagram of $\dot{x} = rx - x^2$

D.1.3 Pitchfork Bifurcation

Pitchfork bifurcations consist in the "birth" or "death" of pairs of equilibria. There are two types of pitchfork bifurcations, *supercritical* and *subcritical*.

Supercritical Pitchfork Bifurcation

A typical supercritical pitchfork bifurcation occurs in the equation:

$$\dot{x} = rx - x^3 \tag{D.3}$$

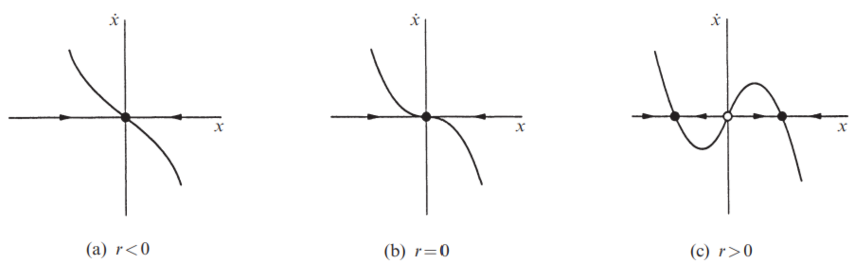


Figure D.5: Plot of $\dot{x} = rx - x^3$ with Varied r

and is depicted qualitatively in Figure D.5. For $r < 0$ there is only one equilibrium and it is stable. The values $r = 0$ is the bifurcation value. When $r > 0$, the fixed point at the origin has become unstable and the two new stable fixed points are symmetric about the origin and located at $x^* = \pm\sqrt{r}$. The bifurcation diagram for this system is given in Figure D.6.

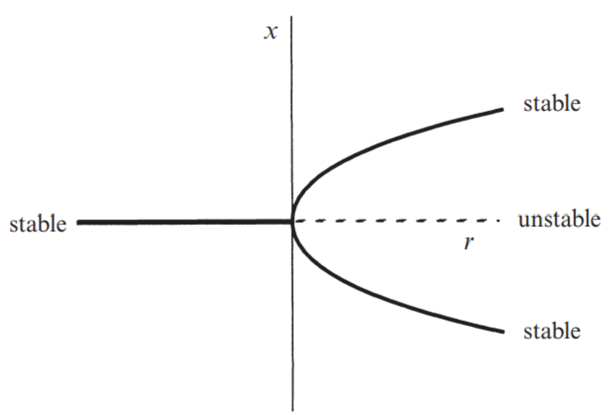


Figure D.6: Bifurcation Diagram of $\dot{x} = rx - x^3$

Subcritical Pitchfork Bifurcation

A subcritical pitchfork bifurcation occurs in

$$\dot{x} = rx + x^3 \tag{D.4}$$

Compared to Figure D.6, the bifurcation diagram of $\dot{x} = rx + x^3$ given in Figure D.7 is inverted. The nonzero equilibria are unstable and only exist below the bifurcation ($r < 0$).

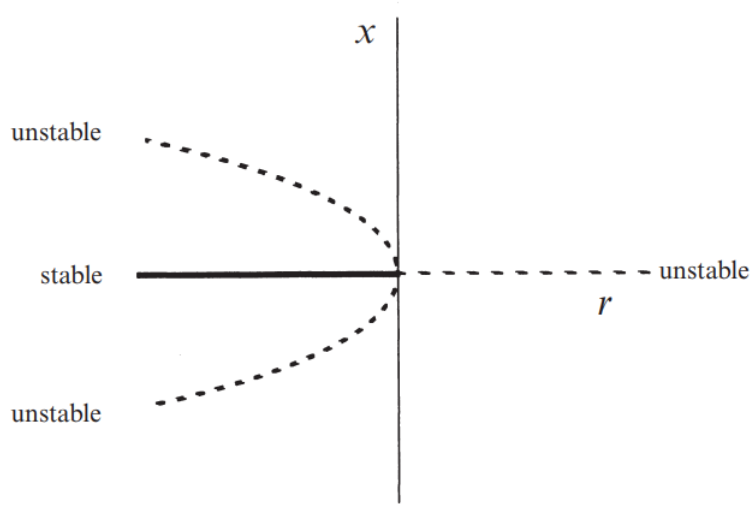


Figure D.7: Bifurcation Diagram of $\dot{x} = rx + x^3$

Appendix E

Geopotential Model of the Earth

This appendix is based on Chapter VI from [Stiefel & al. 1971].

E.1 General Theory

We consider an inertial frame, x_1, x_2, x_3, and a point mass (x_1, x_2, x_3) of mass m and distance r from the origin. The attracting mass M is not necessarily located at the origin but rather is located at (a_1, a_2, a_3) with distance q from the origin. The distance between the mass points m and M is denoted as Δ and the angle between their position vectors is denoted as θ. We have that:

$$\Delta^2 = (x_1 - a_1)^2 + (x_2 - a_2)^2 + (x_3 - a_3)^2 \tag{E.1}$$

or

$$\Delta^2 = r^2 + q^2 - 2rq\cos(\theta) \tag{E.2}$$

Differentiating (E.1), we obtain:

$$\frac{\partial \Delta}{\partial \boldsymbol{x}} = \frac{1}{\Delta}(x_1 - a_1, x_2 - a_2, x_3 - a_3) \tag{E.3}$$

The components of the attraction $\boldsymbol{G}$ of M acting on m are (per unit mass of m):

$$G_1 = -k^2 M \frac{x_1 - a_1}{\Delta^3}, \quad G_2 = -k^2 M \frac{x_2 - a_2}{\Delta^3}, \quad G_3 = -k^2 M \frac{x_3 - a_3}{\Delta^3} \tag{E.4}$$

where k^2 is the gravitational constant. This force can be derived from a potential:

$$U = -\frac{k^2 M}{\Delta} \tag{E.5}$$

according to the rule:

$$\boldsymbol{G} = -\frac{\partial U}{\partial \boldsymbol{x}} \tag{E.6}$$

The scalar function U is the gravitational potential of the mass M.

Theorem E.1.1. *The gravitational potential of a mass point M satisfies Laplace's partial differential equation:*

$$\nabla^2 U \equiv \frac{\partial^2 U}{\partial x_1^2} + \frac{\partial^2 U}{\partial x_2^2} + \frac{\partial^2 U}{\partial x_3^2} = 0 \tag{E.7}$$

at any point that is not M.

Proof E.1.1.

$$\frac{\partial^2 U}{\partial x_1^2} = -\frac{\partial G_1}{\partial x_1} = -k^2 M \frac{\Delta^3 - 3(x_1 - a_1)^2 \Delta}{\Delta^6} \tag{E.8}$$

By computing all partial derivatives in (E.7) *as shown in* (E.8)*, the result is obtained.*

We now consider a sphere centered at the origin and passing through M. We denote $r/q = t < 1$, that is m is located in the interior of the sphere, so that:

$$\frac{1}{\Delta} = \frac{1}{q}(1 + t^2 - 2t\cos(\theta))^{-1/2} \tag{E.9}$$

The binomial expansion of (E.9) is:

$$\begin{aligned}(1 + t^2 - 2t\cos(\theta))^{-1/2} =& 1 - \frac{1}{2}(t^2 - 2t\cos(\theta)) + \frac{1 \cdot 3}{2 \cdot 4}(t^2 - 2t\cos(\theta))^2 \\ &- \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}(t^2 - 2t\cos(\theta))^3 \\ &+ \frac{1 \cdot 3 \cdot 5 \cdot 7}{2 \cdot 4 \cdot 6 \cdot 8}(t^2 - 2t\cos(\theta))^4 - \dots\end{aligned} \tag{E.10}$$

The preceding expansion can be arranged as a power series in t where the coefficient of t^n is a polynomial P_n with $\cos(\theta)$ as its argument. The polynomial P_n is the n-th Legendre polynomial, thus we obtain:

$$(1 + t^2 - 2t\cos(\theta))^{-1/2} = \sum_{n=0}^{\infty} P_n(\cos(\theta)) t^n \tag{E.11}$$

where $(1 + t^2 - 2t\cos(\theta))^{-1/2}$ is the generating function of the Legendre polynomials. The series (E.11) converges if $|t| < 1$, that is if m is located in the interior of the sphere. Then the first five Legendre polynomials are:

$$P_0(\cos(\theta)) = 1 \tag{E.12}$$

$$P_1(\cos(\theta)) = \cos(\theta) \tag{E.13}$$

$$P_2(\cos(\theta)) = -\frac{1}{2} + \frac{3}{2}\cos^2(\theta) \tag{E.14}$$

$$P_3(\cos(\theta)) = -\frac{3}{2}\cos(\theta) + \frac{5}{2}\cos^3(\theta) \tag{E.15}$$

$$P_4(\cos(\theta)) = \frac{3}{8} - \frac{15}{4}\cos^2(\theta) + \frac{35}{8}\cos^4(\theta) \tag{E.16}$$

The potential function when m is contained in the sphere is:

$$U = -k^2 M \sum_{n=0}^{\infty} P_n(\cos(\theta)) \frac{r^n}{q^{n+1}}, \qquad r < q \tag{E.17}$$

Now, if $r > q$, that is m is located outside of the sphere, we set $t = q/r$ and:

$$\frac{1}{\Delta} = \frac{1}{r}(1 + t^2 - 2t\cos(\theta))^{-1/2} \tag{E.18}$$

Then using the same procedure that was previously employed, we obtain the potential function when m is located outside of the sphere:

$$U = -k^2 M \sum_{n=0}^{\infty} P_n(\cos(\theta)) \frac{q^n}{r^{n+1}}, \qquad r > q \tag{E.19}$$

Finally, if m is located on the surface of the sphere, the following functions apply:

$$P_n(\cos(\theta)) r^n, \qquad \frac{P_n(\cos(\theta))}{r^{n+1}}, \qquad n = 0, 1, 2, \ldots \tag{E.20}$$

The functions (E.20) depend only on the position of m and they can be expressed as functions of x_1, x_2, x_3, which leads to the following theorem.

Theorem E.1.2. *The functions* (E.20) *satisfy Laplace's equation.*

Proof E.1.2. *Refer to page 103 in [Stiefel & al. 1971].*

E.2 Spheroids

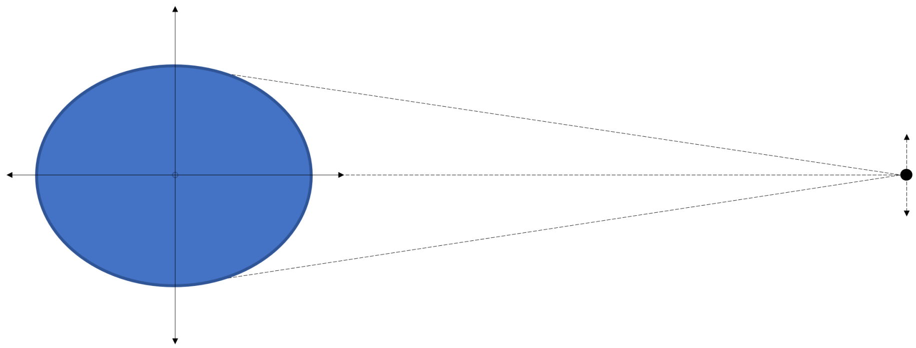


Figure E.1: Spheroid Planet with Orbiting Satellite

We consider an attracting body with its centre of mass located at the origin of a rectangular frame x_1, x_2, x_3. It is assumed that the mass distribution inside the body is rotationally symmetrical about the x_3-axis and symmetrical about the x_1x_2-plane. Such a body is called a spheroid and Earth is nearly shaped as a spheroid where the x_1x_2-plane is the equatorial plane as shown in Figure E.1 with an orbiting satellite and the corresponding forces acting upon it. The angle between the x_3-axis and the position vector of m is denoted θ. Using the theory from the previous section, a potential function is constructed:

$$U = \sum_{n=0}^{\infty} c_n \frac{P_n(\cos(\theta))}{r^{n+1}} \tag{E.21}$$

where r is the distance between m and the origin and c_n is determined by the shape of the spheroid. Since the Legendre polynomials satisfy the following equality:

$$P_n(\cos(\theta)) = (-1)^n P_n(-\cos(\theta)) \tag{E.22}$$

the terms corresponding to odd values of n cancel due to symmetry about the x_1x_2-plane. The coefficient c_0 is determined to be $c_0 = -k^2M$, where M is the mass of the spheroid. In general:

$$U = -\frac{k^2M}{r} + \sum_{n=2}^{\infty} c_n \frac{P_n(\cos(\theta))}{r^{n+1}} \tag{E.23}$$

Then for Earth, (E.23) becomes:

$$U = -\frac{k^2M}{r} + \frac{k^2M}{r} \sum_{n=2}^{\infty} J_n \left(\frac{R}{r}\right)^n P_n(\cos(\theta)) \tag{E.24}$$

where R is the radius from the centre of Earth to the equator and J_n are dimensionless constants. Taking into account only the J_2 term, the following approximate representation is obtained:

$$U = -\frac{k^2 M}{r} + \frac{k^2 M J_2}{2} \frac{R^2}{r^3}(3\cos^2\theta - 1), \qquad \cos(\theta) = \frac{x_3}{r} \tag{E.25}$$

For $\theta = \pi/2$, or equivalently, for $x_3 = 0$, that is at the Equator, the potential reads:

$$U = -\frac{k^2 M}{r} - \frac{k^2 M J_2}{2} \frac{R^2}{r^3}\,. \tag{E.26}$$

and so it is of the form

$$U = -\frac{1}{r} - \frac{\varepsilon}{r^3}\,. \tag{E.27}$$

where without loosing generality, one may re-scale $k^2 M = 1$ and denote $\varepsilon := R^2 J_2/2$. For Earth, J_2 is of order 10^{-3}.

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