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Three Essays On Improving Patient Access To Specialized Services

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By

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Abstract

This dissertation studies how patient access to specialized services in referral networks can be improved. The first study focused on optimizing and coordinating referral and scheduling decisions in a centralized referral network. I proposed a bi-level optimization model which enables the referrer to make optimal decisions for different scenarios based on available capacity in the network and operational competency levels of surgeons. First, I derived optimal scheduling policies for each surgeon in the network. Next, optimal referral decisions for the central referrer were derived for each capacity scenario. Finally, I studied how incorporating fairness in referral decisions can impact patient access to surgeons.

The second study applies deep reinforcement learning (DRL) algorithms in centralized referral networks that help referrers make optimal decisions during the patient referral process while considering different challenges such as distance of the patient from the specialist and wait time. First, I studied the potential impact of using these algorithms in a single centralized referral network. Next, I defined a general framework under which two adjacent centralized referral networks that are applying DRL algorithms can collaborate. Finally, I studied how governments can motivate networks to collaborate and what the impact would be of this collaboration on patient access to surgeons.

The third study focuses on the patient referral process in the Waterloo Cataract referral network. First, I analyzed the data and three different ways that are practiced by the network to refer patients to surgeons. Next, I simulated the whole network and studied how changing current referral policies or adding more surgeons to the network can impact patient access to surgeons.

Acknowledgment

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Finally, I would like to express my gratitude to my parents and my family. Without their tremendous understanding and encouragement in the past few years, it would be impossible for me to complete my study.

Ethics Statement

The author, whose name appears on the title page of this work, has obtained the REB approval (application #7045) for the third chapter of the dissertation which used sensitive patient data.

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Chapter 1

Introduction

On-time access of patients to specialized services is one of the main concerns of decision makers in many health care systems. Waiting too long for specialists not only puts patients' lives in danger, but also results in excessive costs, putting pressure on the healthcare system. In fact, the cost of wait time per patient to the health care system in Canada is estimated to be approximately between $2,254 and $6,838. Exacerbating this situation are imbalanced referral rates to specialists; there are some specialists who are overloaded with referrals, making patient wait times far too long, while there are other specialists who have spare capacities.

Patients can gain access to specialized services through two main referral schemes, namely centralized and decentralized referral networks. Under both referral schemes, the patient is first seen by a general practitioner (GP). Under the decentralized referral scheme, which is the dominant one in most countries, including Canada, when further treatment is needed the GP has access to a limited number of specialists in the network and directly refers the patient to one of them. Under a centralized referral scheme, the GP sends the patient to a referral network. Then, the process of referring them to a specialist is handled by a third party called the central referrer which has access to all specialists in a network.

In recent years, numerous referral networks have moved toward centralizing their referral processes to specialists. However, as indicated by various studies, there are still various questions around centralization: 1) how should centralization be implemented, 2) what policies should be applied by different players of a centralized network, and 3) to what degree can a centralized network actually improve patient access to specialists?

The first chapter of this thesis studies how referral and scheduling policies in a centralized referral network can be coordinated. To ensure the mathematical tractability of the problem, we focused on a centralized referral network with two specialists where different scenarios are defined based on available capacity in the system and the specialists' operational competency level. To coordinate the referral and scheduling decisions, we modeled the system as a bi-level optimization problem and extracted optimal referral and scheduling policies for each scenario. Model extensions were then studied which examine fair-allocation referral models and the benefit of centralization.

In the second chapter of the thesis, we were inspired by how COVID-19 affected referral networks in Canada, so we applied a Deep Reinforcement Learning (DRL) methodology on a centralized referral network which allowed us to study referral rates in a larger scale referral network with multiple specialists. In addition, we applied the methodology on two centralized networks and studied how collaboration between the two networks could further improve patient access to specialists. We show that reinforcement learning methodologies have the potential to significantly help decision-makers in a centralized referral network reduce patients wait times. Our results also indicate that a right amount of incentive from the government can play a great role in motivating networks to collaborate which results in improved access to specialists.

In the third chapter of the thesis, we focus on the centralized cataract network in Waterloo which covers referrals from three major cities, namely Kitchener, Cambridge, and Guelph, and more than 100 townships. We studied the current referral process and its impact on patient average wait time over the next four years and investigated to what degree implementing different referral policies or adding more resources to the network could further improve patient access to specialists. Our results show that the system utilization rate is high and, therefore, implementing new policies without adding more resources to the network does not have a significant impact on patient average wait time .

Chapter 2

Coordinated Referral and Scheduling Decisions for Specialized Healthcare Services

2.1 Introduction

In many health care systems, providing access to specialized services in a timely manner is challenging. Wait times are often long and highly variable leading to inequities in access to specialized care. Many studies have associated poor health outcomes with prolonged wait times (Lawrentschuk et al. 2003a, Haddad et al. 2002a, McKeever et al. 2006) and wait times for specialized services have been recognized as a key impediment to access to quality care (Sanmartin et al. 2000, BA et al. 2005, Bichel et al. 2009, Viberg et al. 2013a, Patel et al. 2018, Bleustein et al. 2014). While these challenges have been acute and persistent in single payer systems such as those in Canada (Barua and Jacques 2018) challenges in access to specialized services are also present in privatized healthcare systems (Penn et al. 2019, Shulkin 2017).

Strategies to manage wait times effectively include attention to the operations of the specialized service as well as improving the referral system which manages the transition of patients from primary care to the provider of specialized care. With respect to the provider's operations, attention has been devoted to increasing resources and improving the efficiency of the operations and delivery of the service. Increasing capacity through increased resources is constrained by costs and availability of human and physical resources which points to improving the efficiency of delivery as a key pathway. Studies like Green (2005) and VanBerkel and Blake (2007) examine the optimal management of capacity and show the potential for dramatic impacts on wait times. Improving the scheduling and operations of specialized services has also been extensively studied both directly by improving policies (Froehle and Magazine 2013, Ahmadi-Javid et al. 2017a) and by providing incentives to surgeons to hit wait time targets (Frank and Brunsberg 1999, Marcus et al. 2009, Viberg et al. 2013a).

This study considers how a referral system can build policies which acknowledge the operational challenges and are coordinated with the provider's incentives. We consider a centralized intake system where a single agent receives all referrals from primary care and allocates these referred patients between the full set of independent providers. Specialized care may include surgical procedures, imaging or other consultations where a referral is needed. The referral is received by a single provider which may represent a unique health care professional (e.g. surgeon or consulting physician) or a centre managing multiple professionals (e.g. working in a centre, hospital or region). The centralized intake system model, also referred to as a pooled referral system, is becoming an increasingly popular replacement for the traditional decentralized model for improving equitable access to specialized medical services (e.g. Saskatchewan-Initiative 2013). In the decentralized model, patients are directly referred from primary care to specialized services. The centralized model can take into account a broader set of information at the time of the referral and has the potential to improve resource utilization and fairness from the point of view of both patients and providers. A number of papers, such as Kinchen et al. (2004a), Barnett et al. (2012b) and An et al. (2018) explore the adverse affects that stem from poor referral decisions. However, these papers are expository in nature and do not provide prescriptive suggestions on how a centralized intake system can rectify these shortcomings.

Our paper takes advantage of this transition to centralized intake which provides the opportunity for increased optimization of the referral process. The broader research question of interest is how to design centralized referral mechanisms that are both optimized and fair. More precisely, the referral should both minimize wait times and ensure that access to low wait time specialized services is equitably distributed throughout the network and patient populations. Despite the increased applied interest in these systems best practices regarding the operations of these systems and realizing these goals remains poorly understood.

We study the problem of designing a centralized referral system which includes the receipt of the referral, the allocation of the referral to the provider and the providers scheduling of the procedure. To permit tractable analysis of this system, we take a different approach from much of the scheduling literature. We avoid focus on the algorithmic and mathematical details of a particular scheduling policy by defining a model for achievable scheduling outcomes associated with the providers' operational capabilities. To motivate this approach we compute the achievable region for a prioritized queueing system and show that the key properties can be modeled via a simple reduced form. This approach allows us to study the implications of particular operational capabilities to the management of the referral system. While this reduced form approach appears to be somewhat novel, it is not the first use of a general achievable region function to derive results which are generalizable to a range of operational settings. Pavlin (2017) for instance uses a general convex function to characterize achievable wait times for prioritized queueing systems. The approach taken in this paper leads to a tractable two-stage optimization problem which can encompass the range of scheduling challenges present in healthcare systems.

The centralized referrer and the providers both seek to maximize the proportion of all patients served within their target wait times. This incentive is considered due to the fact that in many countries including Canada providing patients with on-time access to specialized services has become the key measure performance for assessing access to care. The province of Ontario for instance maintains a public website which shows the proportion of cases seen within procedure specific target wait times for a wide range of specialist services (Government of Ontario 2022). Using access to diagnostic imaging as an example, there are four patient priority levels and the target times for the first and forth priority patients are 24 hours and 28 days respectively. In October of 2021, 96% of first priority and 51% of fourth priority patients were scanned within target time. While we focus on on-time delivery, other incentives such as maximizing revenue or throughput might also be relevant in certain referral networks. Our analysis maps patient demand, system capacity and operational capabilities to an optimal referral policy and a scheduling outcome. The referral policy determines the allocation of patients to providers and the scheduling outcome describes the proportion of each patient stream served within their wait time targets. The key results include the following:

1. When operational abilities are low for both providers, it is optimal for the providers to prioritize one stream of patients. Depending on provider capacities the referral policy may vary. When capacities are heterogenous, the centralized referrer may focus providers on particular patient streams. When capacities are low for both providers, the optimal referral policy may result in providers relegating one of their patients streams to poor quality service where there is zero probability of receiving the service within the wait time targets.

2. When operational abilities vary between providers, the provider with higher operational ability will offer more evenly distributed levels of service. The provider with higher operational ability will also receive more patients allowing the lower ability provider

to offer higher levels of service.

3. Requiring providers to provide equal levels of service with respect to target wait times decreases the overall system performance defined as the number of patients who are served within their wait time targets. However, the negative impact diminishes as operational abilities increase.

The rest of the paper is organized as follows: We first review the related medical and operations management literature. In Section 2 we describe the general setting of our models and define the objective function of our problem. We also discuss the provider problem and optimal scheduling policies. In Section 3, we define the referrer problem and analyze optimal referral policies under different circumstances. Section 4 studies the fair allocation model which extends the standard model by enforcing parity. We conclude the paper in Section 5 with a summary of our findings, managerial insights, and limitations of our study.

2.1.1 Literature and Positioning

Patient referral decisions play a critical role in determining cost and quality of care by acting as the bridge between primary care and access to specialized healthcare services (Barnett et al. 2012b). It is estimated that 30% of patients in the United States and 14% of patients in the United Kingdom are referred to providers each year (Forrest et al. 2002). The referral decision is difficult because of the range of patient specific and systemic factors that need to be taken into account when managing sometimes conflicting objectives such as timeliness and quality. There are several relevant literatures particular to our problem of optimizing referral and scheduling policies which we discuss in turn. Because of the unique methods to derive insights in this paper, we subsequently discuss relevant methodology.

The problem of finding appropriate policies for referring and scheduling patients has been investigated from three perspectives: (1) Gatekeeper (2) Appointment scheduling and (3) Referral Management. The gatekeeper perspective studies the appropriateness of referrals with particular emphasis on ascertaining whether a patient should be treated by a GP or referred to a provider (Shumsky and Pinker 2003). Studies following the appointment scheduling perspective focus on the problem of optimizing scheduling of patients for provider services and typically treat the rate of referrals as exogenous. Stochastic programming and queueing theory approaches have been extensively applied for scheduling patients and allocating physicians capacities with the aim of minimizing costs of wait times and overtime or minimizing the number of patients that exceed waiting-time targets (Castaing et al. 2016, Cayirli and Gunes 2014, Chen and Robinson 2014, Kuiper and Mandjes 2015, Tang et al. 2014, Gocgun and Puterman 2014). Finally, papers taking the perspective of referral management, are focused on analyzing the impact of different interventions in the referral process on patient access to provider services (Braybrooke et al. 2007, Bungard et al. 2009, Akbari et al. 2005). A case-in-point is the study of queue pooling to determine situations under which having a pooled referral system, modeled as a single queue pooling multiple arrival streams, is more advantageous than having several separate referral queues (Dijk and Sluis 2008, Mandelbaum and Reiman 1998). Mahmoudzadeh et al. (2020) and Jiang et al. (2020) are among the papers that have explored optimal decisions in multi-class patients scheduling with wait time targets and Li et al. (2015) has investigated patient pooling optimization to allocate available capacity to different types of patients with different waiting time targets.

Our paper takes a different perspective on this problem. We seek to bridge the gap between appointment scheduling and referral management in healthcare networks. We assume that all the patient arrivals are in real need of specialized services and, considering wait time targets, study the coordination of referral and provider scheduling policies. These features distinguish our paper from most of the papers in the literature. As noted by Ahmadi-Javid et al. (2017a) in their survey of the appointment scheduling literature, most operational papers have explored the problem by doing micro-level analysis which does not provide generalizable insights into the longer term decisions required to design a proper network. The changes in methodology which we introduce in this paper allow for an analytically tractable model where both the referral and scheduling decisions can be studied jointly, allowing the coordination problem to be considered.

The methodology introduced in this paper uses an achievable region to represent the space of possible scheduling outcomes for the provider. The achievable region methodology was first introduced by Coffman and Mitrani (1980) who identified that the space of performance characteristics achievable by allowable strategies is a convex region. By optimization over achievable outcomes rather than available strategies, this methodology allows for a more tractable problem which can often be expressed as a convex mathematical program and sometimes solved analytically. Important extensions of the methodology include Federgruen and Groenevelt (1988), Shanthikumar and Yao (1992) and Bertsimas and Nino-Mora (1996). An achievable region approach has proved useful in a range of managerial studies including the design of optimal pricing of prioritization in queues (Afeche and Pavlin 2016), service differentiation in communication/computer systems (Vanlerberghe et al. 2018) and designing kidney allocation policies (Ata et al. 2020).

We follow in this vein, representing the space of outcomes of the providers scheduling strategies as a constrained region consistent with operational characteristics expected in the real system. As a simple example, the region is larger if the providers capacity is larger. This approach allows for a tractable model where the impact of system parameters on referral and scheduling decisions can be jointly studied.

2.2 Model and Preliminary Analysis

In this section we describe and model a system which allocates specialized healthcare services to heterogeneous patients. The model considers both patient referral and scheduling decisions which are respectively made by a referrer and a set of providers. Patient heterogeneity is defined quite generally and may include patients who are differentiated by different conditions and procedural requirements or simply different levels of severity of the same condition. The model includes a framework for expressing positive and negative complementarities in the ability of the provider to schedule different types of patients. For instance, if the provider is providing two surgical procedures which have different equipment, staffing requirements and substantial changeover time, then the provider would most likely be more productive if they serve a more homogenous set of patients and the complementarity is negative. On the other hand if the provider is delivering two types of consulations that are each partially delivered by different allied health professionals, then the provider may be able to serve more patients when there is a more even mix of the two and more patients can be served in parallel resulting in a positive complementarity. Our model incorporates the scheduling complementarities into the provider's operational constraints and allows both the provider and referrer to take them into account in their operational decisions as these agents act to maximize the number of their patients who are served within their target wait times. We provide the formal description of the model below with respect to the key participating agents before providing an analysis of the providers decision problem.

Patients: To gain generalizable insights into the impact of operational abilities on managing referrals of multiple patient types, we focus on a parsimonious model featuring two types of patients $i \in \{1, 2\}$. λ_i represents the total arrival rate of patients of type i to the central system. In addition, for each type of patient a maximum recommended waiting time is defined by external policy makers (i.e. experts).

Providers: We focus on a system with a pair of providers enumerated $j \in \{1, 2\}$. Each provider has a capacity $m_j \in R^+$. We assume that each provider in the system is eligible to receive both types of patients. The variable λ_{ij} denotes the rate of arrivals of patients of type i at provider j. The provider will select a scheduling strategy which is consistent with their capacity and operational abilities. These operational abilities do not reflect healthcare outcomes. Rather, the operational abilities will reflect managerial skill at scheduling and optimizing patient flows and may also reflect specific operational conditions related to the types of procedures that are being performed. The variable x_{ij} is the probability that patients of type i allocated to provider j are served within their target wait time. For convenience we also define $X_j = (x_{1j}, x_{2j})$ as the vector of target probabilities for patient types assigned to provider j. The providers scheduling strategy will result in a particular proportion of each patient type being served within their target wait times determining x_{ij} for $i \in 1, 2$. Consistent with the achievable region methodology the analysis can ignore the specific scheduling strategy and wait time targets and focus on the achievable outcomes of interest, i.e. x_{ij}. We allow x_{ij} to be a decision variable for the provider j where the achievable region constraints denoted in Equation 2.1 ensure that the target probability for patient type i is consistent with a scheduling strategy available to the provider.

$$x_{ij} \leq H_j(x_{(-i)j}, \lambda_{ij}, \lambda_{(-i)j}, m_j) \tag{2.1}$$

Equation 2.1, shows the target wait time of the focal patient i is bounded by the achievable region function $H_j(x_{(-i)j}, \lambda_{ij}, \lambda_{(-i)j}, m_j)$. This function represents the maximum target probability achievable in the space of available scheduling policies for provider j and depends on the target wait time $x_{(-i)j}$ and arrival rate $\lambda_{(-i)j}$ of the other patient type $-i$ at the provider. In Section 2.2.1 we discuss expected characteristics of H_j and consider a specific functional form for the achievable region function.

Each provider is dedicated to her own slate of patients and makes decisions independently in response to the referrer's allocation decisions. The scheduling problem for provider j becomes a problem of selecting the achievable target wait times as follow:

$$\max_{X_j} \frac{\sum_i \lambda_{ij} x_{ij}}{\sum_i \lambda_{ij}} \tag{2.2}$$

$$S.t.$$

$$0 \leq x_{ij} \leq 1 \qquad \forall i \tag{2.3}$$

$$x_{ij} \leq H_j(x_{(-i)j}, \lambda_{ij}, \lambda_{(-i)j}, m_j) \qquad \forall i \tag{2.4}$$

The objective of provider j, denoted in Equation 2.2, is to maximize the proportion of their patients who are seen within their target wait time. Constraint 2.3 assures that target probabilities are well defined and Constraint 2.4 ensures that x_{ij} is achievable by provider j.

Centralized referrer: Consistent with the centralized intake model, the centralized referrer receives the streams of patients, λ_i, from a group of primary care providers and is the sole pathway of access to the specialized services. The decisions which the referrer makes is the volume of each stream which is allocated to each referrer λ_{ij}. For convenience, we define Λ as the vector of all λ_{ij}. The objective of the referrer is to perform the allocation of patients to providers in a manner which maximizes the number of patients being seen within their target wait times. We assume that the rate of arrivals of patients at the referrer, λ_i, is exogenous and not dependent on the referrer or provider decisions.

We assume that the referrer has knowledge of the providers capabilities and incentives. The resulting decision problem for the referrer is a bi-level optimization problem where the final target probabilities are determined by the decisions of the providers as they serve their particular patient allocations. The mathematical program below shows the complete decision problem for the referrer.

$$\max_{\Lambda} \sum_i \sum_j \lambda_{ij} x_{ij} \tag{2.5}$$

$$S.t.$$

$$\sum_j \lambda_{ij} = \lambda_i \qquad \forall i \tag{2.6}$$

$$\max_{X_j} \frac{\sum_i \lambda_{ij} x_{ij}}{\sum_i \lambda_{ij}} \qquad \forall j \tag{2.7}$$

$$S.t.$$

$$0 \leq x_{ij} \leq 1 \tag{2.8}$$

$$x_{ij} \leq H_j(x_{(-i)j}, \lambda_{ij}, \lambda_{(-i)j}, m_j) \qquad \forall i \tag{2.9}$$

Equation 2.5 denotes the objective function for the referrer. Equations 2.7 through 2.9 denote the providers subproblem. The objective functions of the referrer and provider are aligned such that the problem is coordinated.

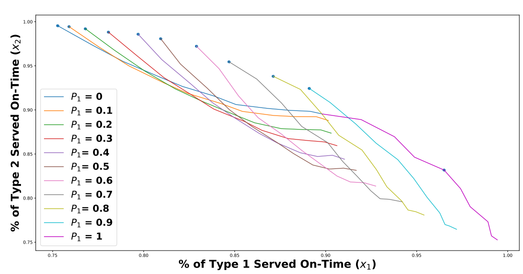
2.2.1 Model of the Achievable Region

The achievable region is the space of performance outcomes corresponding to available scheduling policies for the provider. In this section we discuss the form of the achievable region for an M/M/1 queueing system and how we expect this to generalize to other queueing and scheduling settings. We then present a functional form for the achievable region which is analytically tractable but flexible enough to model a wide variety of possible forms.

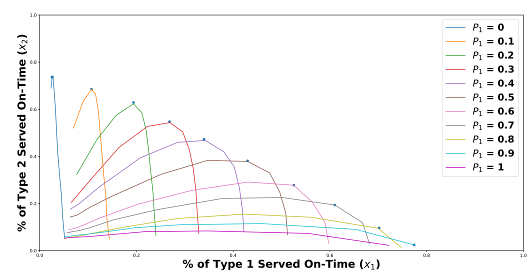
We consider a non-preemptive M/M/1 queueing system where there are two priority classes and patient arrivals are received from two streams which we label Type 1 and Type

2. Both Type 1 and Type 2 patients have fixed target wait times. In addition, the system experiences a switching time when there are consecutive patients of different type. The switching time models operational complexity and is denoted by S. It may reflect different operational challenges such as differences in staffing, location or instrumentation between the procedures for Type 1 and Type 2 patients.

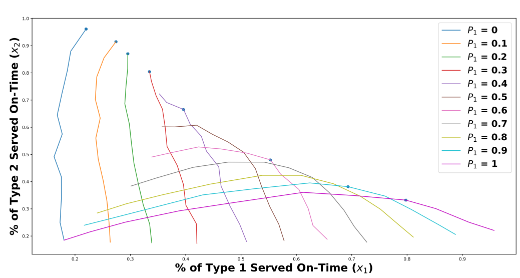
The strategies available to the queueing system are limited to selecting the proportion of each patient stream placed in the high priority class. For simplicity we assume that the scheduling strategy assigns a simple probability to each arriving customer and does not consider historical or future arrivals. The probability of a patient of type i being in the high priority class is denoted by P_i. We determine the achievable region via a search over this space of policies. We use a discrete-event simulation to determine the performance metrics associated with each strategy. The simulations were performed using the SimPy python package and each simulation was run for 20000 units of time and repeated 50 times.



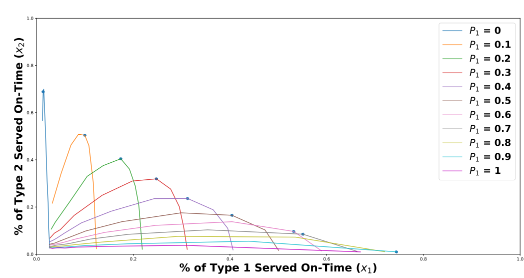
(a) No switching time, low utilization
$S = 0, \mu = 0.5, \lambda_1 = \lambda_2 = 0.1$



(b) No switching time, high utilization
$S = 0, \mu = 0.5, \lambda_1 = \lambda_2 = 0.225$



(c) High switching time, low utilization
$S = 0.25, \mu = 0.5, \lambda_1 = \lambda_2 = 0.1$



(d) High switching time, high utilization
$S = 0.25, \mu = 0.5, \lambda_1 = \lambda_2 = 0.225$

Figure 2.1: Example of achievable region for target probabilities in a prioritized M/M/1 queueing system

Figure 2.1 shows the results of this study for a low utilization (Panels a and c) and a high utilization example (Panels b and d). The two panels show situations where there is no switching time and where the switching time is equal to 0.25. Each line in the figures corresponds to a trajectory of strategies where P_1 is fixed and P_2 is increased from 0 to 1 in increments of 0.1. The achievable region is the region below and to the left of the frontier (emphasized by the black dots).

Figure 2.1 illustrates how the convexity and size of the empirical achievable region varies depending on the operational challenges facing the system. Comparing Figures 2.1a to 2.1b and 2.1c to 2.1d shows that the achievable region decreases in size when the utilization of the service increases. The area decreases primarily through a downward shift of the H function. On the hand, comparing Figures 2.1a to 2.1c and 2.1b to 2.1d shows decreases in size but in a different manner when the switching time is increased. This operational change decreases scheduling complementarities between the two patient types. As a result, the achievable region shrinks primarily where both patient types are prioritized and the H function moves from a concave to a convex function.

While we believe the above example is a useful illustration, it is very simple. In practice enumerating the space of achievable strategies is highly dependent on the particular provider and their operational conditions and it is very difficult to solve for these regions in closed form. For example, a simple scheduling strategy available to the provider is to schedule each patient in the next available appointment slot. More complex strategies are also available such as reserving some proportion of open slots in the next week for higher priority patients. Each of these strategies observed over a long time period with a consistent arrival process will result in steady state performance outcomes. We focus on characterizing an achievable region which is consistent with our simulation studies and reacts in an expected way to changes in key parameters. In particular, we require the achievable region (AR) to have the following characteristics:

1. The AR is a closed region containing target probabilities of 0 for each patient type.
2. Ceteris paribus, the AR is larger for a provider with larger capacity.
3. Ceteris paribus, the AR is larger for a provider with higher operational ability.
4. Ceteris paribus, the AR is smaller for a provider with higher arrival rates.
5. The function H corresponds to the Pareto frontier where any improvement in target probabilities for type i will result in a reduction in the target probability for type $-i$.

The form we select for the achievable region function is:

$$H(x_{(-i)j}, \lambda_{ij}, \lambda_{(-i)j}, m_j) = \left(\frac{m_j - \lambda_{-ij} x_{-ij}^{\alpha_j}}{\lambda_{ij}} \right)^{1/\alpha_j}. \tag{2.10}$$

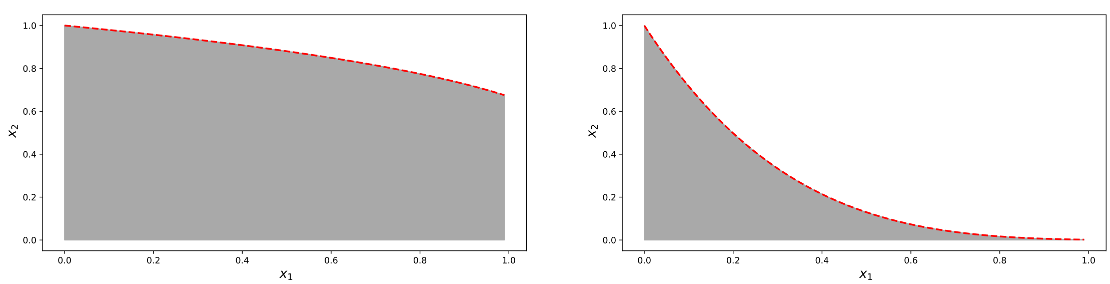
Where we introduce the operational ability parameter α_j. The role of the parameters in determining the achievable region becomes more apparent in the provider's decision problem where the achievable region constraints can be simplified to the following form:

$$0 \leq x_{ij} \leq 1 \qquad \forall i \tag{2.11}$$

$$\sum_i \lambda_{ij} x_{ij}^{\alpha_j} \leq m_j \tag{2.12}$$

It can be verified that each of the required characteristics of the achievable region are satisfied.

Figure 2.2 shows achievable regions (gray region) for target probabilities in a two-type patient referral system under high and low operational ability scenarios where the capacity is insufficient to serve all patients within their target wait-times. In this case there is a tradeoff between the target probabilities for both patient types. In both cases we assume that $m_j = 5$, $\lambda_{1j} = 4$ and $\lambda_{2j} = 5$. The achievable region is between the axes and the dashed boundary line. In the first scenario $\alpha_j = 2$ which corresponds to a higher operational ability. The opposite holds in the second scenario where $\alpha_j = 0.35$. The convexity of the frontier of the achievable region (or equivalently of the H function) depends on a provider's ability. A provider with higher operational ability is able to provide both types of patients with higher target probabilities at the same time. On the contrary, the convexity of the H function when a provider has low operational ability implies that heterogeneous patient populations can be served only at lower target probabilities.



(a) High operational ability provider ($\alpha_j = 2$)

(b) Low operational ability provider ($\alpha_j = 0.35$)

Figure 2.2: Example of achievable region for functional form of $H(\cdot)$ ($m_j = 5$, $\lambda_{1j} = 4$, $\lambda_{2j} = 5$)

Comparing Figure 2.1 with Figure 2.2 shows that the proposed functional form for H can generate achievable regions similar to the achievable regions for target probabilities observed in real systems. The operational ability level in our study (i.e. α) has a similar impact on the convexity of the achievable region as the switching time in the M/M/1 example. This parameter α can reflect both the ability of the provider to optimally schedule new referrals and the inherent challenge in their particular operational setting.

2.2.2 Analysis of the provider's Decision Problem

In this section we study the providers decision problem. The following theorem provides a characterization of the optimal scheduling outcomes for a provider in a centralized system with two types of patients.

Theorem 1. *Given a provider with capacity m_j, operational ability α_j and receiving patient streams λ_{1j} and λ_{2j}, the optimal provider decisions are given in Table 2.1 and depend on the following thresholds:*

$$Th1 = \left(\frac{m_j^{(\alpha_j)^{-1}} - (m_j - \lambda_{1j})^{(\alpha_j)^{-1}}}{\lambda_{1j}}\right)^{\frac{\alpha_j}{1-\alpha_j}} \tag{2.13}$$

$$Th2 = \left(\frac{m_j^{(\alpha_j)^{-1}} - (m_j - \lambda_{2j})^{(\alpha_j)^{-1}}}{\lambda_{2j}}\right)^{\frac{\alpha_j}{1-\alpha_j}} \tag{2.14}$$

Table 2.1 shows the optimal target probabilities decided on by the provider and the conditions for which they are optimal. The operational ability level $\alpha_j = 1$ is important in determining the optimal provider policy. We will refer to providers where $\alpha_j \geq 1$ as having high operational ability or HOC and where $\alpha_j < 1$ as having low operational ability or LOC. When a provider is HOC, the optimal policy is independent of both the arrival rates and the capacity. If the provider is LOC, there are four candidate optimal policies which depend on the arrival rates and provider capacity. The optimality conditions show that P_1 and P_2 are policies which are optimal at higher capacity and P_1 prioritizes Type 1 patients while P_2 prioritizes patients of Type 2. $\bar{P}_1$ and $\bar{P}_2$ are optimal at lower capacity and similarly prioritize patient types. Each of these policies has two conditions either of which is sufficient for the policy to be optimal. The first condition indicates whether the provider has very high or low capacity. The second condition is for intermediate capacities and depends on arrival rate thresholds for the patient types (Equations 2.13 and 2.14).

The optimal policy for a HOC provider sets target probabilities of all types of patients equal to $(\frac{m_j}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}}$. When a provider is LOC, the situation is very different. It is always best for a LOC provider to provide inequitable service by prioritizing one type of patient over the other. It can also be seen that for a LOC provider, the optimal scheduling outcomes are highly dependant on the provider's capacity and the arrival rates. Note that P_1 is identical to $\bar{P}_1$ and P_2 is identical to $\bar{P}_2$ if λ_{ij} is equal to m_j in which case there is full prioritization, i.e. type i receives target probability of one and the target probability of the other type is

Index	(x_{1j}, x_{2j})	Optimality Condition(s)	α_j
1	$S = ((\frac{m_j}{\lambda_{1j}+\lambda_{2j}})^{\frac{1}{\alpha_j}}, (\frac{m_j}{\lambda_{1j}+\lambda_{2j}})^{\frac{1}{\alpha_j}})$	- - -	> 1
2	$P_1 = (1, (\frac{m_j-\lambda_{1j}}{\lambda_{2j}})^{\frac{1}{\alpha_j}})$	$\lambda_{2j} \leq \lambda_{1j} \leq m_j \leq \lambda_{1j} + \lambda_{2j}$ Or $\lambda_{1j} \leq m_j < \lambda_{2j} < \lambda_{1j} + \lambda_{2j}$ *and* $\lambda_{2j} \geq Th1$	< 1
3	$P_2 = ((\frac{m_j-\lambda_{2j}}{\lambda_{1j}})^{\frac{1}{\alpha_j}}, 1)$	$\lambda_{1j} \leq \lambda_{2j} \leq m_j \leq \lambda_{1j} + \lambda_{2j}$ Or $\lambda_{2j} \leq m_j < \lambda_{1j} < \lambda_{1j} + \lambda_{2j}$ *and* $\lambda_{1j} \geq Th2$	
4	$\bar{P}_1 = ((\frac{m_j}{\lambda_{1_j}})^{\frac{1}{\alpha_j}}, 0)$	$m_j \leq \lambda_{1j} \leq \lambda_{2j} < \lambda_{1j} + \lambda_{2j}$ Or $\lambda_{2j} \leq m_j \leq \lambda_{1j} < \lambda_{1j} + \lambda_{2j}$ *and* $\lambda_{1j} < Th2$	
5	$\bar{P}_2 = (0, (\frac{m_j}{\lambda_{2_j}})^{\frac{1}{\alpha_j}})$	$m_j \leq \lambda_{2j} \leq \lambda_{1j} < \lambda_{1j} + \lambda_{2j}$ Or $\lambda_{1j} \leq m_j \leq \lambda_{2j} < \lambda_{1j} + \lambda_{2j}$ *and* $\lambda_{2j} < Th1$	

Table 2.1: The provider problem optimal solutions

zero. When λ_{ij} is not equal to m_j for a high capacity provider prioritization still occurs but in this case, both patients will have positive target probabilities. For a low capacity provider where the prioritized type is above the capacity, the scheduling policy results in a positive target probability for the prioritized type at the expense of a target probability of zero for the second type.

The characteristics of the scheduling policies selected by the provider in response to referral decisions are likely to impact patient outcomes and will be discussed in depth in future sections. The following policy types will be used to frame that discussion.

1. Shared scheduling policy: If the optimal scheduling policy for a provider is to set the target probabilities of both types of patients equal to each other.

2. Partially prioritized scheduling policy: If the optimal scheduling policy for a provider

results in target probability of 1 for one type of patient and a positive but lower than 1 target probability for the other type.

3. Fully prioritized scheduling policy: If the optimal scheduling policy for a provider results in target probability of 1 for one type of patient and 0 for the other type.

Numerical example: The following numerical example illustrates the impact of choosing the optimal policy on the provider objective function value. Consider a high capacity provider with capacity of 6 where the arrival rates of patient of types 1 and 2 to this provider are respectively 5 and 4. The three scheduling outcomes which may be optimal for this high capacity provider are:

Equal Outcome Policy (EOP): Set target probabilities of both types of patients equal to each other ($x_{1j} = x_{2j}$).

Prioritize Type 1 (PT1): Fully serve patients of Type 1 and allocate the rest of the capacity to the second type.

Prioritize Type 2 (PT2): Fully serve patients of Type 2 and allocate the rest of the capacity to the first type.

Figure 2.3 shows $f(X_j)$ under each policy as α_j increases.

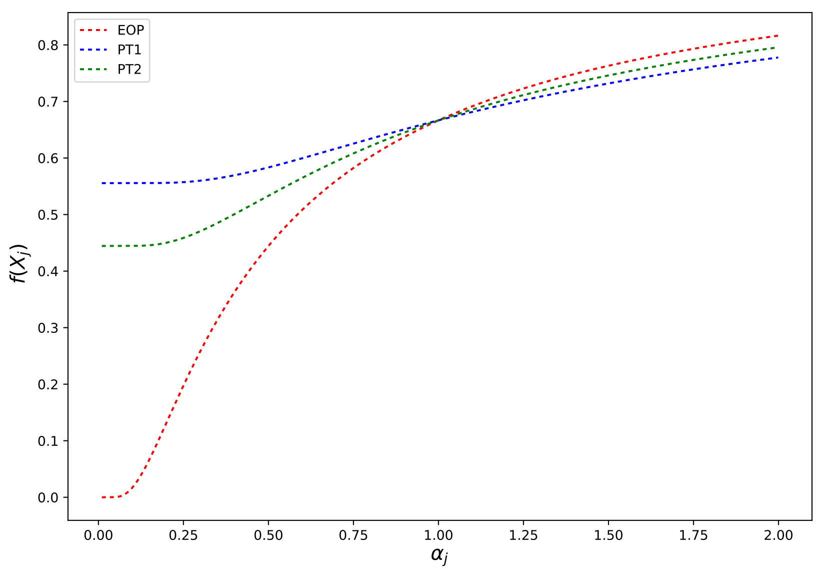


Figure 2.3: Provider objective value under different scheduling policies

It can be seen that if the provider is LOC ($\alpha_j < 1$) then prioritizing Type 1 is the optimal policy. When $\alpha_j = 1$, the value of $f(X_j)$ does not depend on the selected policy. Finally, the equal outcome policy is optimal for the provider if they are HOC ($\alpha_j > 1$) and as α_j increases, the gains from using the equal outcome policy over other policies increases.

2.3 Referrer Problem Analysis

In this section we move on to considering the optimal referral policies for a referrer who takes into account the optimal decisions of the providers. In this Stackleberg game, the referrer is assumed to accurately predict the resulting target probabilities (i.e. x_{ij}) for each patient type given the referral rates the referrer chooses. The provider capacities and total arrival rate of each type of patient to the system are known to the referrer. The referrer's decision problem corresponds to the following multilevel programming problem:

$$\max_{\Lambda} \sum_i \sum_j \lambda_{ij} x_{ij} \tag{2.15}$$

$$S.t.$$

$$\sum_j \lambda_{ij} = \lambda_i \qquad \forall i \tag{2.16}$$

$$\max_{X_j} \frac{\sum_i \lambda_{ij} x_{ij}}{\sum_i \lambda_{ij}} \qquad \forall j \tag{2.17}$$

$$S.t.$$

$$0 \leq x_{ij} \leq 1 \tag{2.18}$$

$$\sum_i \lambda_{ij} x_{ij}^{\alpha_j} \leq m_j \qquad \forall i \tag{2.19}$$

Equation 2.16 ensures that patients are properly partitioned between providers. Equations 2.17 to 2.19 denote the provider subproblems.

In order to ensure that the problem is analytically tractable and the exposition is focused on important cases we make the following series of assumptions:

1. Arrival rate of the second type of patient is less than the arrival rate of the first type (i.e. $\lambda_2 < \lambda_1$).

2. There are only two providers in the system and $m_2 < m_1$.

3. There is enough capacity in the system to serve each type of patient within their target wait times independently but not together (i.e. $\lambda_2 < \lambda_1 < m_1 + m_2 < \lambda_1 + \lambda_2$).

4. Both providers have enough capacity to serve both types of patients independently (i.e $\lambda_{ij} \leq m_j \leq \sum_i \lambda_{ij}$)

5. At least one provider is LOC.

Assumption 1 is without loss of generality. The second assumption allows for heterogeneity in the pool of providers without resulting in an explosion in the number of subcases. Assumptions 3, 4 and 5 ensure that we focus on the most interesting cases where the solution is non-trivial and requires decisions regarding which patients to prioritize. Together, these assumptions lead to the following four feasible capacity scenarios:

1. $m_2 < m_1 < \lambda_2 < \lambda_1 < m_1 + m_2 < \lambda_1 + \lambda_2$ (Low, Low)

2. $m_2 < \lambda_2 < m_1 < \lambda_1 < m_1 + m_2 < \lambda_1 + \lambda_2$ (Mid, Low)

3. $m_2 < \lambda_2 < \lambda_1 < m_1 < m_1 + m_2 < \lambda_1 + \lambda_2$ (High, Low)

4. $\lambda_2 < m_2 < m_1 < \lambda_1 < m_1 + m_2 < \lambda_1 + \lambda_2$ (Mid, Mid)

We name the capacity scenarios as in parentheses above. The first entry in the name corresponds to the capacity of provider 1 and the second to the capacity of provider 2. We call provider j's capacity level "Low" if the provider's capacity is lower than both arrival rates (i.e. $m_j < \lambda_i$, $i = 1, 2$). The capacity of j is "Mid" if the capacity is between the arrival rates (i.e. $\lambda_2 < m_j < \lambda_1$). Finally, the capacity is "High" if provider j's capacity is higher than both arrival rates (i.e. $\lambda_2 < \lambda_1 < m_j$).

In the analysis that follows we will find the optimal referral policies for these capacity scenarios where the pool of providers has both homogeneous and heterogeneous operational abilities (resp. Sections 2.3.1 and 2.3.2).

2.3.1 Referrer Decisions with Homogeneous provider ability

In this section we analyze the optimal referral policies for the referrer where both providers are LOC (i.e. $\alpha_1, \alpha_2 < 1$). Since by assumption, providers have sufficient capacity to serve both types of patients independently, from Theorem 1, P_1 and P_2 are the only possible optimal decisions for each provider. In addition, the assumption $\lambda_2 < \lambda_1 < m_1 + m_2 < \lambda_1 + \lambda_2$ implies that it is impossible to have a situation where both providers select solution P_2 at the same time. So, at least one of the providers will serve Type 1 patients with target probability of 1. Note that in the case that the capacity is entirely utilized by one type, i.e. $\lambda_{ij} = m_j$, type $-i$ may receive a target probability of zero. Theorems 2-5 explain optimal policies for the referrer for each capacity scenario when both providers are LOC.

Theorem 2. [Low provider capacity scenarios] *Consider a referral system where both providers are LOC. For capacity scenario (Low, Low) and (Mid, Low), the optimal referral policies is the set of feasible policies resulting in $G(\Lambda) = m_1 + m_2$. The candidate optimal referral policies are listed in Table 2.2. Of these candidate policies, only Policy 3 is feasible and optimal under all (Low, Low) and (Mid, Low) scenarios.*

Index	Optimal Policy	TP
	$(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$	(x_{11}, x_{21}) (x_{12}, x_{22})
1	$(m_1 + m_2 - \lambda_2, \lambda_2 - m_2)$ $(\lambda_1 + \lambda_2 - m_1 - m_2, m_2)$	$(1, 1)$ $(0, 1)$
2	$(m_1, \lambda_1 + \lambda_2 - m_1 - m_2)$ $(\lambda_1 - m_1, m_1 + m_2 - \lambda_1)$	$(1, 0)$ $(1, 1)$
3	$(m_1, \lambda_2 - m_2)$ $(\lambda_1 - m_1, m_2)$	$(1, 0)$ $(0, 1)$
4	$(\lambda_1 - m_2, m_1 + m_2 - \lambda_1)$ $(m_2, \lambda_1 + \lambda_2 - m_1 - m_2)$	$(1, 1)$ $(1, 0)$
5	$(\lambda_1 + \lambda_2 - m_1 - m_2, m_1)$ $(m_1 + m_2 - \lambda_2, \lambda_2 - m_1)$	$(0, 1)$ $(1, 1)$
6	$(\lambda_1 - m_2, m_1)$ $(m_2, \lambda_2 - m_1)$	$(0, 1)$ $(1, 0)$

Table 2.2: Candidate Policies (both providers are LOC, capacity scenarios (Low, Low) and (Mid, Low))

Theorem 2 shows the candidate optimal policies for the referrer under the scenarios where there is lower capacity in the system and both providers are LOC. The full mapping of scenario to optimal policy is listed in Tables A.2 and A.3 in the Appendix. Each of the candidate solutions listed in the theorem is able to reward the referrer with the full capacity of the two providers and results in a situation where at least one provider fully prioritized one type of patient over the other type, however, depending on the scenario, the policy may not be feasible. For (Low, Low) scenarios each of Policies 1-6 may be optimal and for (Mid, Low) each of Policies 1-4 can be optimal. The only policy which is optimal in all (Low, Low) and (Mid, Low) scenarios is Policy 3 where each provider focuses on delivering a high quality of service to a single patient type and results in a set of patients of each type who have a probability of zero of receiving service within their target wait time.

Theorem 3 shows the optimal referral policy when providers have very different capacity levels which together are close to the full arrival rate.

Theorem 3. [High heterogeneous provider capacity] *Consider a referral system where both providers are LOC. If the capacity scenario is (High, Low) and $2m_2 + m_1 \geq \lambda_1 + \lambda_2$ then the optimal policy for the referrer can be either of the following policies:*

Index	Optimal Policy $(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$	TP (x_{11}, x_{21}) (x_{12}, x_{22})
1	$(m_1 + m_2 - \lambda_2, \lambda_2 - m_2)$ $(\lambda_1 + \lambda_2 - m_1 - m_2, m_2)$	$(1, 1)$ $(0, 1)$
2	$(\lambda_1 - m_2, m_1 + m_2 - \lambda_1)$ $(m_2, \lambda_1 + \lambda_2 - m_1 - m_2)$	$(1, 1)$ $(1, 0)$

Table 2.3: Optimal policies (both providers are LOC, capacity scenario (High, Low) and $2m_2 + m_1 \geq \lambda_1 + \lambda_2$)

Theorem 3 states that in the case that providers capacities are very different but are together sufficiently close to the full arrival rate, then the optimal policies result in a situation where the total number of patients allocated to the first and second providers are m_1 and $\lambda_1 + \lambda_2 - m_1$ respectively. Restricting the arrival rate to the higher capacity provider to its capacity allows this provider to serve both patient types with target probability one. provider 2, who has lower capacity, serves more patients than her capacity but the surplus patients are served with low priority.

Under both policies all types of patients allocated to the the first provider are certain to be served within their wait time targets and the second provider priorities its full capacity to one type of patients (patient Type 2 for policy index 1) and provides low quality service to the remaining patients of the other type. Regardless of policy, the providers objectives are $f(X_1) = 1$ and $f(X_2) = \frac{m_2}{\lambda_1+\lambda_2-m_1}$ allowing the higher capacity provider to provide higher quality service. While the providers are indifferent between the referrer's policy, the patients are not. Policy 1 provides higher quality of service to Type 2 patients and Policy 2 provides higher quality of service to Type 1 patients.

Theorem 4, shown below, explores a similar scenario to that of Theorem 3, except that the total arrival rates are higher than $2m_2 + m_1$.

Theorem 4. [Low heterogeneous provider capacity] *Consider a referral system where both providers are LOC, capacity scenario is (High, Low) and $2m_2 + m_1 < \lambda_1 + \lambda_2$. The optimal referral policy depends on the condition:*

$$m_2(\lambda_2 - m_2)^{\alpha_1^{-1}-1} - (m_1 + m_2 - \lambda_1)^{\alpha_1^{-1}} + (m_1 - \lambda_1)^{\alpha_1^{-1}} > 0 \tag{2.20}$$

The optimal policy and target probabilities are shown in the following table:

Index	Optimal Policy	TP	Condition (Eq. 2.20)
	$(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$	(x_{11}, x_{21}) (x_{12}, x_{22})	
1	$(\lambda_1, \lambda_2 - m_2)$ $(0, m_2)$	$(1, (\frac{m_1-\lambda_1}{\lambda_2-m_2})^{(\alpha_1)^{-1}})$ $(0,1)$	True
2	$(\lambda_1 - m_2, \lambda_2 - m_2)$ (m_2, m_2)	$(1, (\frac{m_1+m_2-\lambda_1}{\lambda_2-m_2})^{(\alpha_1)^{-1}})$ $(1,0)$ *or* $(0,1)$	False

Table 2.4: Optimal policies (both providers are LOC, capacity scenario (High, Low) and $2m_2 + m_1 < \lambda_1 + \lambda_2$)

Theorem 4 shows the only situation where the optimal referral policy results in a provider partially prioritizing one type of patient over the other type. In particular, due to the higher arrival rates than in the scenario for Theorem 3, patients of Type 2 seen by the provider 1 have target probability between zero and one.

The two policies shown in Table 2.4 differ most markedly in that in Policy 1, all patients of Type 1 are allocated to provider 1 allowing provider 2 to focus only on patients of Type 2. This policy results in all patients having a positive probability of being seen within their wait time targets and is optimal only when Equation 2.20 holds. It can be verified that the left hand side of this condition is decreasing in m_1 and increasing in the arrival rates. To put this in plain language, this is due to the fact that with sufficient residual capacity the first provider can see more patients within their target wait time when not overloaded. This congestion effect pushes more patients to be seen by the second provider and results in sub-par service for one of the types allocated to this provider. Notably, this condition does not depend on the second provider's level of operational ability since this provider is only able to offer target wait times above zero to an arrival rate of customers equal to its capacity.

The above theorems have covered all scenarios except for both providers having intermediate capacity levels. Theorem 5 shows optimal referral policy for this scenario:

Theorem 5. [Intermediate provider capacities] *Consider a referral system where both providers are LOC. If the capacity scenario is (Mid, Mid) then either of the following policies can be optimal for the referrer are:*

Index	Optimal Policy	TP
	$(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$	(x_{11}, x_{21}) (x_{12}, x_{22})
1	$(\lambda_1 - m_2, m_1 + m_2 - \lambda_1)$ $(m_2, \lambda_1 + \lambda_2 - m_1 - m_2)$	$(1, 1)$ $(1, 0)$
2	$(m_1, \lambda_1 + \lambda_2 - m_1 - m_2)$ $(\lambda_1 - m_1, m_1 + m_2 - \lambda_1)$	$(1, 0)$ $(1, 1)$

Table 2.5: Optimal policies (both providers are LOC, capacity scenario (Mid, Mid))

The two policies are equivalent from the point of view of the referrer and the patients but differ with respect to the providers objective. The first policy provides higher mean target probabilities to the first provider and the second policy favours the second provider. Under the first policy $f_{policy-1}(X_1) = 1$ and $f_{policy-1}(X_2) = \frac{m_2}{\lambda_1+\lambda_2-m_1}$ and under the second policy $f_{policy-2}(X_1) = \frac{m_1}{\lambda_1+\lambda_2-m_2}$ and $f_{policy-2}(X_2) = 1$. Under both policies the total proportion of patients that have no chance to be served within their wait time targets is $\lambda_1 + \lambda_2 - m_1 - m_2$.

Discussion of optimal referral policies: The results of Theorems 2-5 describe optimal referral policies when both providers are LOC. Prioritization, where at least one provider provides high quality of service to exactly one type of patients, is ubiquitous throughout all capacity scenarios. At low capacities (Theorem 2) all capacity scenarios lead to at least one provider using a fully prioritized scheduling policy. In most cases, intermediate and larger capacities will also have one stream of patients who have a target wait time probability of zero. This is due to the low operational ability which limits the rewards from serving both types of patients with quality service simultaneously. The only exception to this rule is Theorem 4 policy 1, which is optimal when the condition in Equation 2.20 holds. In this case, the first provider partially prioritizes one type of patient over the other type, while the second provider is able to see all patients within the target wait time.

Theorem 4 is also exceptional with respect to the referrers value. In all other cases the referrer receives utility equal to the total provider capacities $m_1 + m_2$. The policies for Theorem 4 use some of provider 1 capacity to provide intermediate service levels to Type 2 patients. This occurs because the residual capacity of provider 1 after serving all the Type 1 patients is insufficient to serve remaining Type 2 patients without compromising the wait time target probability. This required compromise results in the structure of the referral policy being dependent on the operational ability of provider 1. The achievable region for provider 1 is similar to the situation shown in Figure 2.2b.

These results motivate our exploration of fair-allocation objectives in the extension (Section 2.4). In practice it is unlikely to be acceptable to knowingly send patients to a facility where they will certainly suffer from extended wait times.

2.3.2 Referrer Problem with Heterogeneous Operational abilities

In this section we focus on the situations where providers differ in their operational abilities. Specifically, one provider is HOC and thus able to find efficiencies scheduling both patient types together, while the other is LOC. The assumptions from the previous section are maintained which results in the LOC provider utilizing scheduling policies P_1 and P_2, which respectively prioritize patient types 1 or 2. The shared scheduling policy S is the only option for the HOC provider. The theorems below describe the optimal policies for the referrer and providers for the both scenarios where the HOC provider has higher and lower capacity.

Theorem 6. [Higher capacity HOC provider] *Consider a referral system where the provider with the higher capacity (provider 1) is HOC. The optimal referral and scheduling policies are as follows:*

a) In all capacity scenarios except when both the capacity scenario is (Low, Low) and $2m_1 + m_2 < \lambda_1 + \lambda_2$*, all feasible policies are optimal if and only if they satisfy:*

$$\lambda_{12} + \lambda_{22} = m_2 \tag{2.21}$$

The resulting target probabilities from such a policy are: $x_{11} = x_{21} = (\frac{m_1}{\lambda_1+\lambda_2-m_2})^{(\alpha_1)^{-1}}$ *and* $(x_{12}, x_{22}) = (1, 1)$.

b) When the capacity scenario is (Low, Low) and $2m_1+m_2 < \lambda_1+\lambda_2$*, the optimal referral policy depends on the condition:*

$$(\lambda_1 + m_1 - m_2)(\frac{m_1}{\lambda_1 + m_1 - m_2})^{\alpha_1^{-1}} - (2m_2)(\frac{1}{2})^{\alpha_1^{-1}} - (m_1 + m_2 - \lambda_1)((\frac{m_1 + m_2 - \lambda_1}{\lambda_2 - m_1})^{\alpha_2^{-1}-1} - 1) > 0 \tag{2.22}$$

The optimal policy and target probabilities are shown in the following table :

Index	Optimal Policy $(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$	TP x_{i1} (x_{12}, x_{22})	Optimality Condition (Eq. 2.22)
1	$(\lambda_1 - m_2, m_1)$ $(m_2, \lambda_2 - m_1)$	$(\frac{m_1}{\lambda_1+m_1-m_2})^{(\alpha_1)^{-1}}$ $(1, 0)$	True
2	(m_1, m_1) $(\lambda_1 - m_1, \lambda_2 - m_1)$	$(\frac{1}{2})^{(\alpha_1)^{-1}}$ $(1, (\frac{m_2+m_1-\lambda_1}{\lambda_2-m_1})^{(\alpha_2)^{-1}})$	False

Table 2.6: Optimal policies (first provider is HOC - capacity scenario (Low, Low) and $\lambda_1+\lambda_2 > 2m_1+m_2$)

Theorem 6 describes the referrer and provider policies when the HOC provider has higher capacity. There are two cases, where case (b) is the exceptional situation where both providers have very low capacity.

In case (a), the best policy for the referrer is to allocate the LOC provider its full capacity m_2 and the remaining $\lambda_1 + \lambda_2 - m_2$ patients to the higher capacity HOC provider. These arrival rates allow the LOC provider to provide a target probability of one to both patient types. While the HOC provider receives more patients than its capacity, it uses the shared policy which grants both patient types positive target probabilities.

In case (b) there are two referral policies whose optimality is determined by Equation 2.22. Policy 1 has the LOC provider fully prioritizing Type 1 patients. In Policy 2, both providers offer positive target probabilities to all patients. At higher capacity and ability levels of the LOC provider, the referrer moves toward selecting the second policy. As arrival rates increase the referrer moves toward selecting Policy 1. Policy 1 refers a smaller proportion of patients to the HOC provider 1 enabling higher target probabilities for these patients. The additional Type 2 patients referred to the LOC provider receive a target probability of zero.

Theorem 7 describes optimal referral policies in the situation where provider operational capabilities are heterogeneous and the HOC provider has lower capacity:

Theorem 7. [Lower capacity HOC provider] *Consider a referral system where the provider with the lower capacity (provider 2) is HOC. The optimal referral and scheduling policies are as follows:*

a) In all the capacity scenarios if $2m_2+m_1 \geq \lambda_1+\lambda_2$ *then all feasible policies are optimal which satisfy:*

$$\lambda_{11} + \lambda_{21} = m_1 \tag{2.23}$$

The resulted target probabilities are: $(x_{11}, x_{21}) = (1, 1)$ *and* $x_{12} = x_{22} = (\frac{m_2}{\lambda_1+\lambda_2-m_1})^{(\alpha_2)^{-1}}$.

b) The optimal policy and target probabilities when $2m_2 + m_1 < \lambda_1 + \lambda_2$ *depend on the following equations:*

$$(m_1 + m_2 - \lambda_1)((\frac{m_1 + m_2 - \lambda_1}{\lambda_2 - m_2})^{\alpha_1^{-1}-1} - 1) - (\lambda_1 + m_2 - m_1)^{1-\alpha_2^{-1}} m_2^{\alpha_2^{-1}} + 2^{1-\alpha_2^{-1}} m_2 > 0 \tag{2.24}$$

$$\frac{(m_1 + m_2 - \lambda_1)^{\alpha_1^{-1}} - (m_1 - \lambda_1)^{\alpha_1^{-1}}}{(\lambda_2 - m_2)^{\alpha_1^{-1}-1}} - (2m_2)(1 - (\frac{1}{2})^{\alpha_2^{-1}}) > 0 \tag{2.25}$$

The optimal policy and target probabilities are shown in the following table:

Capacity Scenario	Index	Optimal Policy $(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$	TP (x_{11}, x_{21}) x_{i2}	Optimality Condition
(Low, Low) (Mid, Low)	1	$(\lambda_1 - m_2, \lambda_2 - m_2)$ (m_2, m_2)	$(1, (\frac{m_1+m_2-\lambda_1}{\lambda_2-m_2})^{(\alpha_1)^{-1}})$ $(\frac{1}{2})^{(\alpha_2)^{-1}}$	Eq. 2.24 True
	2	$(m_1, \lambda_2 - m_2)$ $(\lambda_1 - m_1, m_2)$	$(1, 0)$ $(\frac{m_2}{\lambda_1+m_2-m_1})^{(\alpha_2)^{-1}}$	Eq. 2.24 False
(High, Low)	1	$(\lambda_1 - m_2, \lambda_2 - m_2)$ (m_2, m_2)	$(1, (\frac{m_1+m_2-\lambda_1}{\lambda_2-m_2})^{(\alpha_1)^{-1}})$ $(\frac{1}{2})^{(\alpha_2)^{-1}}$	Eq. 2.25 True
	3	$(\lambda_1, \lambda_2 - m_2)$ $(0, m_2)$	$(1, (\frac{m_1-\lambda_1}{\lambda_2-m_2})^{(\alpha_1)^{-1}})$ 1	Eq. 2.25 False

Table 2.7: Optimal policies (second provider is HOC and $2m_2 + m_1 < \lambda_1 + \lambda_2$)

Theorem 7 is structurally very similar to Theorem 6. Part (a) describes the first situation where $2m_2+m_1 > \lambda_1+\lambda_2$ and it is always best for the referrer to allocate the LOC provider its full capacity of m_1 patients. Like Theorem 6, the LOC provider is able to provide both types of patients with a target probability of one. The HOC provider again can take advantage of complementarities to offer a shared policy where there is a positive chance for all types of patients to be served within their target wait times.

Part (b) describes optimal policies when there is relatively small capacity in the system. The specific policy depends on the the capacity scenario and the two Equations 2.24 and 2.25. The condition $2m_2+m_1 < \lambda_1+\lambda_2$ excludes the (Mid, Mid) capacity scenario. Equations 2.24 and 2.25 are both decreasing with respect to λ_1 and λ_2 and increasing with respect to m_1. When either condition is true, the same policy (index 1) is used which partially prioritizes Type 1 patients at the first (LOC) provider and utilizes shared capacity at the second (HOC) provider. Policy 2 is used when the LOC provider has lower relative capacity, and results in the LOC provider fully prioritizing Type 1 patients. Policy 3 is used when the capacity scenario is (High, Low). Under Policy 3 the referrer specializes the referral rates, sending all Type 1 patients to the first provider. Of note, Policy 3 is optimal under the same capacity and arrival rate conditions when both providers are LOC (see Theorem 4). This is also the only policy which is optimal for an HOC provider where shared scheduling is not used.

Discussion of impact of provider heterogeneity on optimal referral policies: With the exception of Case (b) Policy 3 of Theorem 7, when there is a HOC provider, the optimal referral policies elicit the use of the shared policy by the HOC provider. In these situations, the referrer allocates patients to the HOC provider above its capacity level. While this allows the system to take advantage of operational efficiencies that this provider is able to gather, it overloads the HOC provider and results in target probabilities below one for both patient types. In many cases this allows for less prioritization on the part of the LOC provider and broadly more equitable patient outcomes. This dynamic is illustrated in Table 2.8 which shows the impact of a HOC provider on the referral policy and target probabilities when the capacity scenario is (Mid, Mid).

Situation	Optimal Policy	TP
	$(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$	(x_{11}, x_{21}) (x_{12}, x_{22})
(LOC,LOC)	$(\lambda_1 - m_2, m_1 + m_2 - \lambda_1)$ $(m_2, \lambda_1 + \lambda_2 - m_1 - m_2)$ *or* $(m_1, \lambda_1 + \lambda_2 - m_1 - m_2)$ $(\lambda_1 - m_1, m_1 + m_2 - \lambda_1)$	$(1, 1)$ $(0, 1)$ *or* $(1, 0)$ $(1, 1)$
(HOC,LOC)	$\lambda_{11} + \lambda_{21} = \lambda_1 + \lambda_2 - m_2$	$x_{i1} = (\frac{m_1}{\lambda_1+\lambda_2-m_2})^{(\alpha_1)^{-1}}$ $(1, 1)$
(LOC,HOC)	$\lambda_{11} + \lambda_{21} = m_1$	$(1, 1)$ $x_{i2} = (\frac{m_2}{\lambda_1+\lambda_2-m_1})^{(\alpha_2)^{-1}}$

Table 2.8: Optimal policies (capacity scenario (Mid, Mid))

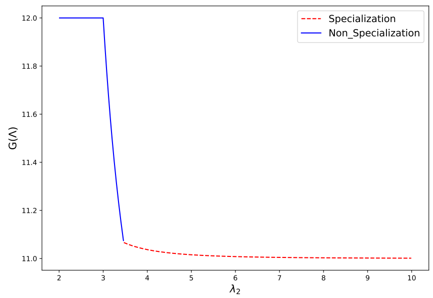
It can be seen that when both providers are LOC there are two optimal policies for the referrer (Situation (LOC,LOC) in Table 2.8). Both policies feature a provider who is overloaded and elects to fully prioritize one patient type. For instance, consider the first policy where provider 2 prioritizes Type 2 patients resulting in target probabilities $(x_{11}, x_{21}) = (1, 1)$ and $(x_{12}, x_{22}) = (0, 1)$. The first provider is referred a total rate to their capacity and is able to fully serve both types of patients within their wait time targets.

Holding arrival rates and capacity equal, improving the first providers operational capabilities from LOC to HOC results in structural changes to the referral and scheduling policies (Situation (HOC,LOC) in Table 2.8). The referrer takes advantage of the ability to efficiently schedule both patient types together and allocates more patients to this provider. This reduces both the target probabilities and the objective value of the first provider. The reduced demand for the second provider allows all patients sent to this provider to be seen within their wait time targets which improves the provider's objective function. In this case, conditioned on the provider, both patient types have the same probability of receiving service within their wait time targets and all patient streams have positive wait time target probabilities. This increased equity is due to the shared scheduling policy favored by the HOC provider 1. The policies have similar structure when the second provider is the unique HOC provider (Situation (LOC,HOC) in Table 2.8). In both situations, The difference in target probabilities between the HOC and LOC provider depends on the total arrival rate and the level of operational ability of the HOC provider.

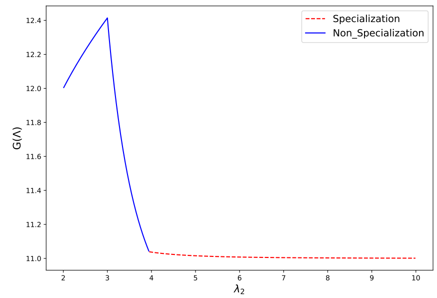
Discussion of impact of patient demand on referral policy: A referral policy is specialized if a provider receives patients of only one type (i.e. there exists $\lambda_{ij} = 0$). Theorems 4 and 7 show that these policies can be optimal only for (High,Low) capacity scenarios. In these scenarios, the lower capacity provider is specialized and receives a referral rate commensurate with the provider's capacity and is able to offer a target probability of 1. Figure 3.2 shows the impact of different parameters on the optimal range of these policies for (LOC, LOC) and (LOC, HOC) scenarios (resp. Figures 3.2a, 3.2c and 3.2b, 3.2d). These figures show the referrers objective function, i.e. the arrival rate of patients receiving treatment within their target wait times, as the arrival rate of Type 2 patients increases. The region where the referral policy is specialized is shown with the dashed line. In all cases, as the arrival rate increases the policy transitions to a specialized policy. At low λ_2, there is a region where the objective function is non-decreasing. For the (LOC, LOC) scenario this region corresponds to Theorem 3 where exactly $m_1 + m_2$ patients are seen within their target wait time. In the (LOC, HOC) scenario this corresponds to Theorem 7 part (a) where patients are served within their wait time targets by the LOC provider and the HOC provider uses a shared policy to serve an increasing rate of patients.

In each scenario, when λ_2 is sufficiently large, the policy transitions to a range where the system remains non-specialized but the number of patients seen within target wait times is decreasing rapidly. These situations correspond to Theorem 4 Policy 2 (Figures 3.2a,3.2c) and Theorem 7 Policy 1 (Figures 3.2b,3.2d). In these cases, provider 1 is using a partially prioritized scheduling policy where the volume of lower priority patients is receiving the additional Type 2 patients. Comparing Figures 3.2a and 3.2c and Figures 3.2b and 3.2d shows how the rate of decline depends on the operational ability of provider 1 in this range.

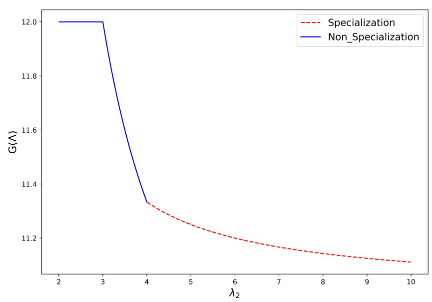
Specialization of referral rates occurs at high arrival rates. When λ_2 is sufficiently large all scenarios transition to a specialized policy where the objective function is decreasing in the arrival rate. Comparing Figures 3.2a with 3.2b and 3.2c with 3.2d show the region of the specialization policy decreases when the second provider is HOC. The specialization policy is also associated with a slower decrease in the objective function. For example for policies in Theorem 4, the following equations show the derivative of $G(\Lambda)$ with respect to λ_2 for specialization and non-specialization policies respectively: $\frac{dG_1(\Lambda)}{d(\lambda_2)} = -(\alpha_1^{-1} - 1)(\frac{m_1-\lambda_1}{\lambda_2-m_2})^{\alpha_1^{-1}}$ and $\frac{dG_2(\Lambda)}{d(\lambda_2)} = -(\alpha_1^{-1} - 1)(\frac{m_1+m_2-\lambda_1}{\lambda_2-m_2})^{\alpha_1^{-1}}$. It can be verified that both formula are negative and $\frac{dG_1(\Lambda)}{d(\lambda_2)} > \frac{dG_2(\Lambda)}{d(\lambda_2)}$. This stems from the greater volume of Type 1 patients that are served at higher priority by provider 1.



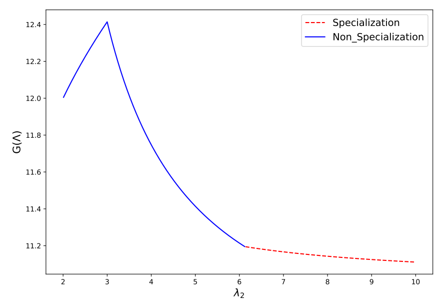
(a) $\alpha_1 = 0.25$, $\alpha_2 = 0.2$



(b) $\alpha_1 = 0.25$, $\alpha_2 = 2$



(c) $\alpha_1 = 0.5$, $\alpha_2 = 0.2$



(d) $\alpha_1 = 0.5$, $\alpha_2 = 2$

Figure 2.4: Impact of λ_2 on the optimal range of the specialization policy. $(m_2, m_1, \lambda_1) = (1, 11, 10)$

2.4 Analysis with Fairness Constraints

Based on results in the previous section we know that in many situations implementing optimal referral policy by the referrer will lead to large differences between the target probabilities of the two referred patient streams. For example in the scenario described in Theorem 2, under Policies 2 and 3, patients of Type 2 referred to provider 1 receive a target wait time probability of zero while the patients of Type 1 referred to this provider have target wait time probability of one. This is unlikely to be acceptable to the referrer despite this policy maximizing the total number of patients served within target wait times. To study the impact of mandating the provider to provide fairness we modify the base model by adding the constraint to the providers scheduling problem that target probabilities of each patient stream are equal to each other (i.e $x_{1j} = x_{2j}$, $\forall j$). This problem is referred to as the fair-allocation scheduling problem:

$$
\begin{aligned}
\max_{X_j} \; & \frac{\sum_i \lambda_{ij} x_{ij}}{\sum_i \lambda_{ij}} \\
St: & \\
0 \leq & \;\; x_{ij} \leq 1 \\
\sum_{i=1}^{n} & \lambda_{ij} x_{ij}^{\alpha_j} \leq m_j \\
x_{ij} & = x_{(-i)j}
\end{aligned}
$$

This additional constraint leads the provider to offer target probabilities of one if capacity allows and $x_{1j} = x_{2j} = (\frac{m_j}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}}$ otherwise. Extensive results on the optimal referral policy are provided in Section A.2.1 of the Appendix. At high operational abilities ($\alpha_i \geq 1$) the shared policy preferred by the providers provides fair scheduling regardless of the constraint. The fairness constraint will lead to lower objective function values when the operational abilities are lower. To illustrate the impact of incorporating fairness on target probabilities and $G(\Lambda)$ we explore the extreme capacity scenario (Low, Low). We derive a precise relationship between the provider level of ability and $G(\Lambda)$. The following table shows optimal referral policies and associated target probabilities when both providers are LOC and $\lambda_1 + \lambda_2 > 2m_1 + m_2$.

Referrer Problem	Candidate Optimal Policies	TP
	$(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$	(x_{11}, x_{21}) (x_{12}, x_{22})
Standard Problem (Thm. 2 Policy 3)	$(m_1, \lambda_2 - m_2)$ $(\lambda_1 - m_1, m_2)$	$(1, 0)$ $(0, 1)$
Fair-allocation Problem	(m_1, m_1) $(\lambda_1 - m_1, \lambda_2 - m_1)$	$x_{i1} = (\frac{1}{2})^{(\alpha_1)^{-1}}$ $x_{i2} = (\frac{m_2}{\lambda_1+\lambda_2-2m_1})^{(\alpha_2)^{-1}}$
	$(\lambda_1 - m_2, \lambda_2 - m_2)$ (m_2, m_2)	$x_{i1} = (\frac{m_1}{\lambda_1+\lambda_2-2m_2})^{(\alpha_1)^{-1}}$ $x_{i2} = (\frac{1}{2})^{(\alpha_2)^{-1}}$

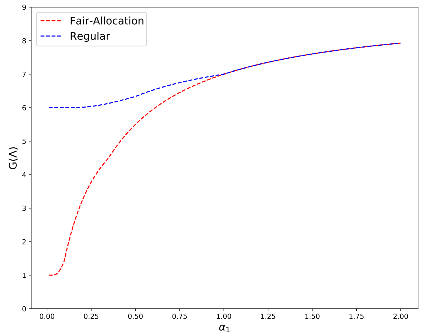
Table 2.9: Optimal policies (both providers are LOC, capacity scenario (Low, Low))

The fair-allocation problem has two candidate optimal solutions. The optimal solution will depend on the relative capacities and operational abilities of the two providers. With respect to the target probabilities, in the regular system the optimal policy results in a fully polarized system where the first and second providers only serve Type 1 and Type 2 patients respectively. However, in the fair-allocation referral system there are positive chances for all patients to be served within their wait time targets. We can explicitly compare the rate at which patients receive service within their target wait times between the standard and fair-allocation problem. This corresponds to the difference between the referrer objective functions and is shown in the following equation.

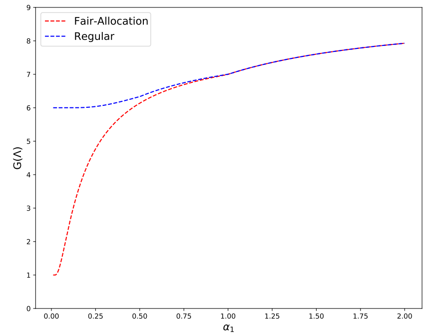
$$\begin{aligned}\mathit{\Delta}(G(\Lambda)) = \min(&m_1(1-(\frac{1}{2})^{(\alpha_1)^{-1}-1}) + m_2(1-(\frac{m_2}{\lambda_1+\lambda_2-2m_1})^{(\alpha_2)^{-1}-1}),\\ &m_2(1-(\frac{1}{2})^{(\alpha_2)^{-1}-1}) + m_1(1-(\frac{m_1}{\lambda_1+\lambda_2-2m_2})^{(\alpha_1)^{-1}-1})). \end{aligned} \tag{2.26}$$

The difference shown in Equation 2.26 is decreasing in operational abilities (α_1, α_2) and approaches 0 as α_1 and α_2 both approach 1. This can be generalized to other situations as well.

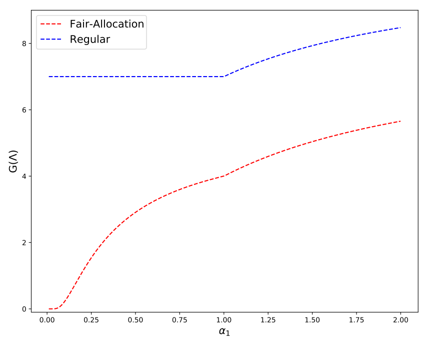
Numerical example: We illustrate the impact of fairness on the system performance and the optimal policy with a series of numerical examples which are shown in Figures 2.5-a to 2.5-d. In these figures, the impact on the optimal system objective function is shown as the operational ability of the first provider is increased. In each of these situations the second provider is LOC, however, we consider the case where $\alpha_2 = 0.01$ (Figures 2.5a,2.5c) and $\alpha_2 = 0.99$ (Figures 2.5b,2.5d). Figures 2.5a and 2.5b show the capacity scenario (High, Low) when $2m_2 + m_1 < \lambda_1 + \lambda_2$ and the second provider has low and moderate operational ability. Figures 2.5a and 2.5-b a similar example but for a (Low, Low) capacity scenario when $2m_1 + m_2 < \lambda_1 + \lambda_2$.



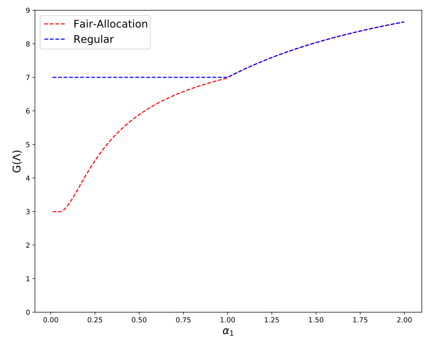
(a) $\alpha_2 = 0.01$, (High, Low)



(b) $\alpha_2 = 0.99$, (High, Low)



(c) $\alpha_2 = 0.01$, (Low, Low)



(d) $\alpha_2 = 0.99$, (Low, Low)

Figure 2.5: Impact of fairness on $G(\Lambda)$. (a)-(b): $(m_2, m_1, \lambda_2, \lambda_1) = (1, 6, 4, 5)$, (c)-(d): $(m_2, m_1, \lambda_2, \lambda_1) = (3, 4, 5, 6.5)$

In all cases as the operational ability of the providers increase the gap between system performance for the fair-allocation and regular policies decreases. When α_1 is small, the gap is substantial in all cases, whereas when α_1 is large the gap depends on the capacity situation. In the (High, Low) situations, there is no gap at $\alpha_1 > 1$ as the optimal policy limits the demand for provider 2 such that all patients can be seen with target probability of 1 (Theorem 6-a). In the lower capacity scenarios, there is a gap even at higher α_1 but the size of the gap depends to a large degree on α_2 and is not visible when $\alpha_2 = 0.99$. The gap is due to the prioritized optimal policies described in Theorem 6-b. When $\alpha_2 = 0.01$ the system is fully prioritized with patients allocated to provider 2 receiving target probability of 1 or 0. When $\alpha_2 = 0.99$ the system is partially prioritized with both types of patients allocated to provider 2 receiving positive target probabilities.

2.5 Conclusion

We studied referral and scheduling policies in a centralized system where both the referring agent and providers delivering the service are attempting to maximize the proportion of their patients seen within target wait times. Our model allowed study of the impact of operational parameters through a novel achievable region methodology where providers may have positive or negative complementarities associated with serving heterogeneous patient populations. We extracted optimal decisions for the referrer and providers in the centralized system for different operational parameters.

When operational abilities are high, scheduling complementarities lead the system to deliver equitable service with high probabilities of patients being seen within their target wait times. When abilities are low, there is a conflict between maximizing the number of patients served and fair access services. In particular, providers may be incentivized to fully or partially prioritize one type of patient over the other. We find that the cost of requiring equitable service between patient types is highly dependent on the operational abilities of both providers.

These results highlight how operational structure may influence equitable access to services for groups of patients with different needs. When scheduling is challenging, there is a strong incentive to simply prioritize one group of patients over the other. In this paper we have investigated a single reward mechanism where referrer and provider maximize throughput of patients receiving service within target wait times. While this is aligned with current practice, rewards and regulations which explicitly take into account fairness goals may be required in more challenging instances.

Chapter 3

The Application of Reinforcement Learning in Patient Referral Networks

3.1 Introduction

On-time access to specialists has become a major challenge for healthcare systems in many countries including the United States, Canada, and most European countries (Jaakkimainen et al. 2014, Alvarez et al. 2019, Viberg et al. 2013b). On-time access to specialists can be measured by wait time targets (WTT), defined as the recommended time within which patients should be treated. It has been widely shown that long wait times exceeding WTT result in deteriorating patient health condition which places additional pressure on healthcare systems (Lawrentschuk et al. 2003b, Haddad et al. 2002b). Wait times have also been significantly impacted during COVID-19. For instance, Mayol and Fernández Pérez (2020) reported that there was reduced capacity and increased wait times for elective surgeries when resources were rationed for COVID-19 patients in the spring of 2020. Finally, Moir and Barua (2020) defined the cost of waiting time as the value of time that is lost while waiting for treatment and estimated that the cost of waiting time per patient in Canada to be between $2,254 and $6,838.

Patients can gain access to specialized services through two main referral schemes: centralized and decentralized referral systems. General practitioners (GPs), specialists, and the referral process are the three main elements of both referral systems. In a decentralized referral system, a patient is first seen by a GP and in the case that further treatment is needed, the patient is referred to a specialist directly by the GP. However, under a centralized referral scheme, if specialized services are needed, the patient is directed by the GP to a central body (the referrer), which has access to all specialists in the network. The referrer is then responsible for allocating the patient to an available specialist.

Research has found that the optimum structure of patient referral networks in many cases is still unclear (Lorant et al. 2017), but there have been moves toward centralization (Scott et al. 1999). Reducing the likelihood of patient readmission to hospitals (Mascia et al. 2015), better continuity of care (Lorant et al. 2017), better equity across the network, wider access to specialists, streamlining of the referral process and information flow (Scott et al. 1999), and higher fairness to physicians through balancing referral rates are some advantages of using a centralized referral scheme.

Despite the potential advantages of a centralized referral network, a referrer's information on the characteristics of the network is very limited and there are high levels of uncertainty in both arrival rates to the network and specialist service times . As a result, finding optimal referral rates to specialists is still a complex and challenging problem. The complexity of the problem significantly increases as the network becomes larger (e.g., the number of specialists in the network increases) or as more factors are incorporated in making referral decisions (e.g., closeness to specialists and wait times for them). Allocating patients to specialists close to them can reduce the burden of travel for patients (Piterman and Koritsas 2005) and therefore, distance between specialists and patients is another important factor besides wait time that can impact referral decisions in a network (Langley et al. 1997).

In this paper, we use deep learning methodology to model a centralized referral network where a referrer assigns patients from different locations to a set of specialists. In particular, we use deep Q-network (DQN), one of the most promising reinforcement learning methodologies (RL), introduced by Mnih et al. (2015). The methodology is used in complex environments where, due to incomplete information of a referring agent on environment characteristics; complexity of interactions within the environment, and a high level of uncertainty in different elements of the environment, it is not possible to find an analytical solution to the optimal referring behaviour for an agent. In our model, patients arrive to the system stochastically and are homogeneous with respect to their health condition. However, the specialists are heterogeneous with respect to the time they require to treat a patient. This setting is a representation of health conditions with standardized care pathways such as cataract surgery. Patients requiring cataract surgery are usually referred to a specialist at a certain stage of disease progression such as when color intensity is reduced or when a patient has difficulty in daytime driving (Allen and Vasavada 2006). In practice, a referrer's information about system characteristics such as specialists' service times and patients' real wait times for specialists are very limited. These make the system complex to solve using traditional analytical models such as Markov Decision Process (MDP) and/or queueing system.

We further extend our analysis to incorporate collaboration among adjacent centralized referral networks. Due to the uncertainty in patient arrivals and service times, a referral network can get congested, in which case patient transfer among adjacent referral networks can improve health outcomes of patients by reducing their wait times (Centre for Substance Abuse Treatment 2000). Collaboration between the networks further increases the complexity of finding optimal referral rates to specialists and thus justifies the implementation of machine learning algorithms. The concept of collaboration between the networks becomes a more urgent issue once we consider how pandemics such as COVID-19 have affected healthcare systems in different countries, including Canada. The pandemic has put unequal pressure on different referral networks, thus resulting in significant increases in arrival rates to some referral networks and decreases in capacity in others (Moir and Barua 2020). It is still unclear under what conditions referral networks can collaborate and what could be the results of this collaboration on patient wait times and referral rates to specialists.

To the best of our knowledge, our study is the first that applies a DRL methodology to investigate the potential impact of using an intelligent referrer in centralized referral networks. Using a state-of-the-art DRL methodology, we seek to answer the following research questions: 1) To what degree can using an intelligent referrer in a single centralized referral network improve average patient wait times for specialists, and 2) How can we define a collaboration mechanism between two intelligent centralized networks and to what degree would collaboration between the networks improve patient access to specialists?

Our study yields several interesting findings that have policy implications. First, in comparison with commonly used referral policies such as the shortest queue policy, we find that depending on a referral's network characteristics, specialist service times and WTT, using an intelligent referrer can significantly improve patients' access to specialists and reduce wait times in the network. In addition, the performance of the intelligent referrer, with respect to wait times, increases as WTT decreases and as specialist service times become more diverse .

Second, we find that it is always better for a referrer to have short-term vision toward optimizing referral rates to specialists. This is similar to situations where investors might choose a short-term investment in a highly volatile market. Due to the high level of uncertainty in arrival and service times, it might be the best practice for a referrer to prioritize short-term rewards over uncertain long-term rewards.

Finally, we find that collaboration has the potential to further improve patients' wait times in the system. However, the impact depends on specific factors such as a compensation scheme for referrers for each transferred patient and WTT. When WTT is high, meaning that there is higher flexibility in patient wait times, it is in the interest of an intelligent referrer to focus on optimizing referral rates to the specialists in its own network and avoid transferring them to an adjacent network. In our model, to incorporate potential government incentives for cross-network collaboration, we allow the payments for a transferred patient to be set differently than payments for within network referrals. However, we show that there is an optimal threshold for the payment for transferred patients. While a small incentive may not motivate the networks to collaborate with each other, a high value of incentive, on the other hand, could result in over-collaboration which occurs when too many patients are transferred between the networks and, consequently, average wait times in the whole system increase. As a result, we show that there is a "sweet spot" for the incentive to improve the overall outcome of the system.

The rest of the study is organized as follows. We first review the related operations management literature. In Section 3.3 we describe different elements of a centralized referral network and introduce a general mechanism under which collaboration between the networks can occur. In Section 3.4 we study how a DQN approach can be applied in a single centralized referral network and in Section 3.5 we study the concept of collaboration between two centralized referral networks. Section 3.6 is devoted to sensitivity analyses we performed on different characteristics of the model. We conclude the paper in Section 3.7 with a summary of our findings, insights, and limitations of our study.

3.2 Literature and Positioning

Convenient and on-time access of patients to specialized services is one of the major concerns in many countries and evidence suggests that the location of a specialist and its proximity to patients is one of the factors that can impact referral rates to the specialist (Piterman and Koritsas 2005, Langley et al. 1997, Olmos et al. 1995). This problem has been investigated from different perspectives. Studies under the facility-location category have focused on optimizing the location of the specialized services considering different sets of constraints. Under this perspective, the main goals are usually to find the optimum number of facilities and optimum locations to locate the facilities to make sure that patients do not wait more than a specific amount of time. For instance, Baron et al. (2008) analyzed the Stochastic Capacity and Facility Location Problem (SCFLP) with the FIFO discipline and general arrival and service processes. Each facility, modeled as single- or multiple-server queue, serves customers within a predetermined radius and it is assumed that customers visit the closest facility. Through decomposition of the problem into three subproblems they showed that arrivals to facilities have Poisson distribution and developed an algorithm to determine the optimum number and location of the facilities. Zhang et al. (2009) also investigate the facility location problem in preventive healthcare with the goal of maximizing the number of clients who participate in the program. They focused on a referral network where each facility is modeled as an M/M/1 queue, and it is assumed that each client favours the facility with minimum expected total time and the number of arrivals to a facility decreases as the expected total time increases.

Although optimizing the location of facilities has the potential to improve patients' access to specialized services, in most cases re-locating or changing the location of a healthcare provider is not practical. For instance, if specialized services are only accessible through specialists, then optimally locating specialists may not be feasible. Therefore, another perspective is to focus on optimizing interactions within a referral network. Optimizing referral rates and scheduling policies has been studied extensively (Cayirli and Veral 2003, Marynissen and Demeulemeester 2019, Ahmadi-Javid et al. 2017b). However, to make the problem tractable, most of the studies in this category make some restrictive assumptions that might not represent what happens in practice. For instance, in studies focused on optimizing scheduling policies, it is usually assumed that the service times of healthcare providers and their distributions are known and referral rates to these providers are considered to be exogenous. On the other hand, in most studies focused on optimizing referral rates to specialists, not only are distance between patients and specialists not considered, but also referral decisions are made based on average wait time for specialists rather than real wait time. The first part of our paper where we study the impact of using DQN methodology in a single referral network falls into the latter category. In our study, a referrer does not have complete information on the specialists' service times. In addition, the referrer not only considers distance between patients and specialists but also takes real wait times for specialists into account when making its referral decisions. Despite the extensive literature on optimizing interactions within a referral network, the on-time and convenient access of patients to specialized services is still one of the challenges yet to be solved.

Another perspective from which optimizing interactions between referral networks has been studied is on a smaller scale and through a qualitative approach by Peng and Bourne (2009). They examined the concept of competition between two healthcare networks, including two core hospitals located next to each other, and their partners, in Taiwan. They proposed that simultaneous competition and collaboration can exist when each organization has complementary but distinctly different sets of resources. Since their study investigated one case study in a single industry, they also argued that their findings might not be generalized. The second part of our paper utilizes a very different approach from Peng and Bourne (2009) and provides complementary results including the level of incentives that are required to initiate collaboration between the networks.

Our paper studies how collaboration between referral networks can improve patient access to specialists by applying a RL technique (i.e., DQN). DQN is a RL technique which is the result of integrating an Artificial Neural Network (ANN) into the Q-learning process. The approach allows us to analyze the problem in a more practical way where an intelligent referrer learns optimal behaviour through interacting with different elements of the network. The methodology has been applied to a variety of topics. Babaev et al. (2019) applied a deep learning approach for credit scoring in the banking industry. The results indicated the superiority of the deep learning algorithm versus benchmarks on historical data where the former resulted in significant higher financial gains for a case study bank. Liu and Shoji (2019) also applied a deep learning algorithm to predict intelligent vehicle mobility. The algorithm outperformed other techniques and resulted in significant improvement in vehicle mobility prediction. Ahn and Park (2020) applied DQN to control balancing between heating, ventilation, and air conditioning (HVAC) systems and Dai et al. (2019) used this methodology to improve the utility of vehicular networks. Deep learning techniques have also been applied widely in medical image analysis to improve both precision and speed of diagnosis processes (Ker et al. 2017). To the best of our knowledge, our study is the first one that examines the application of a deep learning approach in patient referral systems.

3.3 Modelling a Centralized Referral Network

In this section we explain key elements and characteristics of centralized referral networks. Due to the stochastic control nature of the referral system, we explain how the Markov Decision Process (MDP) can be applied to analyze the system. However, the MDP model cannot be analytically solved to find optimal solutions because of the complexity of the system and the uncertainty in many of the variables. As a result, we then propose a DQN approach to analyze the MDP system.

In a centralized referral network (Network M), the patient is first seen by a GP. In the case where specialized services are needed the patient is then referred to the referrer which is responsible for allocating the patient to one of the specialists in the network (i.e., the referrer selects a specialist). Once the patient is referred, the selected specialist is responsible for providing the patient with the required service. Centralized referral networks are widely seen in different healthcare systems. For instance, the centralization of rheumatology referrals in Canada (Hazlewood et al. 2016) is one case which improved patient access to rheumatologists. Another example is the centralization of access to specialized health services in Quebec (Spagnolo et al. 2021). The key elements of the centralized referral Network M and their characteristics are defined as follows:

Patient: Patients are arriving from different locations to the network. We assume that patients are similar in terms of their health risk. This assumption is reasonable for certain health conditions such as cataracts for which patients are recommended to have them removed as soon as they begin interrupting daily activities (Allen and Vasavada 2006). Arrival rates (λ) are stochastic and the location of patient $i \in N$ is represented by (x_i, y_i).

Specialist: There are n_M specialists in Network M that are responsible for providing the required service for the patients (e.g., cataract surgery). Service times of specialists are stochastic and the location of specialist $j \in \{1, ..., n_M\}$ in this network is represented by (x_j, y_j).

Referrer: There is one referrer, Referrer M, in the network that is responsible for allocating arriving patients to specialists. Because all patients need to visit a specialist (as diagnosed by a GP) and no triage or any other processes are done by the referrer, patients are referred to a specialist immediately upon arrival (i.e., no wait time at the referrer stage). We assume that the location (i.e., address) of patients and specialists are known to the referrer. We use Euclidean distance as the measure for the distance between patient i located at (x_i, y_i) and specialist j located at (x_j, y_j).

$$D_{ij} = ((x_i - x_j)^2 + (y_i - y_j)^2)^{0.5} \tag{3.1}$$

Because the distance cost is a standardized cost, the distance measure does not impact the referral policies (see discussion in Section 3.3.1). Upon the arrival of a patient, the referrer seeks to allocate the patient to a specialist that is close to them and will have a low wait time.

3.3.1 Model Structure

We model a centralized referral network (Network M) as a Markov Decision Process (MDP). Table 3.1 shows the model parameters:

Index	Definition
λ_M	Arrival rate to Network M
n_M	Number of specialists in Network M
μ_j	Service rate of specialist j, $j \in \{1, ..., n_M\}$
D_{ij}	Distance between patient $i \in N$ and specialist j
q_{j-t}	Real time queue size for specialist j at state t
p_r	Payment per each patient referred to a specialist
x_{ij}	Binary variable, the value is 1 if patient i is referred to specialist j
s_{ij-t}	Binary variable, the value is 1 if patient i is served by specialist j at time step t
W_{ij}	Patient i wait time for specialist j

Table 3.1: Single Network Parameters

For Network M, the state of the system at time t, s_t, is defined as the number of patients waiting for each specialist (i.e. q_{j-t}, $\forall j$). Therefore, $s_t = \{q_{1-t}, ..., q_{n_k-t}\}$. We assume that decision epochs are short enough such that only one of the following events can occur at each time step:

1. A patient arrives to the network
2. A patient is served by one of the specialists and leaves the network
3. No event

where each event occurs with a specific probability. Note that this implies a memoryless property which is easy to compactly encode into the MDP. However, the deep reinforcement learning which we use and describe in the next section does not require as compact a representation and could be generalized in future work or implementations. The first two events are the only ones that can change the state of the system. A ={Refer,Wait} is the action set for Referrer M. In the case where there is an arrival to the system, the referrer action is {Refer}, otherwise (i.e., if a patient is served by a specialist or no event occurs) the only action is {Wait}. The {Refer} action is itself a set of specific choices of referring the patient to one of the n_M specialists in the network.

Allocating patient i to specialist j at state t results in the following outcomes for Referrer M:

1. Fixed payment (p_r): A government payment per each patient referred to a specialist in the network.
2. Immediate cost: The immediate cost is a distance-dependent cost which is realized by the referrer immediately upon referring a patient to a specialist. In our model, $F(D_{ij})$ represents the immediate cost of referring patient i to specialist j.

3. Delayed cost: The delayed cost depends on the patient wait time for the specialist and is realized by the referrer once the patient is served by the specialist (i.e. when $s_{ij-t'} = 1, t' > t$). In our model, $G(W_{ij})$ represents the delayed cost of referring patient i to specialist j.

Therefore, Referrer M's reward function for referring patient i to specialist j at state t can be defined as follows:

$$R_{M-t}(i,j) = \{p_r\} - \{F(D_{ij})\} - \{s_{ij-t'}G(W_{ij})\} \qquad t' > t \tag{3.2}$$

Note that when there is no arrival to the network there is also no reward for the referrer. More specifically, there is no reward associated with the action {wait}. For our analysis, we set $p_r = 2$ and define $F(D_{ij}) = \frac{D_{ij}}{max_j D_{ij}}$ which is the ratio of the distance between the patient and the selected specialist over the distance between the patient and the farthest specialist from the patient in the network. We also define $G(W_{ij}) = \frac{W_{ij}}{WTT}$ which is the ratio of the patient real wait time for the specialist over the WTT. Therefore, the reward function can be re-written as follows:

$$R_{M-t}(i,j) = 2 - (\frac{D_{ij}}{max_j D_{ij}}) - (\frac{W_{ij}}{WTT}) \tag{3.3}$$

In Equation 3.3 both $F(D_{ij})$ and $G(W_{ij})$ are between 0 and 1 if the patient receives the required service within the WTT. This standardization of costs allows the referrer to equally incorporate the distance and wait time when optimizing its referral decisions. However, if the patient waits more than WTT to see the specialist, then wait time cost (i.e. $\frac{W_{ij}}{WTT}$) becomes greater than 1. This will allow the referrer to balance wait time in the network by prioritizing shorter wait times over closer distance.

Since the reward function of the referrer depends on the distances between patients and selected specialists, the probabilities of patient arrivals are distance-dependent. In practice, it is difficult for the referrer to know these distance-dependent arrival probabilities. In addition, in practice the referrer has limited information on the service times of specialists and their scheduling policies. Therefore, in the transition matrix of the MDP model, the probabilities associated with patients being served by specialists are unknown to the referrer. As a result, it is not possible to get analytical results from this MDP.

3.3.2 DQN Architecture

Sutton et al. (1998) introduced reinforcement learning (RL) as a self-taught process that can be represented by an MDP. The main goal of the RL is to find a policy π, defined as a mapping from states to actions that results in the highest reward. RL algorithms can be divided into policy-based and value-based methods depending on the learning objectives (Rückstiess et al. 2010). To gain the maximum total reward in the future under a policy-based RL method, an agent (e.g., referrer) must learn a policy, only from the data, such that the action executed at each state is optimal. Value-based methods learn the policy by maximizing a value function denoted as $V(s)$ where s is the current state of the system at time step t. Then, the value-based RL methods will find a policy which results in $V^*(s)$ defined as

$$V^*(s) = \max_{\pi} V_{\pi}(s) \tag{3.4}$$

where $V_{\pi}(s) = E\left[\sum_{k=0}^{\infty} \gamma^k R_{t+k} | s_t = s, \pi\right]$ is the expected long-term return of state s under policy π. $\gamma \in (0,1)$ is the discount factor which determines the importance of distant future rewards versus immediate rewards.

Q-learning, introduced by Watkins (1989), is a value-based RL technique that replaces the value function with an action-value function $Q(s,a)$ where $Q : S \times A \rightarrow R$. As a result, the action-value function for taking action a_t in state s_t is calculated as

$$Q(s_t, a_t) = \sum_{s_t} P(s_t, s_{t+1}, a_t).[R_t(s_t, s_{t+1}, a_t) + \gamma \max_{a} Q(s_{t+1}, a)] \tag{3.5}$$

where $s_t \in S$ is the input state, $s_{t+1} \in S$ is a state accessible from s_t, a_t is agent's action in s_t, and $P(s_t, s_{t+1}, a_t)$ is the probability of moving from state s_t to state s_{t+1} given action a_t. $R_t(s_t, s_{t+1}, a_t)$ is the reward the referrer obtains from taking action a_t in state s_t and moving to state s_{t+1}. Initially, all Q(s,a) values, or simply Q values, are set to zero. The Bellman Equation is then applied to update the Q values at each step as follows:

$$Q_{new}(s_t, a_t) = Q_{old}(s_t, a_t) + \alpha[R_t(s_t, s_{t+1}, a_t) + \gamma \max_{a} Q(s_{t+1}, a) - Q_{old}(s_t, a_t)] \tag{3.6}$$

where α is the learning rate, $Q_{old}(s_t, a_t)$ is the old Q-value of the pair (s_t, a_t) and $max_a Q(s_{t+1}, a)$ is the maximum possible future reward considering all possible actions in the new state s_{t+1}. The learning rate determines to what degree old information overrides by the newly obtained information. An episode of the algorithm ends if s_{t+1} is a terminal state. Q-values are then disposed in a table, known as Q-table, and the agent will apply a greedy strategy choosing the optimal choice of action a^* as follows:

$$a^* = \arg\max_{a}(Q(s,a)) \tag{3.7}$$

Although some reinforcement learning methods such as Q-learning allow referrers to extract optimal policies when they have limited information about the environment, these methods can be applied only on small environments and quickly lose their feasibility as the number of states and actions in the environment increase (Hester et al. 2018). To overcome these challenges, Mnih et al. (2015) introduced DQN where the idea is to replace a Q-table with a neural network. The neural network is trained to learn weights in order to approximate the Q-function. A trained network receives a state as input and selects the action with the highest Q-value. For a better understanding of the deep learning process in neural networks readers are referred to Nielsen (2015).

3.3.3 Implementation of DQN For a Referral Network

We use Simpy, a process-based discrete-event framework based on Python, to define the environment where the information about referrers and specialists is stored in a json file. There are three main classes in our model:

1. Stream: This class is responsible for generating the stream of arrivals to the network. Each generated patient i has two attributes: 1) location, and 2) referral time, which shows the time that the patient is referred to a specialist.

2. Referrer: This class is responsible for referring the patient to a specialist (i.e., the referrer selects a specialist) once an arrival to a network is generated.

3. Specialist: This class includes information about the selected specialist. Under this class, the selected specialist provides the patient with the required service and then the patient leaves the system. Each specialist j has two attributes: 1) location, and 2) service time distribution.

Upon the arrival of a patient at time step t the state of the network is s_t . At each time step t, DQN receives the current state s_t as the input that contains information about the number of patients waiting for each specialist in the network. Then, DQN approximates a Q-value for each action that can be taken from that state. For example, if there are three specialists in the network, upon the arrival of a patient, there will be three estimated Q-values which show Q-value for referring the patient to each one of the three specialists. The objective of the network is to find the optimal approximating function.

In each time step the referrer either explores the environment and selects a random action or exploits the environment and selects the action for the given state that gives the highest Q-value. This is called the tradeoff between exploration versus exploitation. It is crucial for the referrer to explore the environment enough so that it can find the optimal referral policies in the environment (Cao et al. 2019). Then, the referrer's experience at each time step is stored in a data set called the replay memory that is used to improve the performance of the DQN over time. For example, executing action a_t at state s_t is an experience that moves the system to state s_{t+1} and results in a reward, R_{t+1}, for the referrer. Thus, we define e_t as the referrer's experience at time t as follows:

$$e_t = (s_t, a_t, R_{t+1}, s_{t+1}) \tag{3.8}$$

All the referrer's experiences over time are stored in the replay memory. The network is then trained on random samples from the replay memory, sampling a specific number of experiences and passing them to the network as input. We define this as the policy-based network where its objective is to approximate the optimal policy through finding the optimal function $Q_*(s, a)$. Forward propagation is then applied and the policy-based network outputs an estimated Q-value for each action that can be taken from the given input state. Then, we calculate the loss which is a mean-squared error function that compares the Q-value outputs from the policy-based network for the actions in the experiences sampled and the corresponding optimal Q-values, called target Q-value, for the same actions. As a simplifying example, assume that at time step k a given sample to the network is $(s_k, a_k, R_{k+1}, s_{k+1})$. The loss then can be calculated as follows:

$$loss = [Q^*(s_k, a_k) - Q(s_k, a_k)]^2 \tag{3.9}$$

Where $Q^*(s_k, a_k)$ is the target Q-value and $Q(s_k, a_k)$ is the estimated Q-value. The policy-based network gives us the value of $Q(s_k, a_k)$. In addition, $Q^*(s_k, a_k)$ satisfies the Bellman requirements. Therefore, it can be written as follows:

$$Q^*(s_k, a_k) = E[R_{k+1} + \gamma \max_a Q^*(s_{k+1}, a)] \tag{3.10}$$

DQN uses a separate network, called the target network, to find the value of $\max_a Q^*(s_{k+1}, a)$. The loss function is then optimized using stochastic gradient descent (Bottou 2012). In short, at the end of each time step, gradient descent is applied to update the weights in the policy network in an attempt to minimize the loss. Also, every certain amount of time steps the weights of the target network are updated to the weights of the policy-based network.

There is a rich literature on different RL algorithms capable of making an agent able to efficiently learn optimum policies in environments where there is only one final reward (e.g., win/lose in chess) (Vecerik et al. 2017, Nair et al. 2018). What makes our problem distinguishable is the fact that every single action that the referrer takes has a delayed reward associated with it. In fact, both the time that a delayed reward is recognized, and its value depend on the selected specialist service time which is stochastic. The learning process becomes much slower and challenging due to the following reasons:

1. Since there is a delay between every action and its associated reward, a referrer always makes decisions during the delay period which might not be optimal.

2. The size of the memory is limited and therefore it is not possible to keep all the state-action pairs in the memory until the referrer's rewards are realized.

In the following section we present our experimental results.

3.4 Single Network Experimental Results

In this section we explain our experimental results obtained from applying DQN methodology on Network M defined in Section 3.3. We began by representing our model using cataract referral data from a local health integration network (LHIN) in Ontario. According to the data, there were six specialists in the LHIN and 2,621 patients entered the system over a year. Because distance is a key parameter in the reward function, to let the referrer optimize referral decision more efficiently and avoid the curse of dimensionality (Poggio et al. 2017), we assumed that patients arrived to the network from two different sources, located at [8,5] and [12,6] which loosely corresponds to the major population centres in the LHIN. These locations were chosen close enough so it would be optimal for the referrer to refer patients from both sources to all specialists; the centralized referral networks are formed such that all specialists are accessible to all patients. We assumed that the arrivals (λ) to the network were from either of these two locations with the same probability.

A set of experiments with the data was performed to calibrate the model and determine the location and service times of the specialists to get similar results as in practice. The estimated service times and locations are shown in Table 3.2. In Section 3.6.1 we present the sensitivity of our results to the service times of the specialists.

Specialist Index	Location	Service Time Mean (Hour)
1	[13,3]	15
2	[14,5]	16
3	[17,7]	17
4	[19,4]	20
5	[18,8]	18
6	[20,3]	14

Table 3.2: Network M Specialists' Characteristics

To better examine how changing the wait time targets (WTT) can impact the performance of the DQN model, we considered three different WTT values [24,168,720] in hours (i.e., 1 day, 1 week, and 1 month). In the following, we explain the steps we took to train and test models for each value of WTT, and the terminology we used in the rest of the paper.

1. We considered five different values for the discount factor (i.e. $\gamma \in (0, 1)$) to study if it is better for a referrer to focus on the short-term rewards (when γ is small) versus the long-term rewards (when γ is large) when it makes referral decisions. The considered values are $[0.1, 0.3, 0.5, 0.7, 0.9]$.

2. In the **learning** phase, for each value of γ we iterated the model ten times. We ran each trial for a million hours using the same random seed to ensure comparability among the trials.

3. The performance measure is, then, the **average performance** of models with the same value of γ taken over the ten trials. For example, when we refer to average expected wait time when $\gamma = 0.1$, we used the average wait time in the network taken over the ten trials with $\gamma = 0.1$. The same concept also holds for the average expected accumulated reward. From now on, we call average expected wait time and average expected accumulated reward as average wait time and average accumulated reward, respectively.

4. The best discount factor is the one which (on average) results in higher accumulated reward and lower average wait time in the network.

5. For each value of WTT, the best trained model is defined as the model which outperforms the rest of the nine models with the same value of γ with respect to accumulated reward and average wait time in the network.

6. We used the best trained models in the testing phase to compare the performance of the DQN methodology with other policies such as shortest queue policy and random allocation policy. To incorporate the effect of randomness on the performance of the models, we performed the test ten times using different seeds. However, to have comparable results across policies, the ten seeds used in the testing phase were the same across all policies.

Next, we presented the results of the training process of Referrer M. Then, we extracted the best trained model for each value of WTT and tested its performance with respect to the resulted average wait time to study to what degree applying DQN methodology in a single referral network can improve patient access to specialists.

3.4.1 Referrer M: Training Phase

We began with the training phase of Referrer M where 150 models in total were trained (3 WTT values $\times$ 5 gamma values $\times$ 10 trials). Figure 3.1 shows the *average performance* of models with respect to the average wait time in Network M for different values of WTT. Because a certain number of patients are served at different times in each trial, we used number of patients served on the x-axis of Figure 3.1 to compare across models.

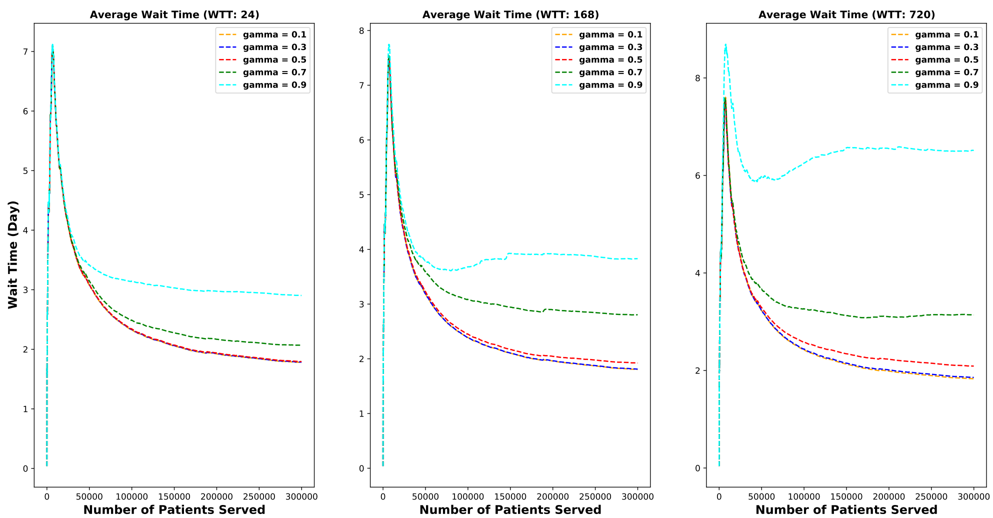


Figure 3.1: Average Wait Time in Network M (Training Phase)

In all cases of γ across different WTTs, the average wait time initially increased but, after a while, started to decrease. This, intuitively, is an indicator that the referrer learned better referral policies over time and was therefore able to improve its performance.

In all WTT scenarios, models with $\gamma = [0.7, 0.9]$ have the worst performances. This suggests that having a long-term vision for optimizing referral rates to specialists may not result in the best outcomes. In fact, Figure 3.2, which focuses on the last 50,000 patients served, shows that in all the three WTT scenarios the best results are gained when $\gamma = 0.1$. In plain words, due to the high level of uncertainty in the system considering a long vision makes it very challenging for the referrer to learn a good referral policy.

In addition, the referrer is able to achieve lower average wait time in Network M as WTT decreases. This is because as WTT decreases the delayed reward term in the utility function of the referrer becomes more important. Therefore, low wait time for specialists becomes the determining factor in the referral decision process. This suggests that small WTT values can help a network achieve lower average wait times if this is the main concern of the network decision makers.

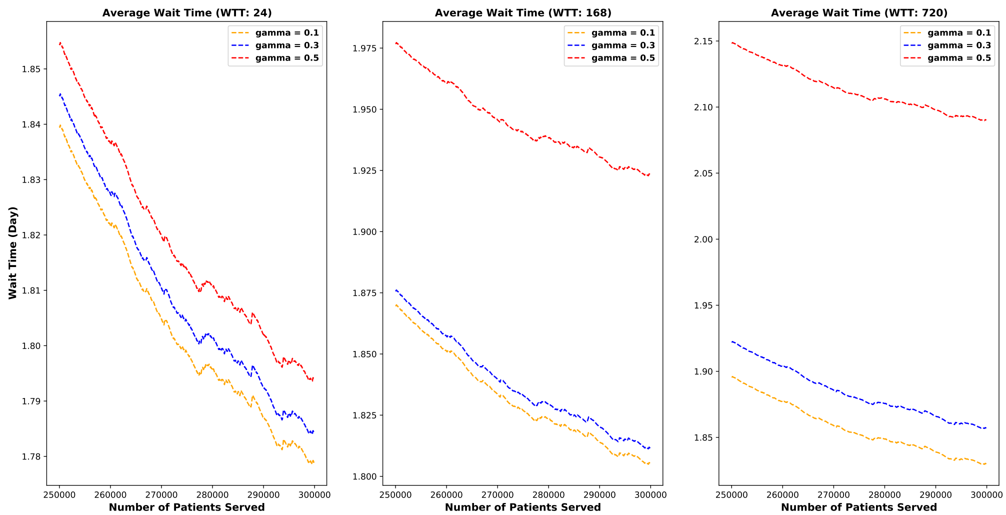


Figure 3.2: Average Wait Time in Network M (Training Phase - Last 50,000 Patients Served)

Figure 3.3 shows Referrer M's total average accumulated reward for different WTT scenarios. When WTT is small (i.e., 24 hours) and $\gamma = [0.7, 0.9]$, the accumulated reward stays negative at the end of the trials. This is because wait time for specialists is the dominant factor in the reward function of the referrer when WTT is small. Figure 3.1 shows that when WTT is 24 hours and $\gamma = 0.9$, the average wait time in the network is three days. Therefore, even if we ignore distance cost in Equation 3.3, on average, Referrer M receives -1 reward for referring each patient to a specialist. This results in a significant negative accumulated reward for the referrer at the end. Comparing Figures 3.1 and 3.3 also shows the tradeoff of wait time and distance. In fact, both wait time and accumulated reward reduce when WTT becomes smaller.

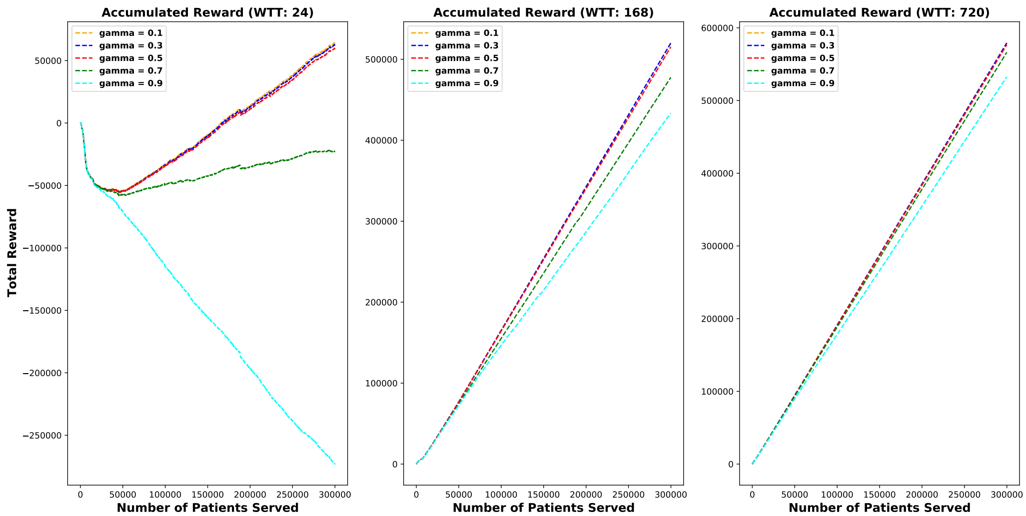


Figure 3.3: Referrer-M Average Accumulated Reward (Training Phase)

Figure 3.4 shows the accumulated reward of the referrer for the last 50,000 patients served. In all scenarios using $\gamma = 0.1$ has resulted in the highest average accumulated reward for the referrer.

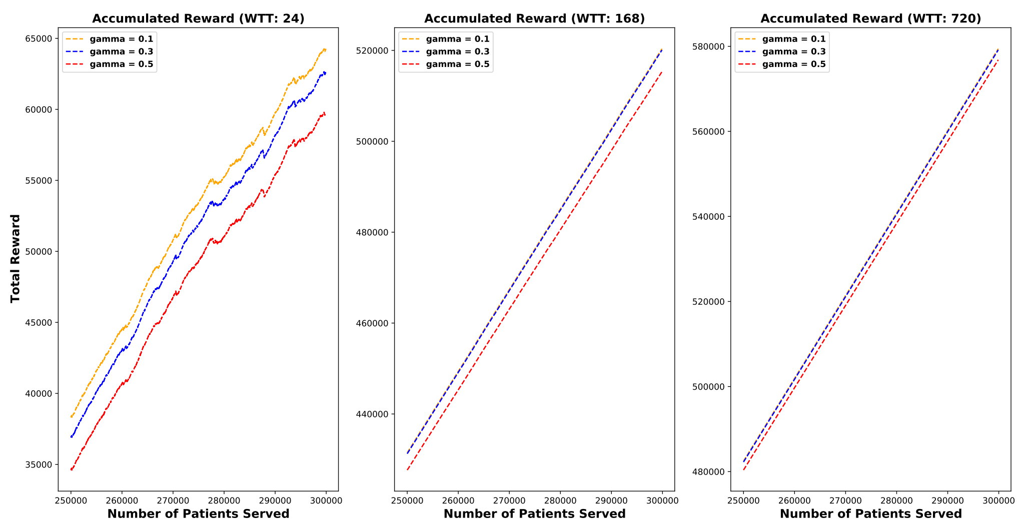


Figure 3.4: Referrer M Average Accumulated Reward (Training Phase - Last 50,000 Patients Served)

The following conclusions are made from results shown in Figures 3.1-3.4. Considering the performance of the network with respect to average wait times and Referrer M's accumulated reward, on average, $\gamma = 0.1$ and $\gamma = 0.9$ result in the best and worst outcomes. As discussed earlier, the better performance of the models with a lower value of γ is due to the existence of high levels of uncertainty in different elements of the model. Having a long vision toward optimizing referral rates in this highly volatile environment results in poor decisions which negatively impacts both wait times and accumulated reward over time.

In addition, we found that as WTT increases the accumulated rewards become more stable and the referrer is able to achieve higher rewards on average. Higher values of WTT allow the referrer to better focus on the tradeoff between wait time and distance. In particular, unlike the situation where WTT is low, referring a patient to a close specialist with a longer average wait time does not necessarily result in negative reward. Mathematically, this occurs due to the fact that when WTT is high, the importance of the delayed reward term in Equation 3.3 decreases. This intuitively means that the impact of uncertainties in specialist service times on accumulated reward becomes less significant. As a result, the referrer's performance becomes more stable, and it can achieve higher accumulated rewards.

Figures 3.5 and 3.6 compare the average wait time and accumulated reward of the best trained models with different values of gamma for different WTT scenarios. Note that the best trained model is defined as the model which outperforms the rest of the nine models with the same value of γ with respect to accumulated reward and average wait time in the network. While we use the average performance of the training set for each gamma to choose the best gamma, we need to choose the best trained model from that optimal gamma for the testing phase. Earlier we showed that on average models with $\gamma = 0.1$ resulted in both lower average wait time and higher accumulated reward. We now show that the best trained model also has $\gamma = 0.1$. We then use this best trained model in the testing phase.

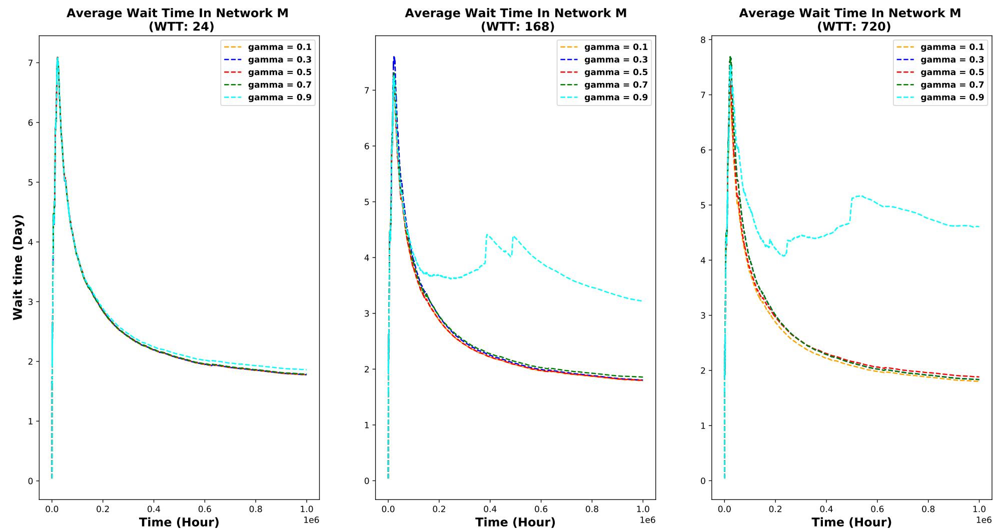


Figure 3.5: Average Wait Time in Network M (Best Model Comparison)

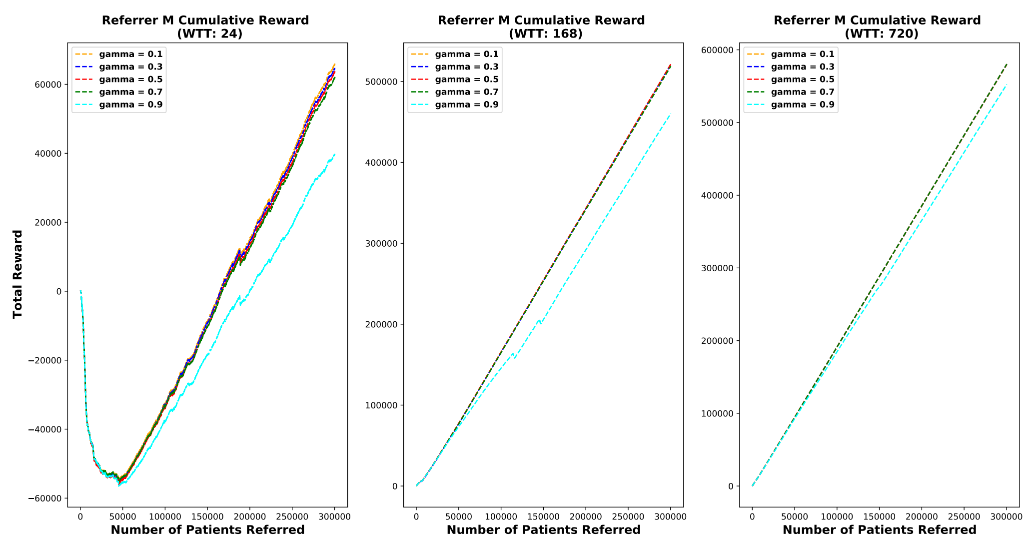


Figure 3.6: Referrer M Average Accumulated Reward (Best Model Comparison)

In all three WTT scenarios, $\gamma = 0.1$ has the best outcomes with respect to wait times and accumulated reward. Therefore, out of 50 models trained for each value of WTT, the best model with $\gamma = 0.1$ is selected as the best trained model for Referrer M.

To assure that Referrer M has learned a referral policy we analyzed the accumulated reward it received for every 1,000 patients referred to specialists. If the referrer has in fact learned and implemented a policy, we expect to see that the stability of the reward increases as time passes. The following figure shows accumulated reward for every 1,000 patients referred using the best trained model for different values of WTT.

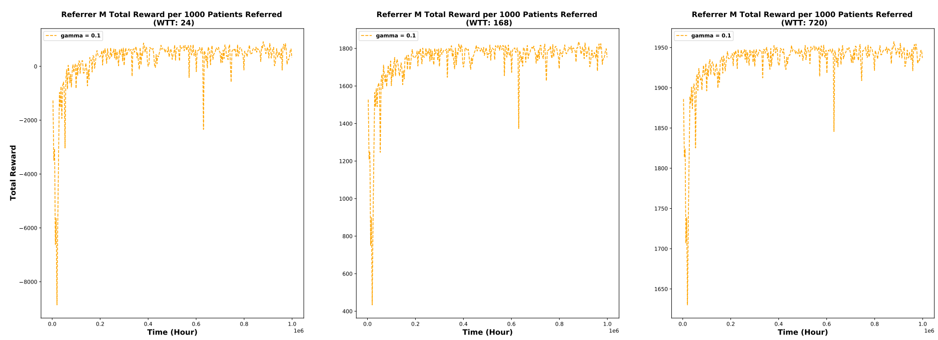


Figure 3.7: Referrer M Accumulated Reward for Every 1,000 Patients Referred Using Best Trained Model

The fluctuations of the reward per 1,000 patients are expected and caused by the stochastic nature of the environment. However, the relative stability of the rewards indicate that Referrer M has learned a referral policy for each scenario.

3.4.2 Referrer M: Testing Phase

In this section we first analyze how the best trained DQN models obtained in the previous section can handle wait times in the network. We then test the performance of these DQN models versus shortest queue and random allocation policies.

In order to analyze the performance of the best DQN models, each model was tested 10 times with different random number seeds, each time for 100,000 hours. Each trial tests our model performance for a different patient arrival and service time scenario. We used the same seeds in other referral policies to make their results comparable with the DQN model result. Figure 3.8 shows average wait time in Network M for each run.

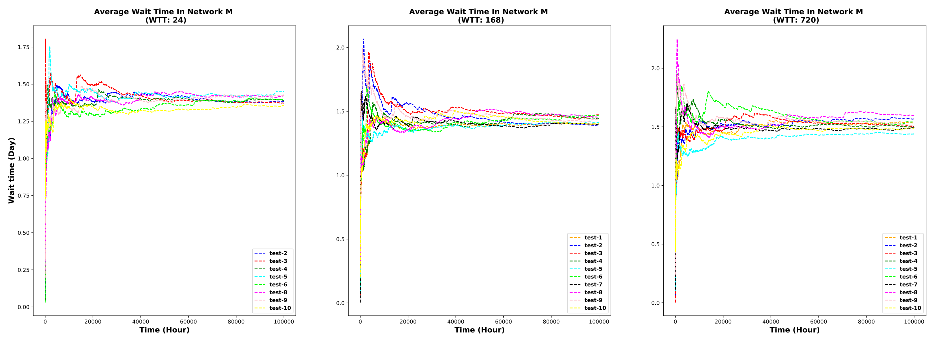


Figure 3.8: Average Wait Time in Network M (Testing Phase)

At the beginning of the simulation there were significant fluctuations in the wait times; however, in all three scenarios the referrer was able to reach a long-term stable situation with respect to the average wait time in the network. Similar to the training phase, increasing WTT resulted in increased average wait time in the network.

Figure 3.9 compares the performance of the DQN model with the performance of random allocation policy. As shown, in all the scenarios the difference between the performance of the two policies is significant and the DQN model was able to reduce average wait time in the network by almost 80% percent (five days in this particular simulation).

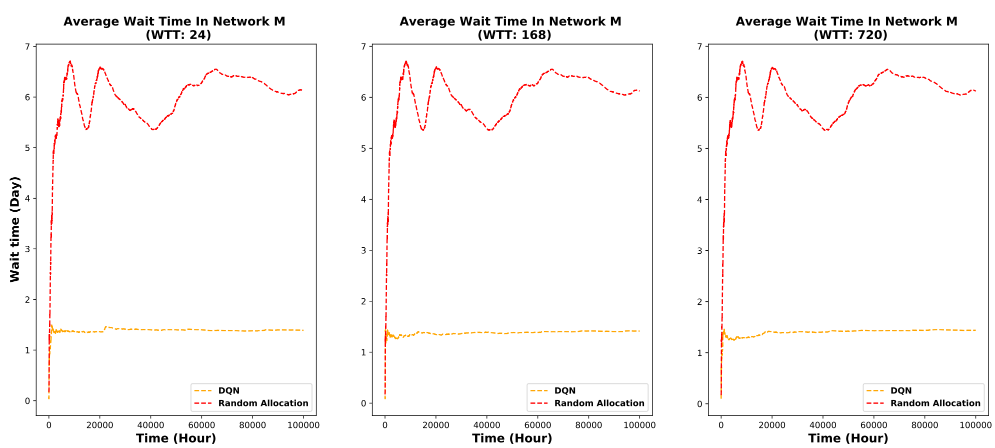


Figure 3.9: Average Wait Time in Network M (Best DQN Model vs Random Allocation Policy)

Figure 3.10 compares the performance of the DQN models with the performance of the shortest queue policy resulted from 10 trials using the same random number seeds.

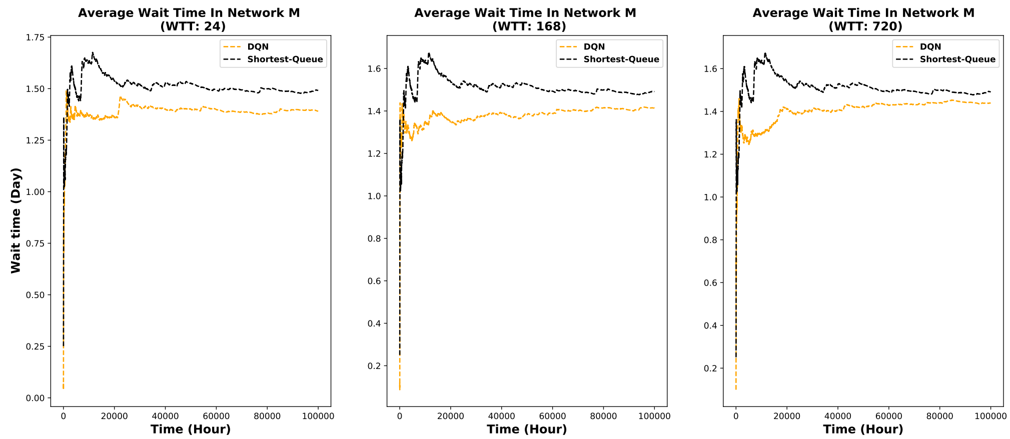


Figure 3.10: Average Wait Time in Network M (Best DQN Model vs Shortest-Queue Policy)

We show that in all three WTT scenarios the DQN model outperforms the shortest queue policy, but the performance of the two policies become closer as WTT increases. The underlying reason is that when WTT is low, wait time for specialists is the key factor in making referral decisions. Since the DQN model is able to recognize the fastest specialists in the system the referral policies applied result in lower average wait time in the network. In particular, upon the arrival of a patient to the network, if the referrer applies shortest queue policy the patient will be referred to the specialist with the shortest wait list no matter what the service time of the specialist. Therefore, it is possible that the patient is allocated to a specialist with a very low service rate (or equivalently with a high service time). In contrast, the recognition of the fastest specialists in the network allows the DQN model to make better decisions with respect to patient wait time especially when WTT is restricted.

Our comparisons show that depending on a network's characteristics, applying the DQN approach can improve patient access to specialists. In particular, in a network with restricted target wait times using DQN models can greatly reduce patient wait time over time.

3.5 Collaborative Networks Analysis

In the previous section we showed how applying DQN can improve the performance of a centralized referral network. However, in practice, to reduce the wait times of a referral network, patients can be referred to specialists in an adjacent network. Therefore, in this section we study collaboration between two centralized referral networks (Networks N and M) where patients from one network can be served by the specialists in the other network.

The stochastic nature of patient arrival and service times and the fact that arrival probabilities are distance-dependent make it almost impossible to study the concept of collaboration using well-known mathematical approaches such as queuing systems. To make the problem mathematically tractable researchers usually either assume that servers are identical or combine servers into one or two servers (Wen et al. 2019). It can be shown that pooling two identical parallel exponential servers would lead to a reduction factor of at least 50% for the mean wait time. Using an approximate formula, van Dijk and van der Sluis (2008) show that similar reduction factors of at least 50% can also be found for larger groups of servers to be pooled if 1) the service characteristics are the same, and 2) the workloads are equal.

Our approach can not only overcome these difficulties, but it is also capable of incorporating patient (realized) wait times for specialists in referral decisions. In the following we explain how collaboration between the networks can be modelled and what incentives can be considered for the networks to collaborate. We assume that Network N is experiencing higher arrival rates and therefore seeks collaboration from Network M.

Upon an arrival of a patient to Network N, Referrer N can either allocate the patient to one of the specialists in the network or send a transfer request to Network M. Referrer M on the other hand can reject or accept the transfer request. If the request is accepted, the patient will be transferred to Network M and will be allocated to one of the specialists in this network. Otherwise, Referrer N must refer the patient to one of the specialists in Network N. As discussed earlier, no queue is allowed for either of the referrers and patients are immediately referred upon their arrivals to the networks. Once the patient is referred to a specialist, they will be added to the specialist's waiting list (queue). Figure 3.11 shows a general scheme of collaboration between the networks:

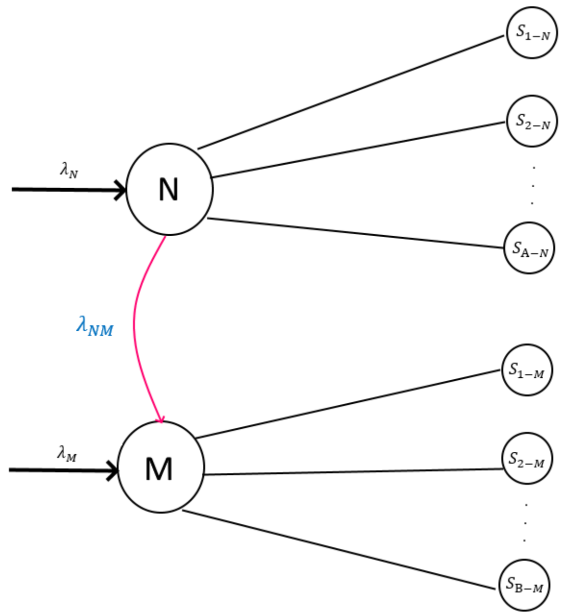


Figure 3.11: Collaboration Between Referral Networks

First, we explain the MDP model for Referrer N and how collaboration between networks can impact the dynamics of the model presented in Section 3.3.1. Number of patients waiting for each specialist in Network N defines the state of the network and we assume that decision epochs are short enough such that only one of the following events can occur at each time step:

1. A patient arrives to the network (or a transfer request is rejected by Network M)

2. A patient is served by one of the specialists and leaves the network

3. No event

$A = \{\text{Refer, Transfer, Wait}\}$ is the set of actions for Referrer N. In the case where there is an arrival to the system, the referrer actions are {Refer, Transfer}, otherwise (i.e., if a patient is served by a specialist or no event occurs) the only action is {Wait}. For example, define sp_j as the jth specialist in Network N. Assuming that there are k specialists in Network N (i.e., $n_N = k$), upon the arrival of a patient to the network, the referrer's set of actions is {"refer to sp_1", "refer to sp_2", ..., "refer to sp_k", "transfer to Network M"}. We assume that Referrer N receives no reward (i.e., reward equal to zero) for transferring a patient to Network M.

There can be two streams of patient arrivals to Network M: 1) Patients who arrive directly to the network, and 2) transferred patients accepted from Network N. One of the following events can occur at each time step for Network M:

1. A patient arrives to the network (direct arrival)

2. Network N requests to transfer a patient and Network M accepts the request (indirect arrival)

3. A patient is served by one of the specialists and leaves the network

4. No event

We assume that upon the transfer request, Referrer N provides Referrer M with the location of the patient. Then, for each specialist in Network M, Referrer M calculates the utility function presented in Equation 3.11.

$$R_{M-t}(i,j) = p_{tr} - (\frac{D_{ij}}{max_j D_{ij}}) - (\frac{W_j}{WTT}) \quad \forall j \in M \tag{3.11}$$

Where W_j is the average wait time for specialist j. If for at least one of the specialists in the network the utility value is positive, then Referrer M accepts the request. Otherwise, the request is rejected. Once the request is accepted, the transferred patient is immediately referred to a specialist, called j, in Network M and Referrer M receives the following reward:

$$R_{M-t}(i,j) = p_{tr} - (\frac{D_{ij}}{max_j D_{ij}}) - (\frac{W_{ij}}{WTT}) \tag{3.12}$$

Equation 3.11 is equivalent to Equation 3.3 except for the monetary reward that the network receives for each patient referred to a specialist. Equation 3.11 shows that p_{tr} plays an important role in motivating Network M to collaborate and accept transfer requests from Network N. Therefore, p_{tr} can be interpreted as a government incentive for collaboration among networks. We initially assumed that $p_{tr} = 4 > p_r = 2$, which indicates that Referrer M receives a higher payment per patient for accepting and allocating a transferred patient to a specialist in its network. However, in Section 3.6.2 we perform a sensitivity analysis on the value of p_{tr} to better understand the impact of government incentive on network collaboration.

Complicating the model presented in Section 3.3.1, when networks are collaborating, Referrer M has no information about the probability of receiving a request from Network N at each state. Therefore, there is uncertainty in the indirect arrivals to this network and as a result, the MDP approach cannot be solved analytically to analyze the impact of collaboration among networks on different performance metrics.

In the following, we explain our experimental results obtained from applying the DQN methodology on the two referral networks. According to the data, there are six specialists in

Network N and 2,883 patient arrivals over a year. Therefore, Network N is facing a higher arrival rate (around 10% higher) than Network M and therefore seeks collaboration from this network. The following table shows specialists' characteristics in Network N:

Specialist Index	Location	Service Time Mean (Hour)
1	[0,10]	16
2	[1,2]	15
3	[3,5]	14
4	[4,0]	17
5	[6,6]	15
6	[2,7]	14

Table 3.3: Network N Specialists' Characteristics

3.5.1 Referrer N: Training Phase

First, we focus on the training phase of Referrer N and then analyze how much collaboration between the networks impacts patient access to specialists. We trained 300 models, for which half the referrer had the option to transfer patients to the adjacent Network M. Using the best models with different values of γ for each value of WTT, Figure 3.12 shows average wait time in Network N when patient transfer between networks is allowed. Note that the general terms used in the rest of the paper as well as the steps taken to train and test the performance of the models are the same as the process we explained in Section 3.4.

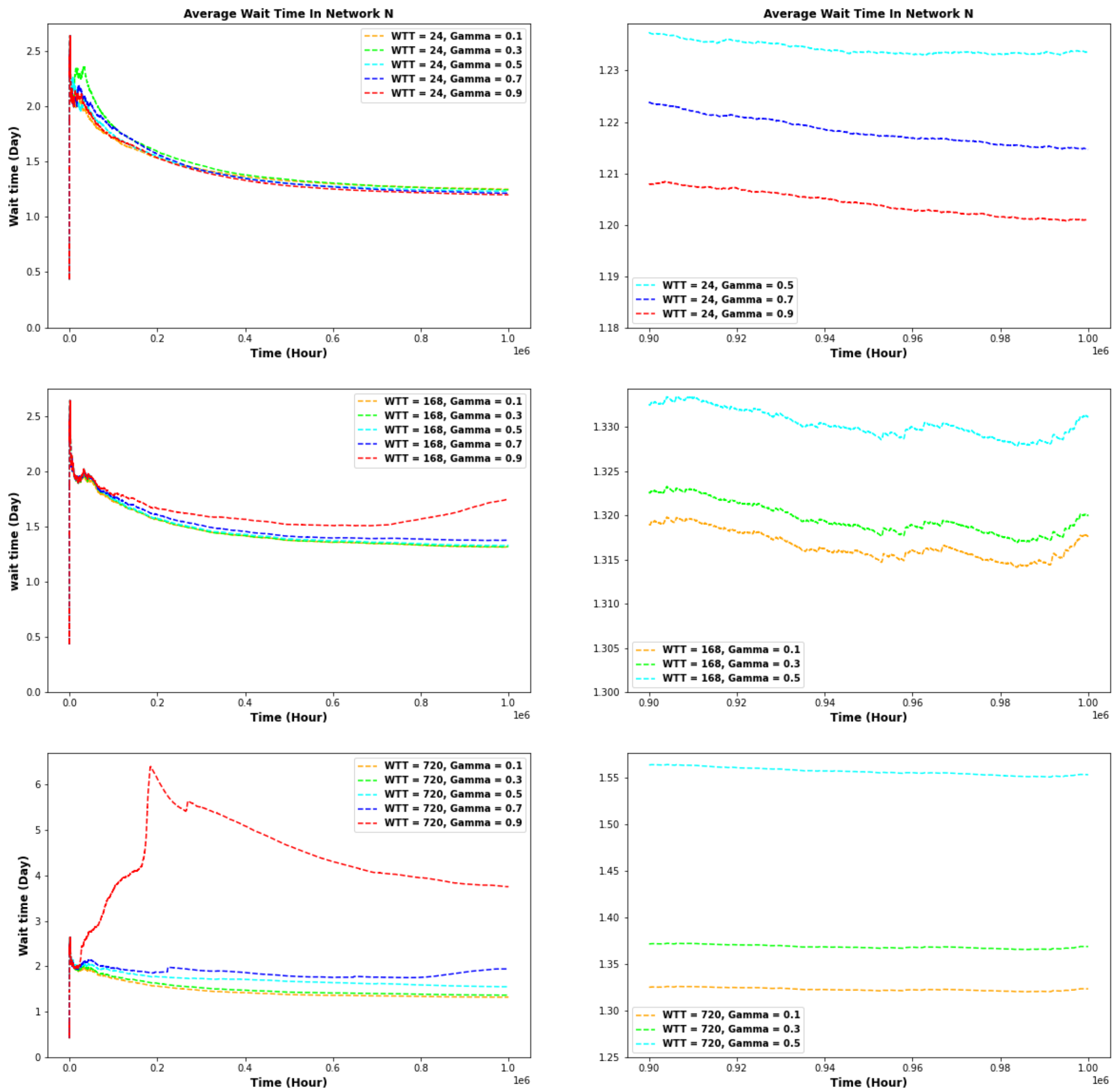


Figure 3.12: Average Wait Time in Network N (Training Phase - Best Models Comparison)

The right-hand side of Figure 3.12 shows the performance of the best model for each value of WTT for the last 100,000 hours. When WTT is large (i.e., 168 and 720) $\gamma = 0.1$ and when WTT is small (i.e., 24) $\gamma = 0.9$ slightly outperform the other models with respect to the average wait time in Network N. However, to select the best trained model we need to consider the average wait time in the whole system (i.e., both networks). In fact, a model with low average wait time in Network N may not be the best option as it may have simply transferred more patients to the other network which results in high average wait time in Network M. This, in fact, is the case when WTT is 24 hours and is shown in Figure 3.13. Figure 3.13 shows average wait time in the whole system using the best trained models for each value of γ while Referrer N has the option to transfer patients to the other network:

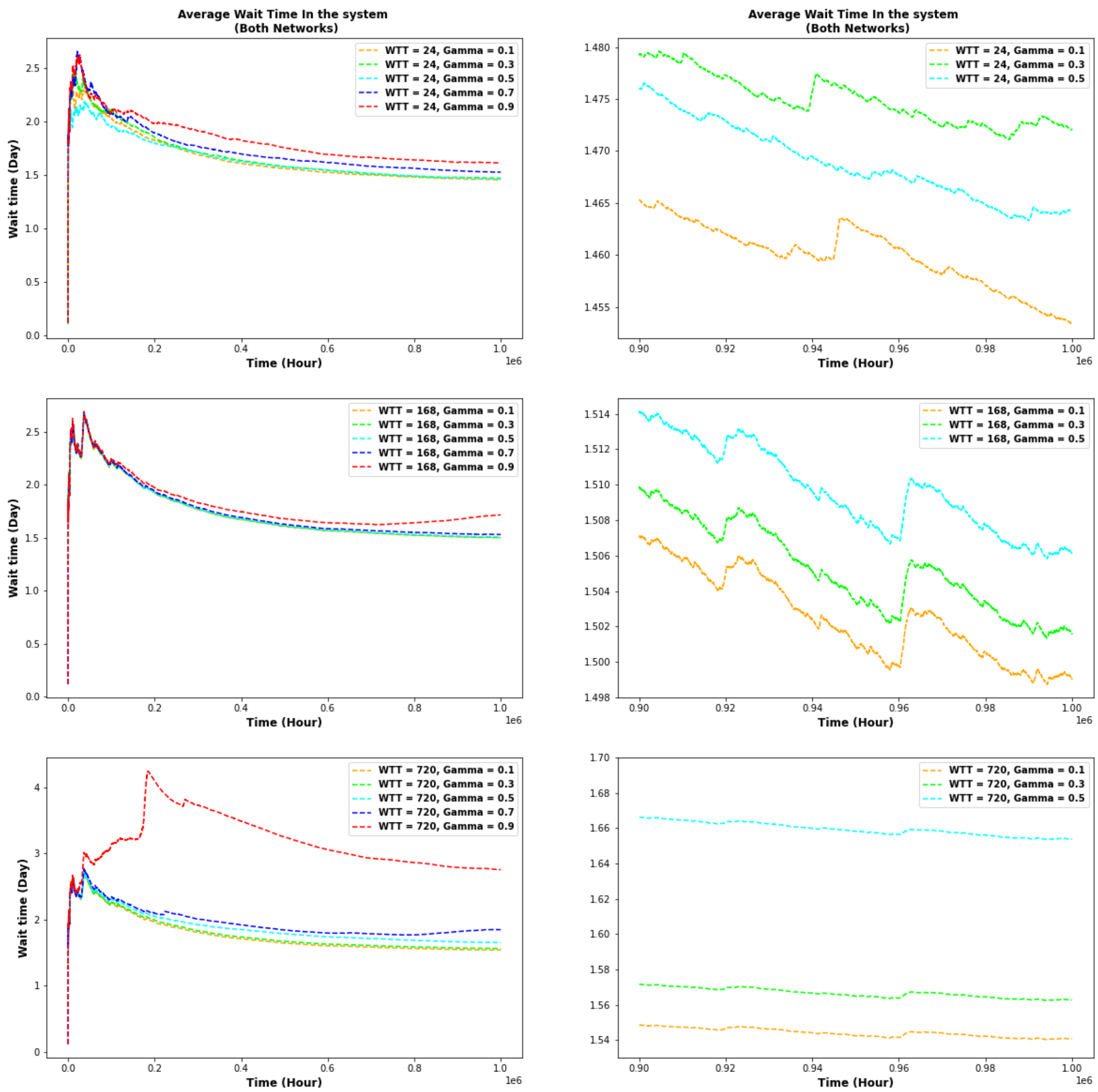


Figure 3.13: Average Wait Time in The System (Training Phase - Best Models Comparison)

It is clear from 3.13 that in all WTT scenarios, best models with $\gamma = 0.1$ has resulted in the lowest average wait time in the whole system.

Table 3.4 shows the percentage of patients transferred to the other network for different WTT-γ scenarios:

	WTT = 24	WTT = 168	WTT = 720
γ	Avg % Transferred	Avg % Transferred	Avg % Transferred
0.1	4.54	3.45	3.41
0.3	4.74	3.47	3.43
0.5	5.19	3.50	3.43
0.7	5.80	3.61	3.44
0.9	9.1	4.00	3.50

Table 3.4: Average Percentage of Patients Transferred to Network M (Training Phase)

Table 3.4 shows the percentage of patients transferred to Network M in different scenarios. We show that higher values of gamma are associated with a higher rate of patient transfer. In particular, as gamma increases Referrer N transfers more patients to the Network M. Though this policy results in better outcomes for Network N when WTT is small, it can result in higher average wait time in the system (Figure 3.13). In addition, Table 3.4 shows that higher values of WTT are associated with lower transfer rates. The underlying reason is that as WTT increases the wait time cost becomes less important and thus there is no need for Referrer N to send patients to the other network. By reducing the transfer rate Referrer N can better balance the tradeoff between the distance and wait time costs in the reward function.

To understand the implications of the collaboration between networks, Figure 3.14 compares the performance of Referrer N for different values of WTT when it has the option to transfer patients to the other network versus when it does not have that option.

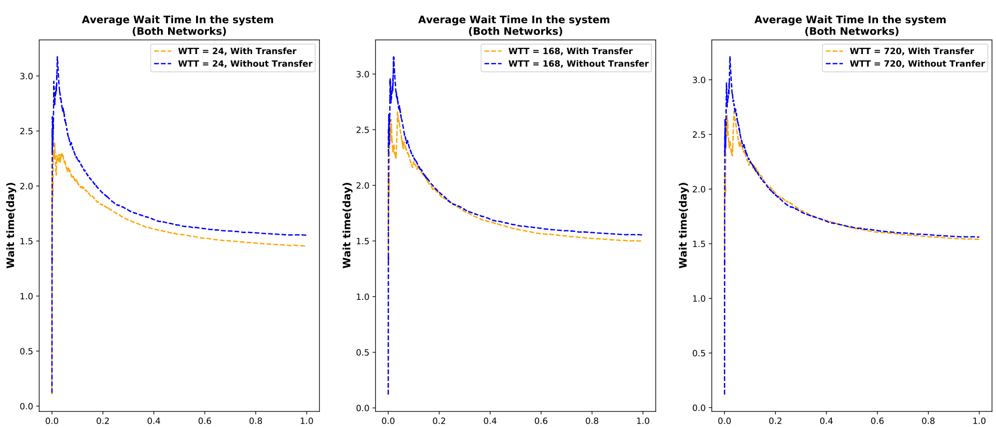


Figure 3.14: Average Wait Time in The Whole System (Collaboration vs Isolation - Best Model Comparisons)

Figure 3.14 shows that in all scenarios collaboration between the networks has resulted in lower average wait time in the whole system. In comparison with the isolated situation, collaboration between the networks has reduced average wait time in the system (in the training phase) by approximately 12%, 6%, and 1% when WTT is 24, 168, and 720 hours, respectively. As expected, the impact of collaboration decreases as WTT increases. In fact, when WTT is high, a referrer can perfectly handle arrivals to the network without the need to transfer them to the other network.

Note that in all three cases, there is a huge gap in performance of the two models at the beginning. This occurs because during this period, Referrer N was mostly exploring the environment and referrals were mostly made randomly. However, as time passes, the gap in performance of the two models first decreases and then starts increasing. As Referrer N became adept at the referral policy, it matched the performance of Referrer M. However, once in the collaborative situation, Referrer N added the transfer policies to its knowledge and, as it started to take advantage of the transfer option, the gap in performance between the two networks increased again.

3.5.2 Testing Phase: Both Networks

In this section we test the performance of the best trained models by the two referrers. Each model is tested 10 times, each time for two years ($\approx$ 18000 hours). To be able to compare the performance of the referrers when different allocation policies are applied, in each run we use a random generated seed to generate arrival rates to the networks. Moreover, none of the random number seeds used to test the referrers' performance was the same as the seeds used to train the models.

Table 3.5 shows the percentage of patients transferred from Network N to Network M in the testing phase.

	WTT = 24	WTT = 168	WTT = 720
Run #	% Transferred	% Transferred	% Transferred
1	2.26	0	0
2	1.83	0	0
3	1.43	0	0
4	1.76	0	0
5	1.68	0	0
6	1.77	0	0
7	1.86	0	0
8	2.26	0	0
9	1.57	0	0
10	1.95	0	0
Avg	1.84	0	0

Table 3.5: Percentage of Patients Transferred to Network M (Testing Phase-Both Networks)

On average, around 2% of patients were transferred when WTT was low (i.e., 24 hours) and no patient was transferred when WTT was high (i.e., 168 and 720 hours).

Similar to the training phase, it can be seen that WTT can greatly impact transfer rates to the adjacent network. Comparing Tables 3.4 and 3.5 shows a huge difference between transfers in training and testing phases. The underlying reason is that in the training phase most transfers occurred for the learning purpose of the model (exploration) because the referrer was analyzing the potential impact of this option. However, in the testing phase the referrer is no longer exploring and therefore it only transfers patients if necessary. This is evident in the testing phase as no patient is transferred when WTT is 168 and 720 hours. Therefore, when WTT is high it is in the interest of Referrer N to focus on optimizing referral rates to specialists in its network rather than transferring them to the other network.

Referrals to Specialists

In this section we focus on the referral rates to specialists made by Referrer N during the testing phase. Figure 3.15 shows the average proportion of patients referred to each specialist in Network N for different values of WTT. Note that for each value of WTT the average is taken over the 10 runs.

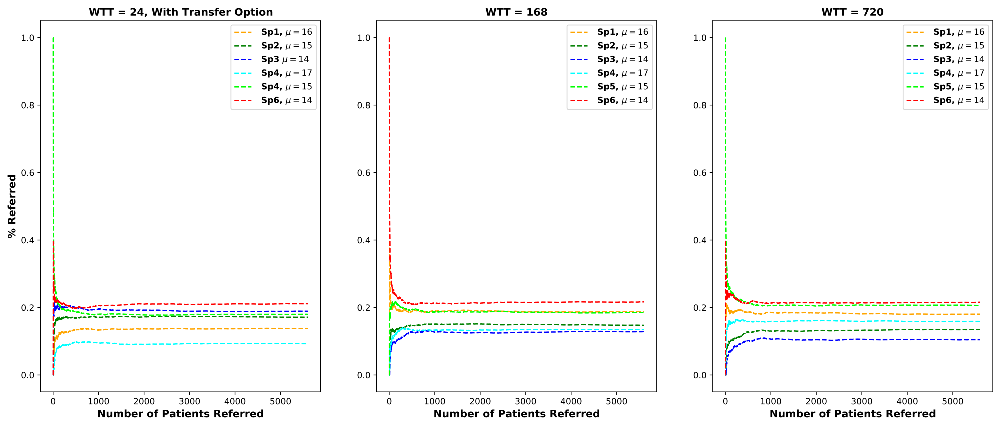


Figure 3.15: Average Proportion of Patients Referred to Each Specialist by Referrer N (Average Performance)

In all three cases, Specialist 6 is the first choice of the referrer, receiving, on average, around 21% of the referrals. Based on values in Table 3.3, Specialists 3 and 6 have the fastest service times. When WTT is 24 hours, on average, around 40% of patients are referred to these two specialists. However, as WTT increases the rate of referrals to Specialist 3 decreases and it even becomes the last choice for the referrer when WTT is 720 hours. On the other hand, referral rates to some specialists, such as Specialist 5, increase as WTT increases. The underlying reason is that as WTT increases, the weight of the delayed reward in the referral decision function reduces and closeness to the selected specialist becomes a more important factor. This is represented in Figure 3.16 which shows the percentage of patients that are referred to the closest specialists under different WTT scenarios. According to the figure, When WTT is low only 13% of patients are referred to specialists close to them. This amount increases to 19% and 22% when WTT is 168 and 720 hours, respectively.

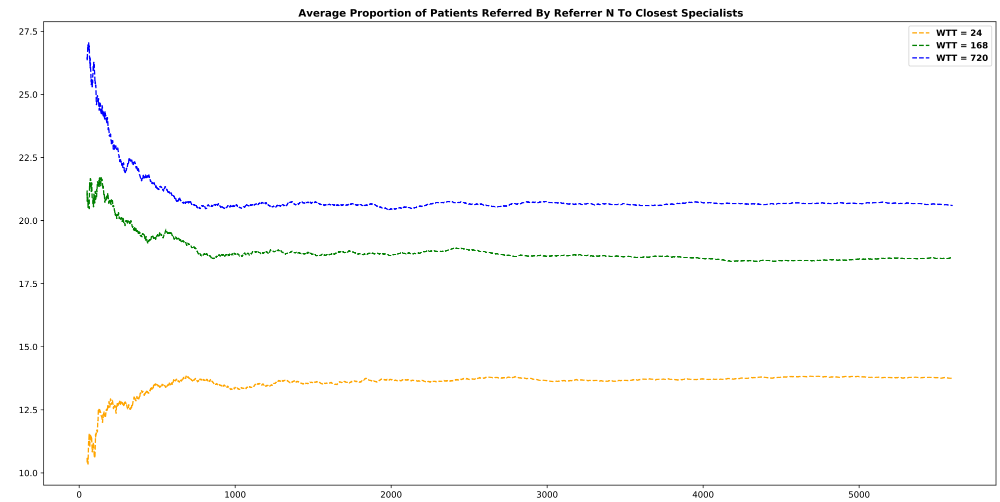


Figure 3.16: Average Proportion of Patients Referred to Closest Specialists by Referrer N

DQN vs Shortest Queue

In this section, we test the performance of the DQN model with collaboration between networks in the case where each network applies a shortest queue policy for its referrals (Figure 3.17).

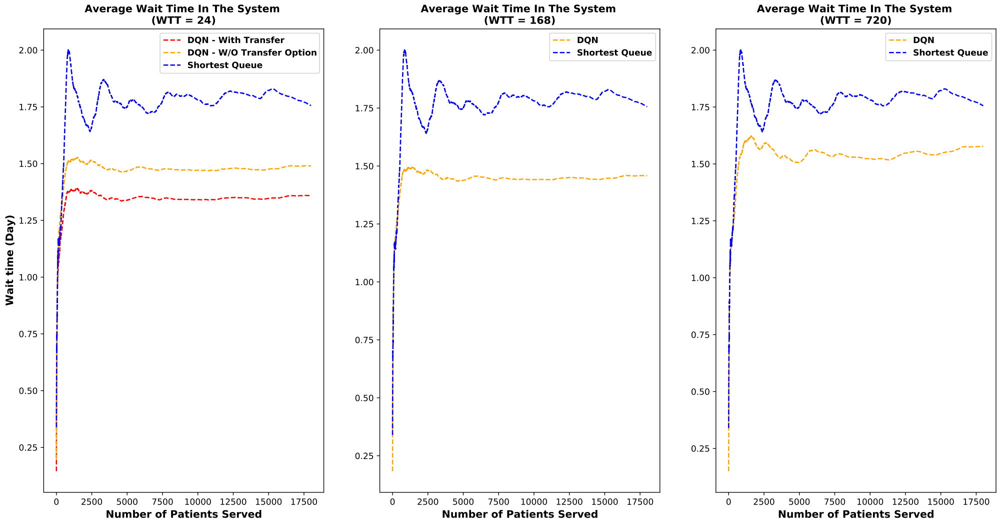


Figure 3.17: Average Wait Time in The Whole System (Best-Trained DQN Models vs Shortest Queue)

In all scenarios DQN models outperform the shortest queue policy and the difference in performance becomes more significant as WTT decreases. When WTT is low, the best trained DQN model with the transfer option is able to reduce average wait time in the system by 13% and 26% in comparison with the DQN model without the transfer option and shortest queue policy, respectively. Note that patients experienced an average wait time of 30 hours under the best trained model which is still higher than their 24-hour wait time threshold. We further analyze this in the next section where we conduct a sensitivity analysis on specialists' service times.

The DQN models are able to reduce the average wait time in the system by 14% and 7% when WTT is 168 and 720 hours, respectively. On average, when the DQN model is used to find the referral and transfer policies, patients waited for around 36 and 39 hours when WTT is 168 and 720 hours, respectively. In these scenarios, no patient was transferred between the networks and the performance of the models with and without the transfer option were barely different from each other. This is already shown in 3.5.

3.6 Sensitivity Analysis

In this section, we first investigate how much changing the initial values of service times presented in Table 3.2 can impact the performance of the DQN model versus the shortest queue policy. Next, to understand the impact of government incentives on motivating collaboration between the networks, we perform a sensitivity analysis on the ptr parameter introduced in Section 3.5.

3.6.1 Sensitivity Analysis on Specialist Service Times

In the initial model presented in Section 3.4.1, we used data from the cataract referral network to estimate the service times of the specialists. However, the variation in service times may impact the performance of the DQN model. Therefore, in this section we perform the following two sets of sensitivity analyses on the service times of specialists in Network M:

1. Homogeneous Services Times: We assume that all specialists in Network M have equal service times.

2. Heterogeneous Service Times: We consider a wide range of service times for the specialists.

Homogeneous Service Times

We set service times of all specialists in Network M equal to 12 hours. Then, for each value of $\gamma = [0.1, 0.3, 0.5, 0.7, 0.9]$ we train the model 10 times, each time for 1,000,000 hours. Overall, 50 models are trained and then the best trained model is selected through the same process explained in Sections 3.4.1 and 3.5.1. The performance of the best trained model is then tested against the shortest queue policy. In the testing phase, for each policy we run the model 10 times (using 10 specific seeds), each time for two years ($\approx$ 18000 hours). In order to make the results comparable we use same seeds in all scenarios to generate the arrival rates to the network.

Figure 3.18 compares the average performance of the DQN model versus shortest queue policy when WTT is 24 and 720 hours.

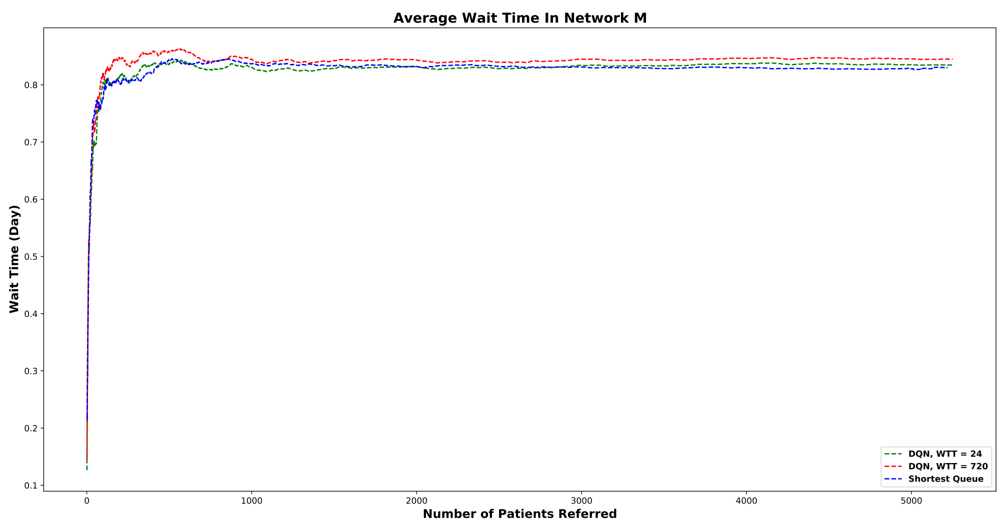


Figure 3.18: Average Wait Time in Network M (Best DQN Models vs Shortest Queue Policy - Average Performance)

From Figure 3.18 we conclude that specialist service times can greatly impact the performance of the DQN models. When specialists are identical in terms of their service times and WTT is low, the performance of the DQN model and the shortest queue policy become very similar. However, as WTT increases the shortest queue policy becomes the better option with respect to the average wait time in the network. This is because when WTT is high the closeness of patients to the selected specialists becomes the dominant force in referral policies for the DQN model. In other words, wait times for specialists become less important for the referrer. As a result, more patients are allocated to the specialists that are close to them, though these specialists may not be the best options in terms of patient wait time.

Heterogeneous Service Times

Using the same training and testing procedure described in Section 3.6.1, we trained our model using service times, presented in Table 3.6, that have higher variability compared to the original model's service times.

Specialist Index	Location	Service Time Mean (Hour)
1	[13,3]	1
2	[14,5]	7
3	[17,7]	10
4	[19,4]	14
5	[18,8]	12
6	[20,3]	4

Table 3.6: Specialist Service Times in Network M

Figure 3.19 shows the average performance, taken over the 10 runs, of the DQN models versus the shortest queue policy with respect to average wait time in Network M when specialist service times are heterogeneous.

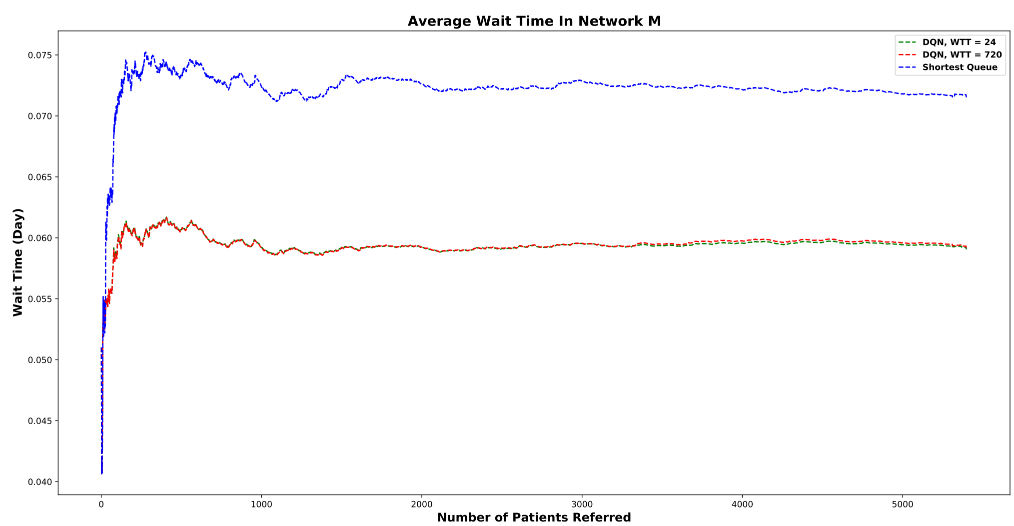


Figure 3.19: Average Wait Time in Network M (Best-Trained DQN Models vs Shortest Queue- Avg Performance)

As the variability in service times increases, the DQN model becomes the better option even at a high level of WTT. Specifically, in comparison to the shortest queue policy, the DQN models have improved wait time in Network M by about 25%. The significant performance improvement is because in the DQN models, the referrer is able to recognize the difference between fast-service specialists (such as Specialists 1 and 6 in Table 3.6) with those who are slower (such as Specialists 4 and 5). This recognition allows the referrer to allocate patients in an efficient way which results in a significantly lower average wait time in the network when compared to the shortest queue policy.

Moving from homogeneous to heterogeneous service times shows us that the DQN models are outperforming the shortest queue policy in environments with higher variability in service times.

3.6.2 Sensitivity Analysis on Payment Per Transfer

Previously we assumed that Referrer M receives four units of payment if it accepts a patient from the other network. Even with this high amount of payment (compared to the two units of payment in the main model), in Section 3.5.2 we show that no patients are transferred between the networks when WTT is high.

In this section, we focus on the situation where WTT is 24 hours and study how changing the value of the p_{tr} can impact collaboration between the networks. This will determine to what degree government incentives can motivate collaboration between networks and what would be the impact on patients' access to specialists. The training and testing phases are similar to the procedures used in 3.5.

Overall, 200 models were trained and then for each value of p_{tr} the best trained model was extracted and tested. Table 3.7 shows the average percentage of patients transferred between the networks for different values of payment per transfer. The average is taken over the 10 runs.

Payment Per Transfer	Average Percentage Transferred
2	1.3
4	1.8
6	2.1
8	2.6

Table 3.7: Percentage of Patients Transferred (Testing Phase)

Table 3.7 shows that increasing the value of payment per transfer results in higher levels of collaboration between the networks and consequently higher transfer rates to Network M. Specifically, increasing the value from 2 to 8 has doubled the transfer rate to Network M (from 1.3 to 2.6).

Figure 3.20 shows the impact of changing payment per transfer on average wait time in the whole system. In all scenarios, collaboration between the networks has resulted in lower wait time in comparison with the isolated scenario where patient transfer is not allowed.

Our sensitivity results show that the impact of transfer payment on system performance is not monotonic. In particular, the average wait time decreases as the transfer payment increases to $p_{tr} = 6$. However, a higher transfer payment ($p_{tr} = 8$) results in higher average wait time compared to payments 4 and 6. The 95% confidence intervals for the average wait time when payment is 6 and 8 are [1.3166, 1.3193] and [1.3921, 1.3951], respectively.

Based on the intervals the difference between the average wait times is significant and this indicates the fact that an uncontrolled amount of payment can result in over-collaboration between the networks which, in turn, negatively impacts patients' access to specialists.

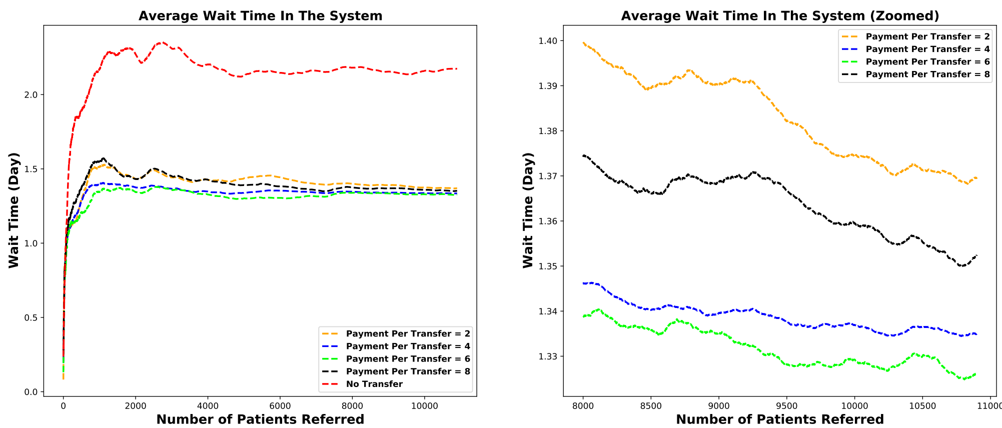


Figure 3.20: Average Wait Time in The Whole System (Average Performance)

3.7 Conclusion

We study the impact of using intelligent referrers in centralized referral networks and collaboration between such networks on improving patients' access to specialists. Our approach allows us to focus on medium-size referral networks where both arrival rates and specialist service times are unknown to the referrers. On-time access to specialists (i.e., wait time for specialists) and convenient access to them (i.e., proximity to specialists) are considered the key factors in making referral decisions. Overall, we trained 750 models, each for 1,000,000 hours, and the performance of the selected models with respect to different metrics were tested enough to assure the robustness of the results.

We found that using an intelligent referrer in a single centralized referral network can greatly reduce average wait time in the network and improve patients' access to specialists.

In our study, the referrer was able to learn a referral policy which outperformed two currently used allocation policies in the real world (random allocation and shortest queue policies). In comparison with the shortest queue policy, our DQN model for Network M was able to reduce average wait time in the network by 17%, 14%, and 10% when WTT was 24, 168, and 720, respectively. The sensitivity analysis results also revealed that the performance of the DQN model increased as the variability in specialists' service times increased. In comparison with the shortest queue policy, our model was able to reduce average wait time by 25% in a centralized system with widely varied service times. In addition, we show that it is always beneficial for a referrer to set short-term vision toward optimizing referral rates to specialists. We show that the intelligent referrer is able to recognize the fastest specialists in a network and this helps the referrer to significantly improve patient wait time in the network, specifically when wait time target (WTT) is low. The referrer is also able to adapt itself to distance-dependent arrival rates and take advantage of this information in its referral decisions when WTT is high.

Further, we find that collaboration between networks has the potential to further improve wait times and thus patients' access to specialists. However, we show that wait time targets (WTT) and government incentives can play a major role in motivating networks to collaborate. Referral networks tend to collaborate more as WTT decreases and government incentives increase. We show that when WTT is low, in comparison with the shortest queue policy, the DQN model with collaboration was able to reduce average wait time in the whole system by 28%. The impact can further increase given that the right incentives are provided by the government. In fact, sensitivity analysis results show that there is a critical threshold for government incentives. While a low value for incentive may not motivate the networks enough to collaborate, a high value for incentive can result in over-collaboration which negatively impacts patients' average wait time in the whole system.

Chapter 4

Analyzing Patient Access to Surgeons in a Cataract Centralized Referral Network in the Waterloo Region

4.1 Introduction

On-time access of patients in need of cataract surgery to surgeons is one of the major concerns of health care systems in different countries including Canada (Rachmiel et al. 2007). It is projected that the number of cataract operations in Ontario face a 128% growth from 2006 to 2036 (Hatch et al. 2012). While an aging population has affected all areas of the healthcare system, ophthalmology is estimated to face the greatest growth in demand for services among surgical specialties (Etzioni et al. 2003, Roos et al. 1998). Considering the impact of the aging population on demand, Taylor (2000) estimated that 50% of people will need to have cataract surgery. In a study of over 4,900 patients, Klein et al. (2002) also found that cataracts are age-related and that about 50% of people between 55 and 64 years old, and 85% of people over 75 years old will develop cataracts.

Patients in need of cataract surgery in the Waterloo region can gain access to surgeons through a centralized referral network named "Ocean". The network connects referrals in three cities, namely Cambridge, Guelph and Kitchener, and referral decisions are made based on a patient-choice process. Upon the arrival of a patient to the network, the patient is first seen by a primary care provider (PCP). In the case where a cataract surgery is needed, the patient will be provided with a referral form with three options to choose from: 1) the closest surgeon, 2) the first available surgeon, or 3) a specific surgeon. Once the referral decision is made by the patient, the referral form will be sent to the centralized intake system by the

PCP and then it is the central referrer who is responsible to allocate the patient to one of the surgeons in the network.

Two different types of wait times, called wait time 1 (WT1) and wait time 2 (WT2), are defined in the system where WT1 represents how long it takes for a patient to get in to see the selected surgeon in the office, and WT2 shows how long it takes after that to get in for surgery. The total wait time is then defined as the sum of WT1 and WT2. When the patient chooses the first available surgeon option, WT1 is the primary decision factor used by the referrer as it is challenging to estimate and use WT2 information due to the following reasons:

1. WT2 not only depends on the scheduling policy of the surgeon but also the availability of operation rooms, the number of beds in the hospitals and even each hospital practice of allocation of its capacities to surgeons.

2. For each patient, WT2 can only be calculated once the patient is served by the selected surgeon. At this point, for the majority of surgeons, WT2 is more than a year. As a result: 1) it is challenging for the central referrer to monitor WT2 for each individual patient and therefore, the central referrer information on WT2 is provided by surgeons, and 2) it is better to use WT1 to determine the first available surgeon as the impact of patient aggregation in the surgeons' queues on wait times can be realized much sooner using WT1 rather than WT2.

Both testimonials from the decision makers in the referral network as well as the data itself support the fact that WT1 is the main factor used to determine the first available surgeon in all three major cities. For instance, while one of the surgeons in one of the major cities had a total wait time of around 300 days, the majority of patients allocated to the first available surgeon in this city were referred to a surgeon with the lowest WT1 but a high WT2, making their total wait time around 800 days.

Long wait times for surgeons and potential inequity in patient access to specialized services are amongst the major concerns of the decision makers in the system. The negative impact of long wait times for surgeons on patient satisfaction has previously been shown by Dunn et al. (1997). Gimbel and Dardzhikova (2011) also found that waiting for more than six months for cataract surgery is associated with negative outcomes such as vision loss, reduced quality of life and even depression. Laidlaw et al. (1998) and Harwood et al. (2005) found that expedited cataract surgery resulted in better outcomes for patients who received it.

We use Simpy, a process-based discrete-event simulation framework based on Python, to simulate the system and study how changing current referral processes and introducing new policies can impact patient access to surgeons. In particular, we introduce two models named the base-model and the multinomial-model. In the base-model, the values of every parameter in the model, including patient preferences, are directly derived from real data from the network. We use the base-model to analyze the system performance in the near future. On the other hand, in the multinomial-model, a multinomial logistic regression is applied to determine patient preferences over time.

To the best of our knowledge our study is the first that studies a patient-choice centralized network and investigates how modifying this process can impact patient access to surgeons over time. The following is a list of key findings:

1. The system currently has a high utilization rate and limiting patient choices does not have any significant impact on patient average wait times to see surgeons (WT1) in the system.

2. If patient options are limited to the first available surgeon, the majority of arrivals to the network would be seen by surgeons within 180 days.

3. Eliminating patients' option to travel to another city for the first available surgeon does not necessarily improve average patient wait time to see surgeons (WT1) in the network.

4. Adding a new surgeon to the network can significantly reduce patient wait time to see surgeons (WT1) in the network. Kitchener and Guelph are also the best options for the new surgeon to be located.

The rest of the study is organized as follows. We first review the related operation management literature. In Section 2 we analyze the data in detail. In Section 3 we introduce key parameters of the network and how they are modeled. In Section 4 we present our results for the base-model and multinomial-model and study how introducing new policies can impact patient access to surgeons. We conclude the paper in Section 5 with a summary of our findings, insights, and limitations of our study.

4.1.1 Literature Review and Positioning

Cataracts are considered the most common eye disorder in most countries including the USA and Canada (Hatch et al. 2012). While on-time access of patients in need of cataract surgery can result in better outcomes such as improved quality of life (Olson et al. 2017), long wait times for surgeons can deteriorate patient conditions (Freeman et al. 2009). In addition, Conner-Spady et al. (2004) found that patient satisfaction decreases as wait time for cataract surgery increases.

Numerous reasons for long wait times for cataract surgery are considered in the literature. Rachmiel et al. (2007) compared cataract surgery referral data from 1992 to 2004 and found a significant negative correlation between the number of surgeons per million people and cataract surgery rates. However, Wormald and Foster (2004) argued that the number of surgeons is not the limiting factor, and cataract surgical rates need to be increased. Hopkins et al. (2008) also estimated that an annual increase of 4% in treatment volume is needed to meet the stated wait time targets for cataract surgery in Ontario. Boisjoly et al. (2010) were able to double the rate of cataract surgery at a hospital in Montreal through the implementation of a cataract efficiency program where surgical technicians were trained and new technologies were used.

In addition to long wait times, equity in access to surgeons is another major concern in the healthcare community of most countries including the USA and Canada (Johnston et al. 2020, Hong et al. 2016, Gauer et al. 1994). While the need for cataract surgery is estimated to increase (Hatch et al. 2012), due to the aging population, the number of surgeons per million people in Ontario decreased by around 14% from 1992 to 2004 (Rachmiel et al. 2007). This raises the question of whether there is equity in access to surgeons in the current referral scheme and to what degree adding more resources to the system can balance access to surgeons.

Simulation has been used to design and analyze patient flows in referral networks and hospitals (Gibbons and Samaddar 2009, Donker et al. 2010). For instance, Yao et al. (2020) used simulation to analyze referral rates between two hospitals in Taiwan. To the best of our knowledge, however, our paper is the first that 1) applies simulation to study a centralized referral network, and 2) uses real data from a centralized referral network and analyzes the impact of a patient-choice referral process on patient access to surgeons. We also study how adding more surgeons to the network can further increase the proportion of patients seen (WT1) within a specific target wait time.

4.2 Data Description

For our study, we gained access to data from the Waterloo Region cataract referral network which is a centralized network with 16 surgeons that covers arrivals from three major cities, namely Kitchener, Guelph and Cambridge, and more than 140 townships. For each individual patient, the data contains information on their time of arrival to the network, preference for surgeon, city, age, gender, selected surgeon and wait time to see the selected surgeon (i.e., WT1). Due to the high values of WT2 for the majority of surgeons, WT2 data for each individual patient was not available at the time of receiving the data. However, surgeon cities and the average value of WT2 for each surgeon are provided on the network website (Waterloo-Eye-Program 2021). Therefore, with respect to the wait time, for each individual patient we have WT1 and the average value of WT2 for the selected surgeon.

Upon the arrival of a patient to the network, the patient is first seen by a PCP and if further treatment is needed the patient is provided with a referral form with three options to choose from: 1) A specific surgeon, 2) the closest surgeon, or 3) the first available surgeon. If a patient selects a specific surgeon, then they are required to select one of the 16 available surgeons in the network.

Any arrival patient is also asked to specify whether they are willing to travel to another city for service. In the rest of the paper, we use term "locally referred" if a patient is referred to a surgeon in their city. Therefore, if a patient selects the first available surgeon and is not willing to travel to another city, they are locally referred to the first available surgeon. The central referrer then receives the completed form and refers the patient to the selected option.

During an eight-month period of time, 1,671 patients in total arrived to the system where the majority of patients (more than 98%) were 50+ years old. Categorizing arrivals by city show that 594 patients came from townships, 517 came from Kitchener, 473 came from Guelph, and 87 came from Cambridge. Figure 1 shows a summary of the total number of arrivals in each month:

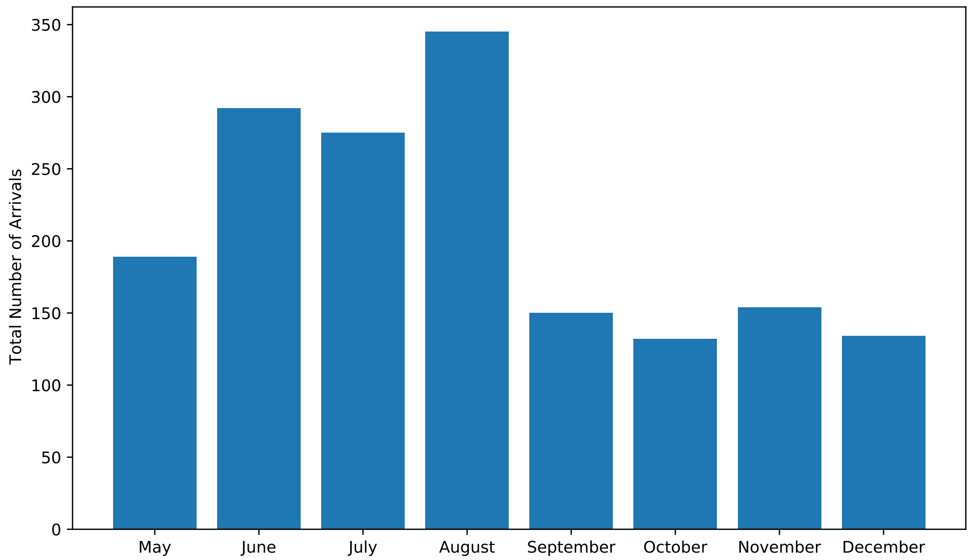


Figure 4.1: Monthly Arrival Rate to the Network

From May to August 2021 (i.e., prior to the discovery of the Omicron variant of Covid-19), the average monthly arrival rate to the network was 275.25 patients and reduced to 142.5 from September to December.

Analyzing patient preferences revealed that around 33% of patients chose the first available surgeon, 19% chose the closest surgeon, and 48% chose a specific surgeon. Figure 2 shows how patient preferences change over time. While the number of patients who chose the closest surgeon did not change significantly over time, changes in the other two options is noticeable and both options follow a similar trend from May to September.

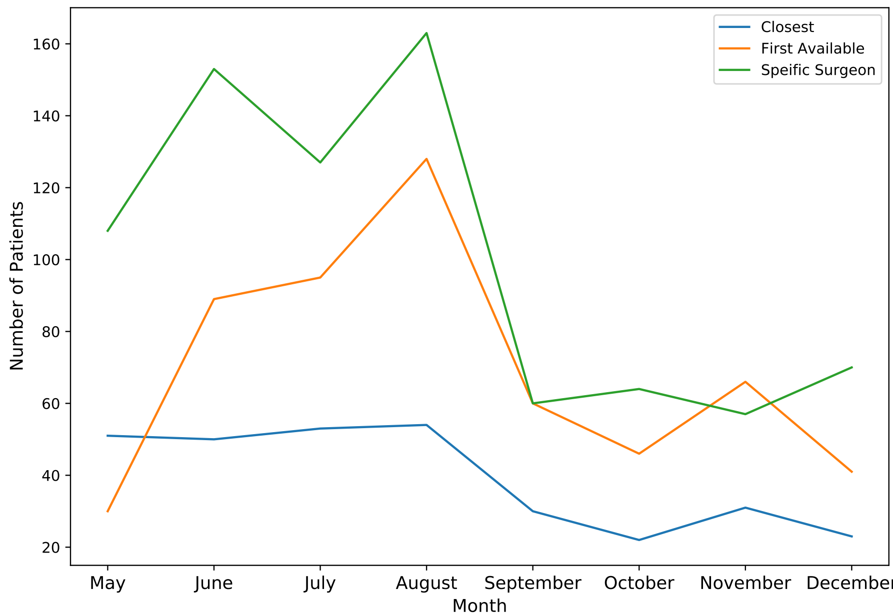


Figure 4.2: Patient Preferences Over Time

Table 4.1 shows how many patients of each gender selected each one of the options and Figure 4.3 shows the proportion of each gender by their choices.

Gender	Preferences		
	Closest	First Available	Specific Surgeon
Female	183	298	480
Male	131	257	322

Table 4.1: Patient Preferences by Gender

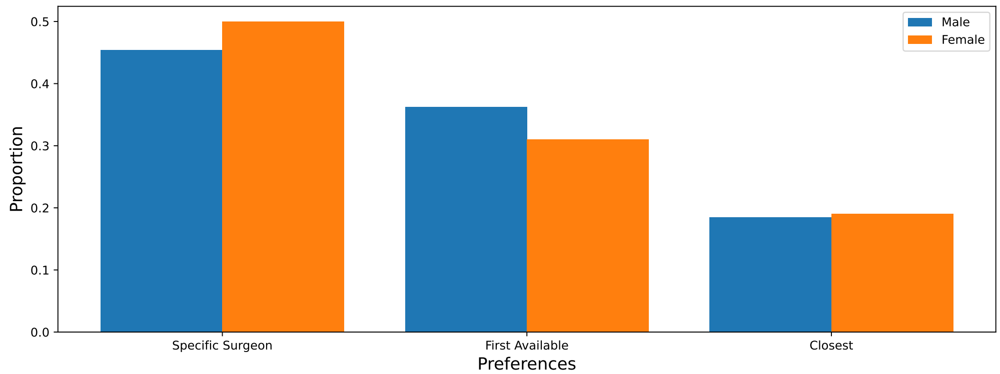


Figure 4.3: Proportion of Patient Preferences by Gender

Figure 4.3 shows that the majority of patients selected a specific surgeon option followed by the first available surgeon. In both genders, the closest surgeon is selected by roughly 18% of patients and men choose the first available surgeon more than women.

Figure 4.4 shows the proportion of patients who choose each option in each city.

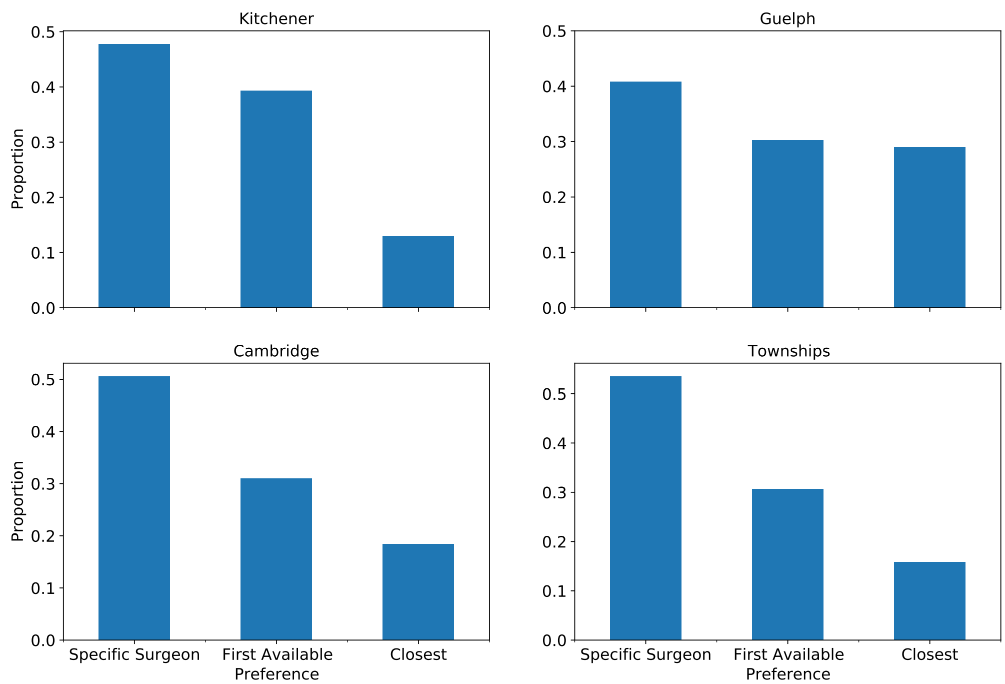


Figure 4.4: Patient Preferences by City

In all cities the majority of patients have selected the specific surgeon option. In addition, the closest surgeon option is the least selected option in all cities except Guelph where around 30% of patients selected this option.

The following table shows the percentage of patients in each city referred to surgeons in the same city:

Patient City	Surgeon City	Percentage
Kitchener	Kitchener	93%
Guelph	Guelph	89%
Cambridge	Cambridge	70%

Table 4.2: Percentage of Arrivals Allocated to Surgeons in the Same City

Most arrivals from these three major cities are locally referred. Further analysis shows that 53% of those patients who are not locally referred chose the first available surgeon and 45% had a preferred surgeon. Finally, 53% and 45% of arrivals from townships were referred to surgeons in Guelph and Kitchener, respectively.

To analyze the relationship between patient age and preferences we defined two age groups where patients in the first and second groups are under and over 70 years old, respectively. The first group contains 637 patients and the second group contains 1,034 patients. The following figure shows patient preferences by their age group.

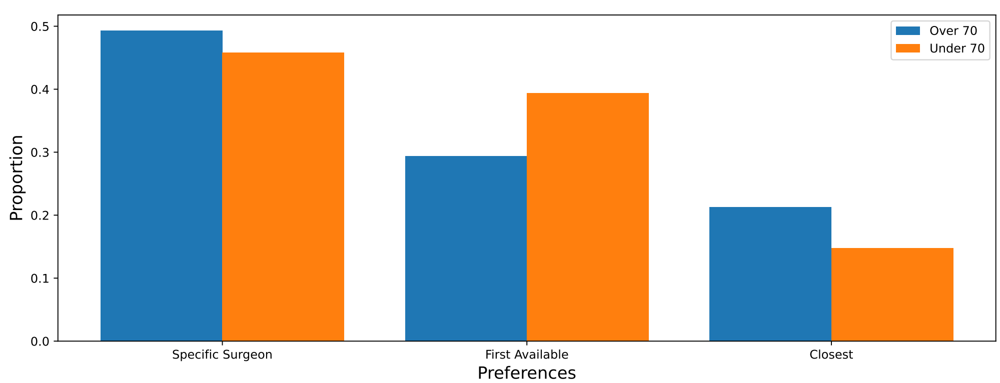
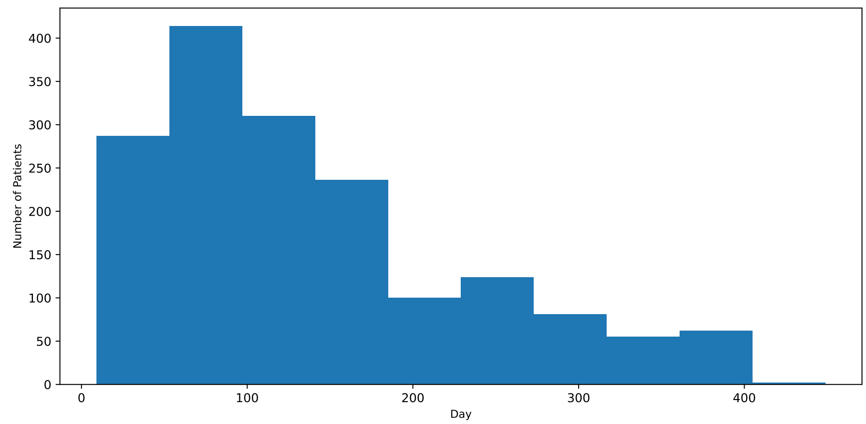


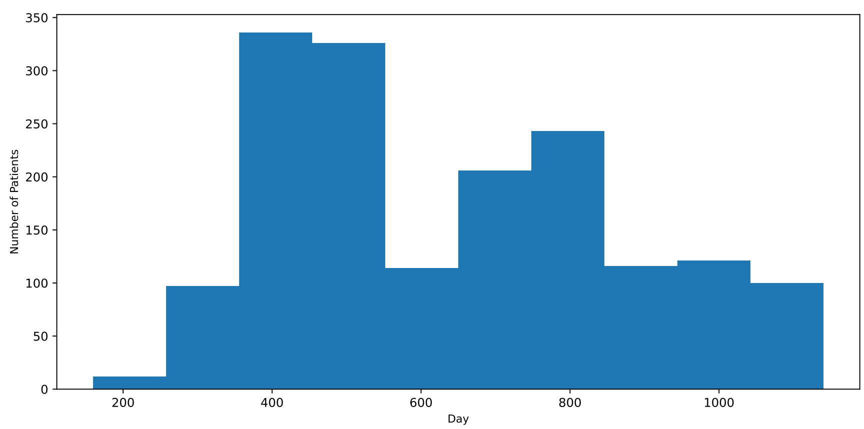
Figure 4.5: Average Wait Time by City and Patient Preferences

In both age groups, around 50% of patients selected the specific surgeon option. However, the difference in the other two options is noticeable. While more than 20% of patients over 70 years old selected the closest surgeon option, only around 10% of patients in the other group selected this option.

We also analyzed patient average WT1 for surgeons in the network. Figure 4.6a shows a histogram of patients' WT1 to see surgeons and Figure 4.6b shows patients' approximated total wait time to receive the surgery. Define $WT1_{ij}$ and $W\tilde{T}2_j$ as the real wait time of patient i to see surgeon j and the average $WT2$ for the surgeon. The approximation of the total wait time for patient i is estimated by adding the real value of WT1 to the $W\tilde{T}2_j$ (i.e., $WT1_{ij} + W\tilde{T}2_j$). For instance, if patient i waited 100 days to see surgeon j (i.e., $WT1_{ij} = 100$) and the average WT2 for the surgeon was 150 days (i.e., $W\tilde{T}2_j = 150$), then the total wait is approximated to be 250 days. We used the term "total wait time" instead of "estimated total wait time" in the rest of the Figures.



(a) WT1



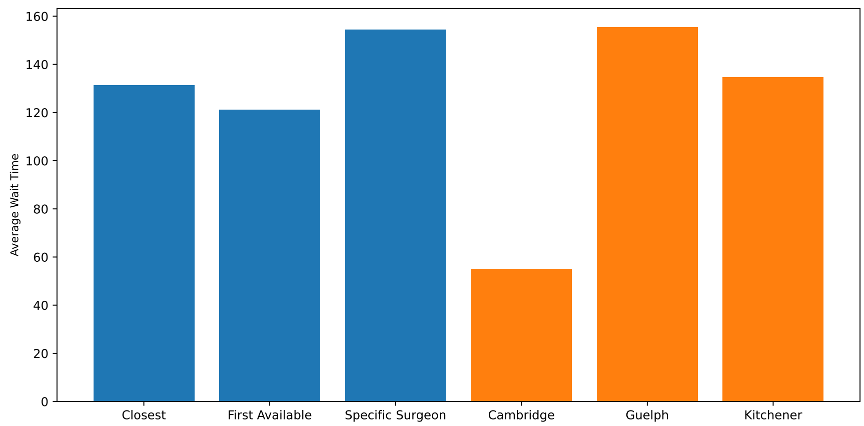
(b) Total Wait Time

Figure 4.6: Patient Average Wait Time

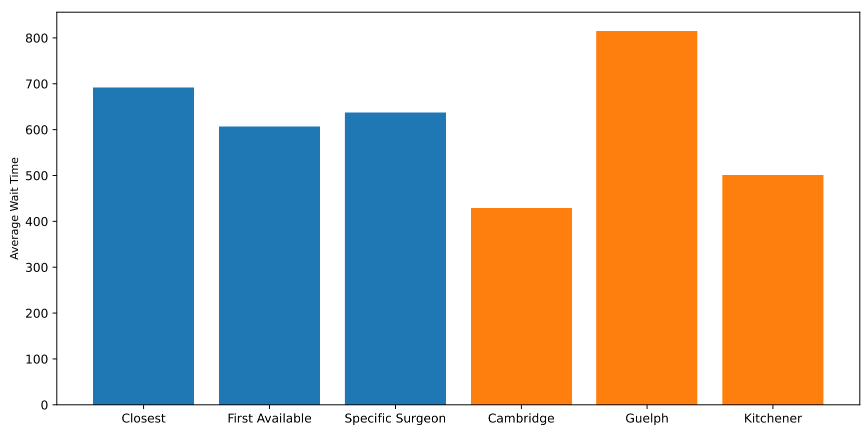
On average, patients waited more than 130 days to see surgeons in the network. Average WT1 for surgeons in Kitchener, Guelph, and Cambridge were 145, 169, and 56 days, respectively. The majority of patients ($\approx 73\%$) waited less than 180 days to see surgeons in the network and around 8% waited more than 300 days.

According to our estimation, only 7% of patients will receive cataract surgery within one year and more than 50% of patients will have to wait more than 600 days to receive the surgery.

We also further studied wait times for different types of patients in each city. Figure 4.7a shows how long each type of patient from each city waited, on average, to see surgeons (WT1) and Figure 4.7b shows average total wait time for different types of patients to receive service from surgeons in the network.



(a) Average WT1



(b) Average Total Wait Time

Figure 4.7: Average WT1 and Total Wait Time by City and Patient Preferences

Overall, patients who chose the first available surgeon experienced lower average WT1 and total wait time than the other two types of patients. With respect to the patient city, patients from Guelph experienced higher wait times than patients from the other two cities.

Due to the unavailability of WT2 for individual patients as well as the reasons mentioned in Section 4.1 in our simulation study, we only focused on patient access to surgeons in the network and in particular on WT1 rather than WT2. Therefore, in the rest of the paper all wait times and analyses done on the wait times are only about WT1 and not WT2.

4.3 Analysis of System Design Decisions

In this section we introduce different elements of the cataract centralized referral network in Waterloo and how they are incorporated in the simulation process.

According to the data on patient cities and townships we considered four major cities in our simulation. Each city is considered as a circle in a coordinate plane defined by the following equations:

$$Kitchener: (x-8)^2 + (y-15)^2 = 25 \tag{4.1}$$

$$Guelph: (x-30)^2 + (y-25)^2 = 9 \tag{4.2}$$

$$Cambridge: (x-22)^2 + (y-4)^2 = 4 \tag{4.3}$$

$$Townships: (x-21)^2 + (y-21)^2 = 36 \tag{4.4}$$

The following figure shows the cities on a coordinate plane:

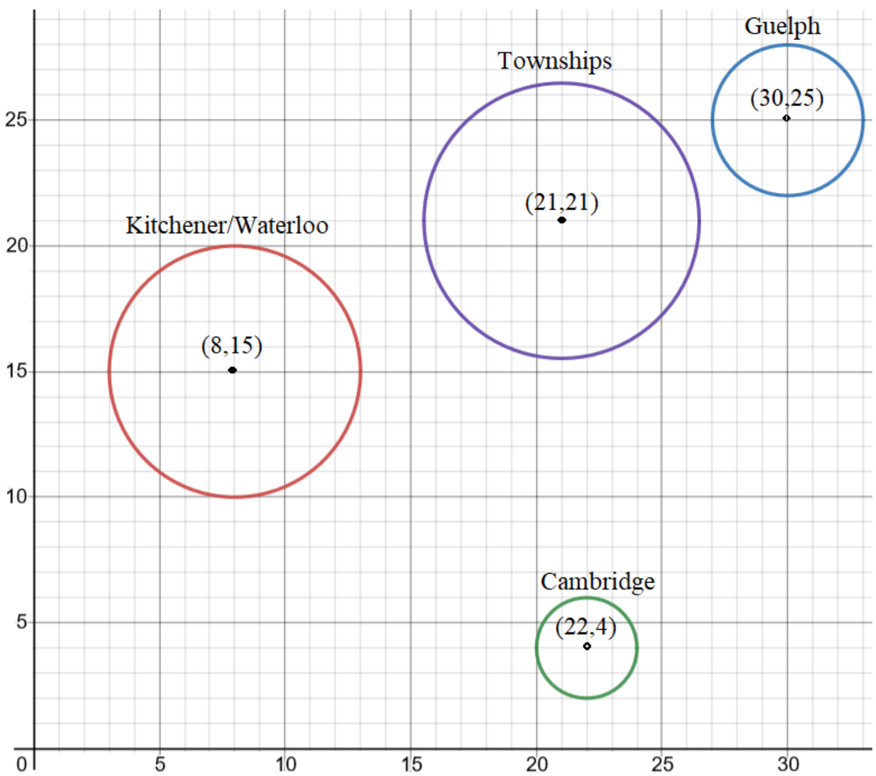


Figure 4.8: Cities Covered by The Centralized Cataract Network

Note that distances between the centers of the circles represent an average distance between the cities in the real world. In addition, the radius of each circle is a representation of the arrival rate from the city that the circle symbolizes. Finally, we assume that townships are between Kitchener and Guelph and closer to Guelph as, according to the data, 70% and 29% of patients in the townships who chose the closest surgeon were referred to surgeons in Guelph and Kitchener. Four main elements of the referral network and their characteristics are listed below:

Patient A patient can come to the network from any point inside a circle. Patient characteristics, including patient gender and age, are generated based on the information derived from the data.

PCP As we only focus on those patients who are in need of cataract surgery and to maintain our focus on the impact of the patient-choice referral process on the performance metrics of the system we represent the population of PCPs with three main ones in each city (in total, nine PCPs) except the townships. According to the data, no surgeon in the townships is connected to the centralized referral network. In each city, the first PCP receives the stream of patients who have a preferred surgeon. The second PCP receives the stream of patients who choose to go to the closest surgeon to their home. Finally, the third PCP receives the stream of patients who choose to go the first available surgeon.

Central Referrer The central referrer is responsible for allocating patients to surgeons based on their choice and has information about wait times and locations of surgeons.

Surgeon Surgeons are located in different places in the network and they each receive a stream of patients who are in need of cataract surgery. Once a patient receives the required service from a surgeon, they leave the system immediately. Note that surgeons in each city are placed on the city circle's perimeter.

Figure 4.9 shows the histogram plot of daily interarrival times in an hour.

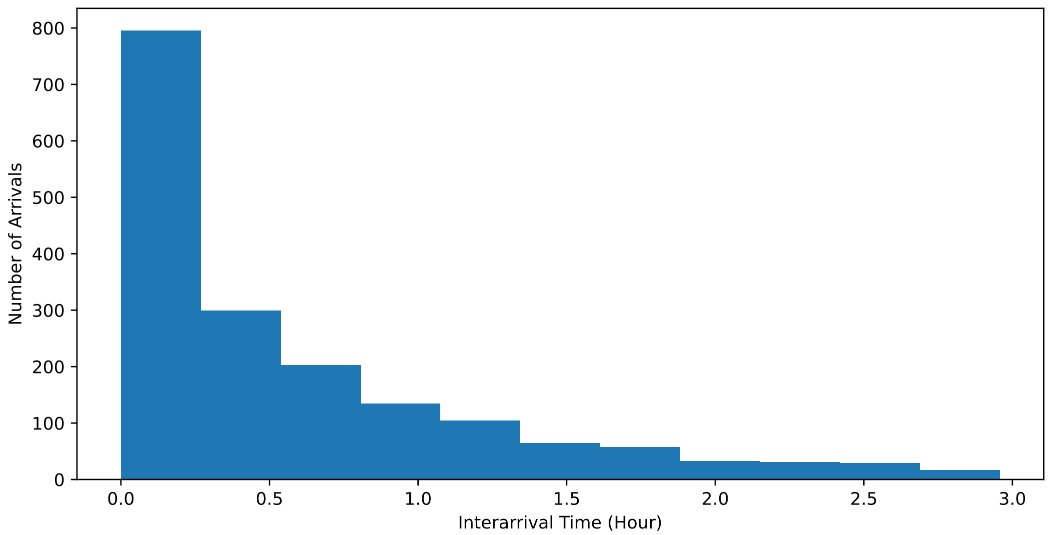


Figure 4.9: Patient Interarrival Times (Hour)

The daily interarrival for most of the arrivals to the network ($\approx 95\%$) are less than three hours. In addition, approximately 57% of daily interarrivals are lower than 30 minutes. Using statistical analysis, in Figure 4.10 we show that exponential distribution is a good fit for the interarrival times.

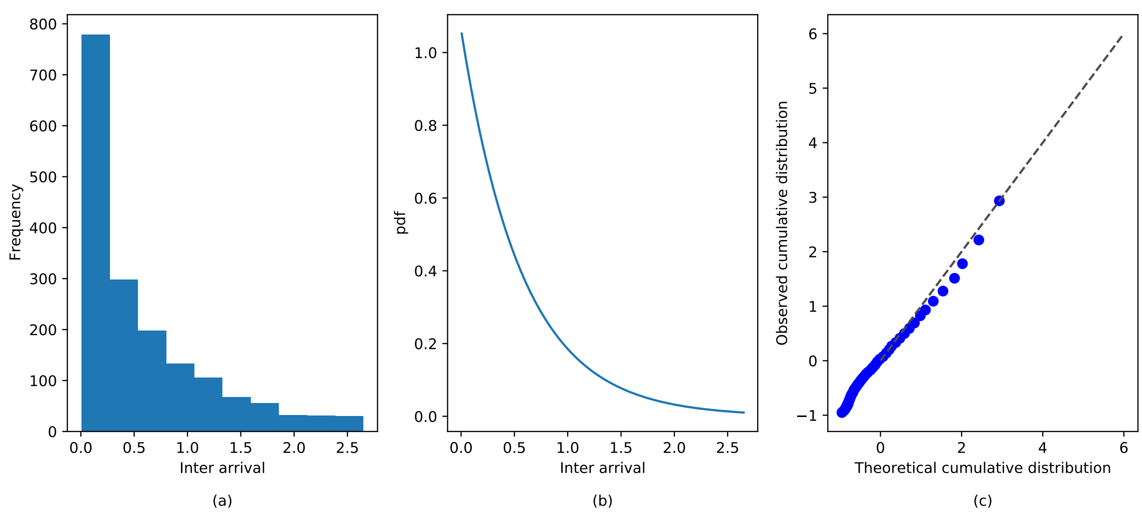


Figure 4.10: Interarrival Distribution

Figure 4.10(b) shows that exponential distribution performs fairly well in representing interarrival times shown in Figure 4.10(a). Chi-square statistics and the QQ plot (shown in Figure 4.10(c)) also suggest that exponential distribution is a good fit in approximating "interarrival" data. Therefore, we assume that arrivals to the network have a Poisson distribution with the means extracted based on the real data from the network.

Finally, in order to extract surgeon service rates, we modeled each surgeon as an M/M/1 queuing system. Using the number of referrals to the surgeon and their associated waiting time from the data, we estimated the service rate of each surgeon in the system. As discussed earlier, based on testimonial data from decision makers in the system there have been times when surgeons have rejected referrals to keep their waiting list short. We incorporated this feature into our model by considering specific capacities for surgeons. The service rates and capacities were then further adjusted in the simulation model to make sure that the results of the model for each individual surgeon was close to what we observed in practice. Table 4.3 compares the results of the base-model that we ran for eight months with the results from the available data.

	# of Referrals to Surgeons		Avg Wait Time		Wait Time 75%	
Data/Simulation	D	S	D	S	D	S
All Network	1671	1677	132.5	132	97	80
Kitchener	819	800	145	142	73	67
Guelph	749	758	167	169	110	97
Cambridge	103	119	56	59	25	35

Table 4.3: Base-Model Comparison with Data (Where D Stands for Data and S Stands for Simulation)

We further assured that the results of the simulation model with respect to the number of referrals and resulted average wait time for each individual surgeon in each city were close to what we observed in practice.

In total, there are 16 surgeons in the system: three surgeons in Cambridge, four in Guelph, and nine in Kitchener. However, due to the low referral rates to three specific surgeons in the Kitchener area, we combined the data on these surgeons and used one surgeon in our simulation model instead of these three surgeons. Table 4.4 shows surgeon locations in the network:

City	Surgeon	Location (x, y)
Kitchener	1	(8,20)
	2	(13,15)
	3	(8,10)
	4	(12,12)
	5	(4,12)
	6	(4,18)
	7	(3,15)
Guelph	8	(27,25)
	9	(33,25)
	10	(30,28)
	11	(30,22)
Cambridge	12	(22,2)
	13	(22,6)
	14	(24,4)

Table 4.4: Surgeon Locations

Note that surgeon locations are determined based on the available data on the referral rates between the four cities.

4.4 Results

In this section we analyze the behaviour of the current system and estimate how different policies impact patient access to surgeons. We considered two scenarios. In the first scenario we used a model, called a base- model, where model parameters including arrival rates of each type of patient from each city, referral rates to surgeons, and patient preferences are derived directly from the available data. The base-model mimics the behaviour seen over eight months in practice very closely. This model is then used to analyze the future of the current system over the next two years and examine how policies such as adding more capacities to the system can improve patient access to surgeons.

To better understand how the patient choice referral system can impact important performance metrics such as average wait time in the system, in the second scenario we used a simulation model, called the multinomial-model, where we applied a multinomial logistic regression in a simulation model to determine patient preferences. This makes referral rates to surgeons dependable on different characteristics of both surgeons and patients and allows us to analyze the behaviour of the system for a more distant future.

In the following sections we first present our results for the base-model and then focus on the multinomial-model.

4.4.1 Base-Model Results

In this section we study the future of the cataract referral system using the base-model. The model was run 100 times and values shown in the rest of the papers are the average values taken over these 100 runs. Note that the base-model is simulated for only two years as one of the main assumptions in this model is that system characteristics, including arrival rates to the system, surgeon service rates, and patient preferences, remain unchanged over time. The following figure shows average wait time for surgeons in the network over the next two years:

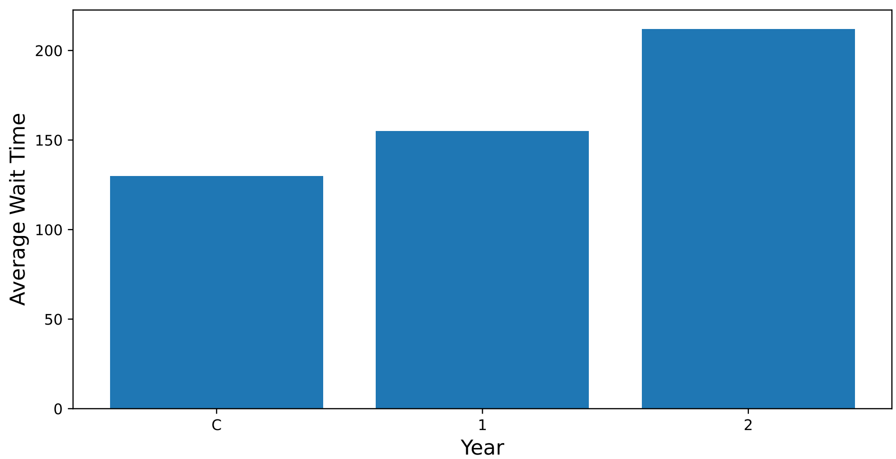


Figure 4.11: Expected Average Wait Time

where "C" is the indicator of the current year. If system characteristics, including arrival and service rates, remain unchanged, the average wait time in the network is expected to increase from 130 days to around 200 days over the next two years. Figure 4.12 gives a better understanding of how average wait times for surgeons will be impacted in each city:

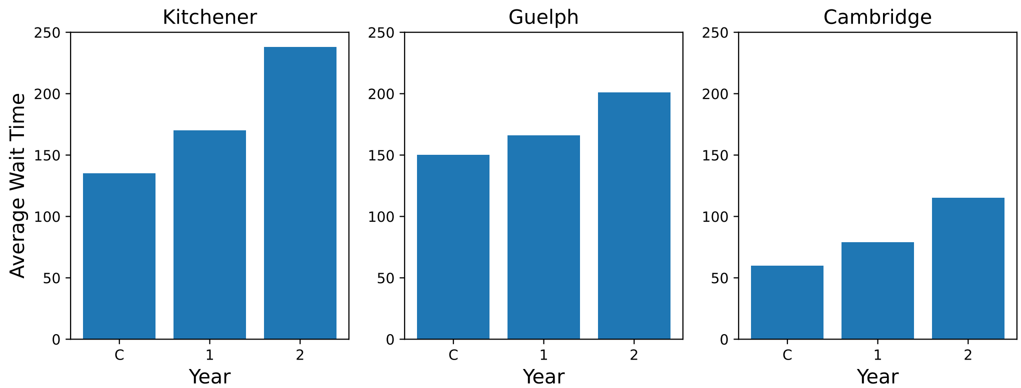


Figure 4.12: Expected Average Wait Time by City

Wait times for surgeons are expected to increase in all cities over the next two years if no new policy or capacity is added to the network. In comparison with other cities, surgeons in Kitchener are expected to experience higher wait times at the end of the second year.

Finally, Figure 4.13 shows average wait times for different types of patients.

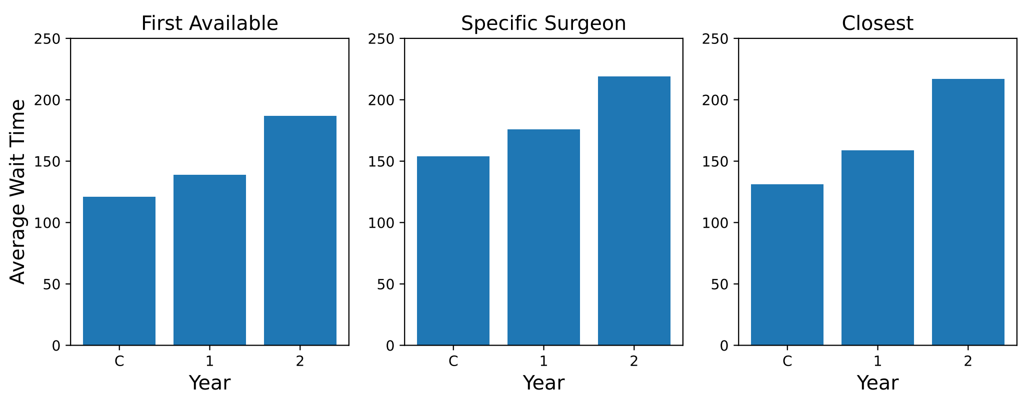


Figure 4.13: Expected Average Wait Time by Type

Figure 4.13 shows that over the next two years, patients who choose the first available surgeon are expected to experience lower average wait times for surgeons in the network. On the contrary, patients who choose a specific surgeon will experience higher wait times than the other types. This result is comparable with what is observed in practice where patients who chose a specific surgeon waited, on average, 154 days to see surgeons while others who chose the first available surgeon and closest surgeon waited for 121 and 131 days, respectively.

Base-Model: Policy Analysis

The base-model application is limited. One limitation is that there is no mechanism in this model that allows changes in referral rates to surgeons if system characteristics change over time. Another limitation is the fact that patient preferences are directly derived from the data and therefore they are fixed. As a result, we can only study how patient access to surgeons will be affected over the next two years if patient options are reduced to one option. For instance, if we only eliminate the first available surgeon option, then there is no mechanism in this model that allows us to understand what proportion of patients who had chosen this option would now prefer each one of the other two options. Since wait times are a main concern in the network, in our study we analyze what would happen to patient wait times if the only option is the first available surgeon. We considered two scenarios. In the first scenario we incorporated the percentage of patients from each city who were locally referred. This information was derived directly from the data. In the second scenario, which we used as a benchmark, patients were allocated to the first available surgeon in the whole network no matter where the locations of the patient and selected surgeon are. Figure 4.14 shows the impact of this policy for each scenario on patient average wait time in the network and across each city.

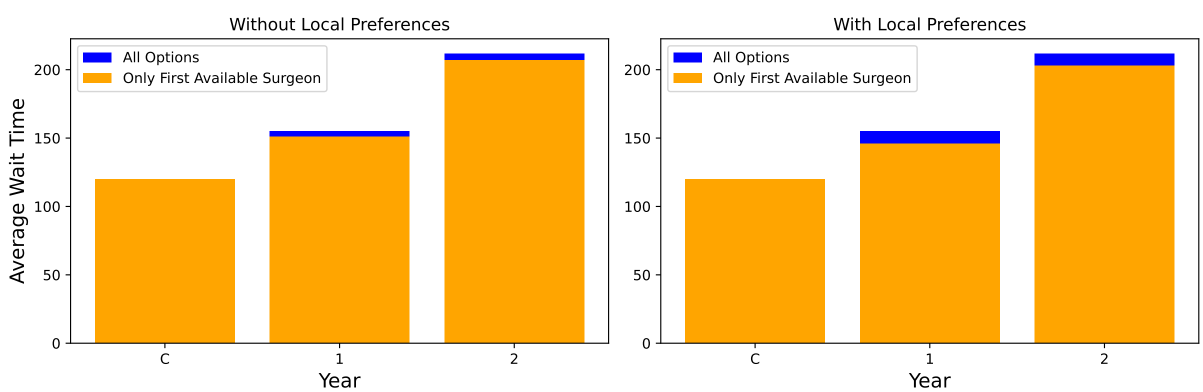


Figure 4.14: Policy Comparison: Expected Average Wait Time in the Network

Figure 4.14 shows that limiting patient options to only first available surgeon did not reduce patient average wait time significantly. More interestingly, the first scenario, in which we incorporated the percentage of patients from each city who were locally referred, resulted in lower average wait times than the second scenario. The main underlying reason for this is that the first available surgeon is not necessarily the fastest surgeon in the network . This is also the reason why patients who chose the first available surgeon in Kitchener and Cambridge experienced longer wait times, on average, than the other two types of patients in practice.

We now examine how much adding a new surgeon to the system improves patient access to surgeons. Note that the surgeon is added at the beginning of year 1 in the simulation. For the location of the added surgeon, we consider two scenarios, named L1 and L2, for each city. In the L1 scenario the new surgeon is located in the center of each circle, shown in Figure 4.8, and in the L2 scenario the surgeon is located close to the other cities to increase the chance of receiving patients from outside the city. Finally, for the service rate of the new surgeon we consider three scenarios, named S1, S2, and S3. In S1, the service rate of the new surgeon is the same as the fastest surgeon in the network. In S2, the service rate of the added surgeon is the average of all the surgeons in the network. In S3, the service rate is the average of all the surgeons in the city in which the surgeon is added. Since the surgeon is newly added to the network, we assume that 1) the surgeon capacity is not limited, and 2) none of the patients who have a preferred surgeon select the new surgeon. For scenario L2, the surgeon location in each city is provided in the following table:

City	New Surgeon Location
Kitchener	(12,18)
Guelph	(27.6,23.2)
Cambridge	(20.8,5.6)

Table 4.5: New Surgeon Location in Scenario L2

Figure 4.15 shows the impact of adding a new surgeon to Kitchener on patient average wait time in the next year.

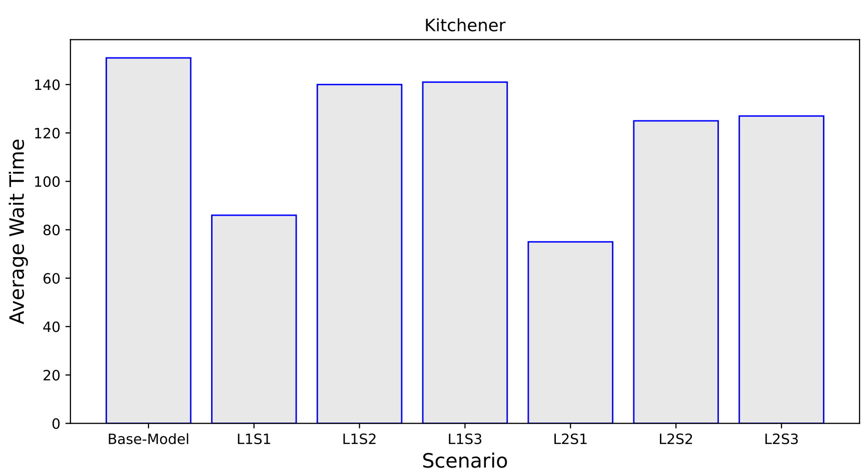


Figure 4.15: Impact of Adding a Surgeon to Kitchener

The average wait time in the system will significantly reduce if the added surgeon has a fast service rate. In addition, adding the surgeon to the border of the city has resulted in lower wait times than the L1 scenario. The underlying reason is that in Scenario L1 the surgeon is located in the middle of the city and therefore the majority of patients who choose the closest surgeon option are being referred to the new surgeon. This intuitively makes referral rates imbalanced and as a result average wait time in the network is increased. The same trend can be seen in Figure 4.16 which shows the results for Guelph and Cambridge.

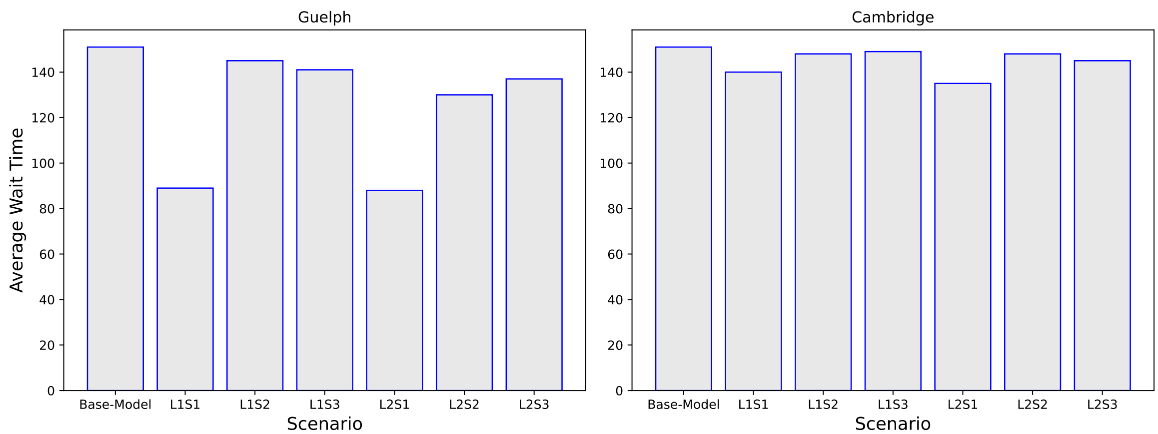


Figure 4.16: Impact of Adding a Surgeon to Guelph and Cambridge

Comparing Figures 4.15 and 4.16 shows that in all scenarios adding a new surgeon to the system can reduce patient average wait time over the next year. However, the impact depends on the surgeon location and service rates. Since the majority of patients prefer a local surgeon, adding the new surgeon to Cambridge has not resulted in any noticeable changes in average wait times. Finally, adding the surgeon to Kitchener has resulted in slightly better outcomes in comparison with Guelph.

4.4.2 Multinomial-Model Results

In this section we introduce the multinomial-model used to analyze how patient preferences impact their access to surgeons over the next four years. In contrast to the base-model where we derived patient choices directly from the available data, in the multinomial-model we applied a multinomial regression model in the simulation process to determine patient preferences. The details of the regression model can be found in the Appendix.

We used patient age, gender, city and three variables associated with wait time, namely average wait time, fifth percentile, and 95th percentile, as the independent variables and patient preference as the dependent variable in the model. In particular, upon the arrival of a patient from a city to the network the model calculates the average wait time, the 5th percentile, and 95th percentile for the past ten arrivals from that city. Considering the patient age and gender, the model then estimates three probabilities, each associated with one of the patient preferences. In order to improve the performance of the model, patient choices are derived from the probability distribution of the regression model. Define p_F as the estimated probability for the first available surgeon option. Similarly, define p_C and p_S for the closest and specific surgeon options and assume that $p_F > p_C > p_S$. In order to determine the patient preference, we generate a random floating point number, called rv, in the range $[0, 1]$. The following shows how the model determines a patient preference.

Random Variable Value	Patient Choice
$rv <= p_S$	Specific Surgeon
$p_S < rv <= p_S + p_C$	Closest Surgeon
$rv > p_S + p_C$	First Available Surgeon

Table 4.6: Patient Choice Using Regression Model Probability Estimates

It has been shown that previous experience of a PCP with a surgeon can impact the choice of surgeon by the PCP (Barnett et al. 2012a, Kinchen et al. 2004b). Since our data lacks information about patient history and PCPs, it is difficult to understand why a patient might select a specific surgeon. Therefore, in our model if a patient choice is a specific surgeon, the surgeon is selected based on the probabilities derived from the data.

Using the multinomial regression process to determine patient choices has the following advantages over the base-model: 1) we can analyze the performance of the system for a more distant future, 2) to study the impact of patient preferences, we are not obligated to reduce patient options to only one; in fact, we can estimate what the impact would be of removing one of the options on patient access to surgeons, and 3) we can study how surgeon characteristics impact patient preferences. For instance, if average wait time for a surgeon reduces, it might impact the referral rate to the surgeon as more patients who choose the first available surgeon will be directed to them. However, as discussed earlier, the base-model is better when the main concern is how changing different parameters of the system impacts patient access to surgeons in the near future.

We considered two scenarios for the arrival rates to the network. In the first scenario we assumed that arrival rates remain unchanged over time. Using real data from the cataract referral network in Ontario, Hatch et al. (2012) predicted that demand for cataract surgery would see a 128% growth from 2006 to 2036. Using this information, in the second scenario we considered a 2.79% annual increase in arrival rates to the network. Each scenario was run 100 times and the same seeds (each run with a different seed) were used in both scenarios to make the results comparable. For each scenario, all the results presented in the rest of the paper are the average values taken over the 100 runs.

Depending on patient condition, various target wait times, ranging from seven days to six months, are mentioned in the literature (Hodge et al. 2007, Government of Ontario 2022) for cataract surgery. For the multinomial-model we considered two performance metrics, namely average wait times for surgeons and number of patients seen within 180 days. This gives us a better understanding of how different policies impact patient access to surgeons and their satisfaction over time.

The following figure shows patient average wait time over the next four years and the percentage of patients served within 180 days for each scenario.

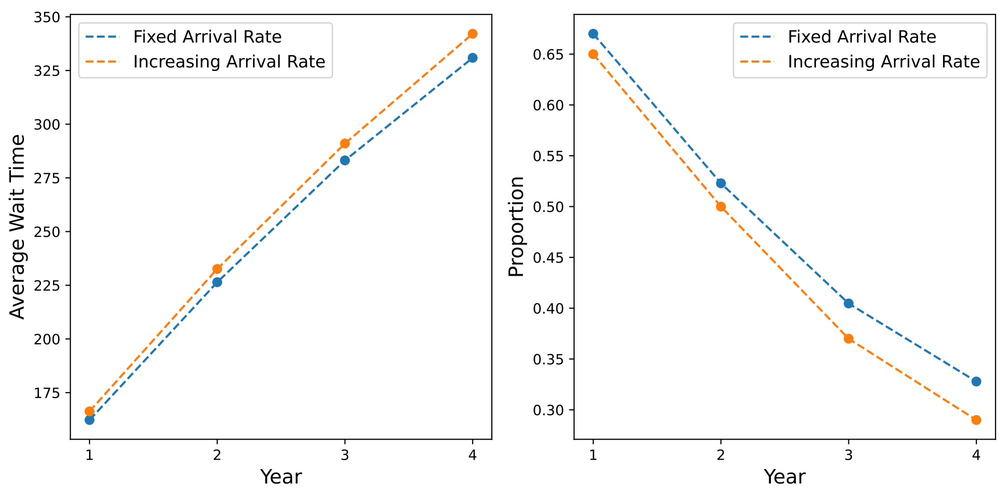


Figure 4.17: The Expected Performance of the Network Over the Next Four Years (Multinomial-Model)

The average wait time in the network is expected to increase by around 13% yearly and reaches over 320 days four years from now. Since surgeon service rates are not changing and no new capacity is added to the network, the gap between the average wait times in the first and second scenarios increases over time and reaches 14 days in the fourth year. Under the fixed arrival rate scenario and over the four years of simulation, patients on average waited around 250 days to get served by surgeons and 75% of patients received service within 326 days. Finally, in both scenarios, the proportion of patients who receive service within 180 days is expected to halve and reaches around 30% at the end of the fourth year.

Figure 4.18 shows average wait time for surgeons in each city.

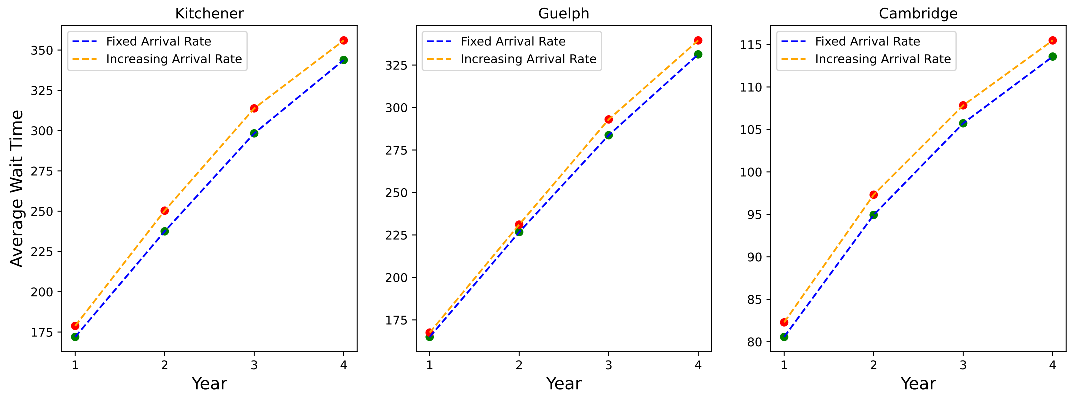


Figure 4.18: Average Wait Time for Surgeons in Each City (Multinomial-Model)

In both scenarios, the average wait time for surgeons in Kitchener is higher than the other two cities and will double after four years if the annual increase in arrival rate is considered. Higher arrival rates to this city and the fact that the majority of patients prefer a local surgeon are the main reasons why wait times in Kitchener are higher than the other two cities.

Multinomial-Model: Policy Analysis

In this section we present our policy analysis results for the multinomial-model for the second scenario where arrival rates to the network increase annually. In addition to the policies introduced in Section 4.4.1 we study how eliminating either of the "closest surgeon" or "specific surgeon" options can impact patient access to surgeons. Figure 4.19 compares patient average wait time in the network when different policies for patient options are implemented.

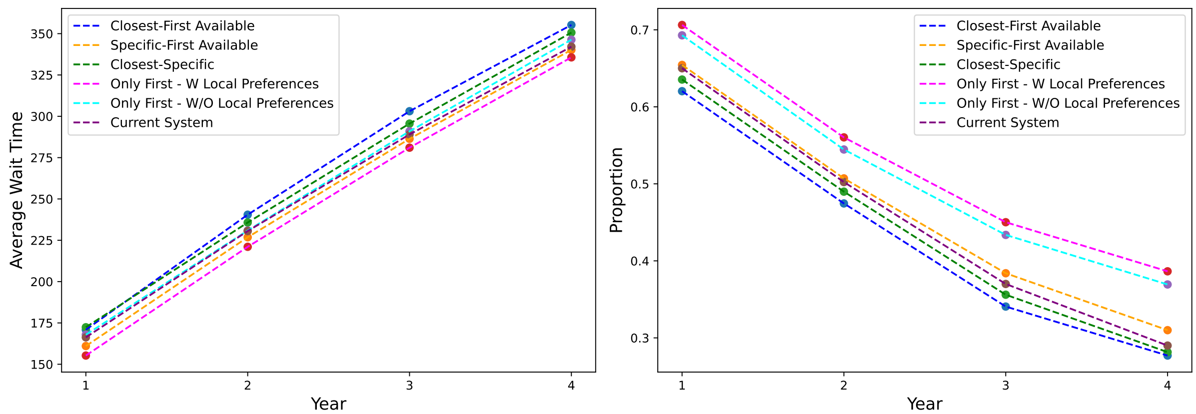


Figure 4.19: Impact of Different Patient Choice Policies on Patient Average Wait Time (Multinomial-Model)

Limiting patient options to only the first available surgeon option and incorporating patient local preferences has resulted in the best outcomes with respect to the patient average wait time. An additional noticeable advantage of reducing patient options to only first available surgeon is the higher proportion of patients seen by surgeons within 180 days in the end of the fourth year (40% compared with ≈ 30% under other policies). However, from the figure it can be seen that all policies have resulted in very close average wait times in the network. This indicates that the utilization rate of the current system is very high and therefore modifying current policies has limited impact on improving patient access to surgeons over the next four years. This leads us to the next policy where we study how adding a new surgeon to the network can reduce patient wait time in the system. The same scenarios explained in Section 4.4.1 are considered here for the location and service rate of the new surgeon. For each city we then calculate the average wait time over all six defined scenarios.

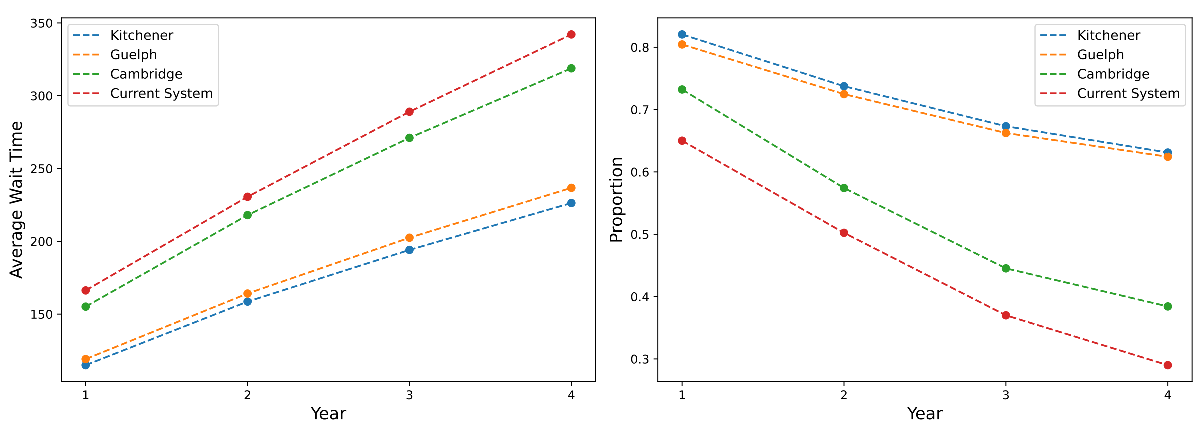


Figure 4.20: Impact of Adding a New Surgeon to the Network (Multinomial-Model)

Figure 4.20 compares the impact of adding a new surgeon to the network if surgeon characteristics remain unchanged. Like the base-model, adding a surgeon to Kitchener results in the best outcomes; it reduces patient average wait time by up to 43% at the end of the fourth year and makes the chance of being seen by a surgeon within 180 days near 70%.

4.5 Conclusion

We study the referral process in the cataract referral network in the Waterloo region using real data from the network and examine how different policies, including reducing variations in patient preferences and adding a new surgeon to the network, could impact patient access to surgeons.

We introduce two specific models, called the base-model and the multinomial-model, which allow us to analyze system performance in the future. In the base-model, the values of all parameters are fixed and directly derived from the data. Due to this limitation and the fact that no element of the model can impact patient preferences the base-model is used to analyze system behaviour in the near future. In the multinomial-model, we consider two arrival scenarios and to determine patient preferences we apply a multinomial regression model in the simulation. This allows patient preferences to be affected by different parameters of the system including patient wait times to see surgeons. Although the structure of the models is different, the results from both models are consistent.

We find that limiting patient options does not significantly improve patient average wait time to see surgeons (WT1) in the system. This indicates the high utilization rate of the system where changing referral policies does not improve the tradeoff between demand and supply. However, in comparison with other modifications of patient choices, our results from the multinomial-model suggest that more patients could be seen within 180 days if the only available option is to allocate patients to the first available surgeon. We also find that eliminating patients' option to travel to another city for those patients who choose the first available surgeon does not necessarily improve patient average wait time to see surgeons in the network.

Finally, we examine the impact of adding a new surgeon to the network. Different scenarios for the location and service rate of the new surgeon are considered. Our results from both models suggest that Kitchener and Guelph, respectively, are the best options to locate a new surgeon. Our results from the multinomial-model also suggest that adding a new surgeon to Kitchener can reduce patient average WT1 by up to 43% by the end of the fourth year.

The models and insights gained from the analysis presented in this paper are a first step in understanding the performance of various referral policies on managerial and health outcome measures. In particular, due to the limited WT2 data for individual patients in this paper we study how implementing new referral policies or potentially adding more capacity to the network could improve patient access to surgeons (reduce WT1). However, on-time delivery of service (in our case, cataract surgery) to patients is yet another important topic that requires further investigation. Analyzing the impact of hospital operation room availability and WT2 for each surgeon (and understanding the relationship between hospital bed availability and surgeon scheduling policies) would be another interesting line of research for future exploration. We would like to thank all the people from the Waterloo Centralized Intake System and the surgeons in the network who supported us and provided us with reliable data resources throughout the whole study.

Appendix A

First Chapter Proofs

A.1 Online Appendix

The Appendix consists of three sections. In the first section we present our proofs for the standard centralized referral system and theorems discussed in the main text (Sections 2.2.2 and 2.3, Theorems 1-7). The second section is dedicated to the optimal policies and proofs for the fair-allocation referral system (Section 2.4).

A.1.1 Proofs of Results for the Standard Centralized Referral System

The proofs presented in this appendix follow the order in the text.

A.2 Results for Section 2.2.2 (optimal provider scheduling)

We begin with the results characterizing optimal behavior of the provider. The following lemma is an intermediate result for proving Theorem 1. This result allows us to consider only a discrete number of candidate optimal scheduling policies for the provider.

Lemma 1. *Consider the providers optimization problem shown in Equations 2.17-2.19. An interior point* (x_{1j}, x_{2j}) *where* $0 < x_{ij} < 1, \forall i$*, may be optimal only if* $x_{1j} = x_{2j} = (\frac{m_j}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_1)^{-1}}$.

Proof of Lemma 1. Based on the Karush–Kuhn–Tucker (KKT) conditions for the provider problem under the proposed achievable region we have:

$$\frac{\lambda_{1j}}{\lambda_{1j}+\lambda_{2j}} = u\alpha_j\lambda_{1j}x_{1j}^{\alpha_j-1} \Longrightarrow \begin{cases} \lambda_{1j}=0 \\ \frac{1}{\lambda_{1j}+\lambda_{2j}} = u\alpha_j x_{1j}^{\alpha_j-1} \quad (i) \end{cases}$$

$$\frac{\lambda_{2j}}{\lambda_{1j}+\lambda_{2j}} = u\alpha_j\lambda_{2j}x_{2j}^{\alpha_j-1} \Longrightarrow \begin{cases} \lambda_{2j}=0 \\ \frac{1}{\lambda_{1j}+\lambda_{2j}} = u\alpha_j x_{2j}^{\alpha_j-1} \quad (ii) \end{cases}$$

$$u(\lambda_{1j}x_{1j}^{\alpha_j} + \lambda_{2j}x_{2j}^{\alpha_j} - m_j) = 0 \quad (iii)$$

Variable u is a Lagrangian multiplier. From Equations (i) and (ii), u cannot be 0. In addition, since $0 < x_{ij} < 1, \forall i \Rightarrow \lambda_{ij} \neq 0$. Therefore, from equations (i) and (ii) we have:

$$u\alpha_j x_{1j}^{\alpha_j-1} = u\alpha_j x_{2j}^{\alpha_j-1} \Rightarrow x_{1j}^{\alpha_j-1} = x_{2j}^{\alpha_j-1} \Rightarrow \begin{cases} x_{1j} = x_{2j} \\ or \\ \alpha_j = 1 \end{cases}$$

Therefore, if $\alpha_j \neq 1$ then x_{1j} must be equal to x_{2j}. Since $u \neq 0$, for equation (iii) we have $\lambda_{1j}x_{1j}^{\alpha_j} + \lambda_{2j}x_{2j}^{\alpha_j} = m_j$. Substituting $x_{1j} = x_{2j} = x$ in equation (iii) results in $x_{1j} = x_{2j} = (\frac{m_j}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}}$. □

Proof of Theorem 1. Table 2.1 shows the five candidate optimal solutions, the interior point S and the boundary points $\{P_1, P_2, \bar{P}_1, \bar{P}_2\}$ and the scenarios where they are optimal. We will go through each of these cases to confirm this correspondence.

Case 1: $\alpha_j > 1$.

In Lemma 1 we show that $x_{1j} = x_{2j} = (\frac{m_j}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}}$ is the necessary condition for an interior point to be optimal. Now, we prove that this point is the optimal solution for a provider if $\alpha_j > 1$. First, let's compare $f(S)$ with $f(\bar{P}_1)$ and $f(\bar{P}_2)$. Since $\alpha_j > 1 \rightarrow 0 < \frac{1}{\alpha_j} < 1$. Therefore, we have:

$$(\frac{\lambda_{1j}}{\lambda_{1j}+\lambda_{2j}}) < (\frac{\lambda_{1j}}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}} \rightarrow (\frac{\lambda_{1j}}{\lambda_{1j}+\lambda_{2j}})(\frac{1}{\lambda_{1j}})^{(\alpha_j)^{-1}} < (\frac{1}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}} \xrightarrow{\times (m_j)^{(\alpha_j)^{-1}}}$$
$$(\frac{\lambda_{1j}}{\lambda_{1j}+\lambda_{2j}})(\frac{m_j}{\lambda_{1j}})^{(\alpha_j)^{-1}} < (\frac{m_j}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}} \Rightarrow f(\bar{P}_1) < f(S)$$

$$(\frac{\lambda_{2j}}{\lambda_{1j}+\lambda_{2j}}) < (\frac{\lambda_{2j}}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}} \rightarrow (\frac{\lambda_{2j}}{\lambda_{1j}+\lambda_{2j}})(\frac{1}{\lambda_{2j}})^{(\alpha_j)^{-1}} < (\frac{1}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}} \xrightarrow{\times (m_j)^{(\alpha_j)^{-1}}}$$
$$(\frac{\lambda_{2j}}{\lambda_{1j}+\lambda_{2j}})(\frac{m_j}{\lambda_{2j}})^{(\alpha_j)^{-1}} < (\frac{m_j}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}} \Rightarrow f(\bar{P}_2) < f(S)$$

Now, let's compare $f(S)$ with $f(P_1)$. When $m_j < \lambda_{1j}$ then $f(P_1)$ is undefined. Therefore, $f(S)$ and $f(P_1)$ are comparable when $\lambda_{1j} \leq m_j < \lambda_{1j} + \lambda_{2j}$. Define $\Delta_1 f = f(S) - f(P_1)$. If we set $m_j = \lambda_{1j}$ then $f(P_1) = \frac{\lambda_{1j}}{\lambda_{1j}+\lambda_{2j}}$ and $f(S) = (\frac{\lambda_{1j}}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}}$. Since $\alpha_j > 1$ and $0 < \frac{\lambda_{1j}}{\lambda_{1j}+\lambda_{2j}} < 1$ we can conclude that if $m_j = \lambda_{1j} \Rightarrow \Delta_1 F > 0 \Rightarrow f(S) > f(P_1)$. In addition, if we set $m_j = \lambda_{1j} + \lambda_{2j}$ then $f(S) = f(P_1) = 1$. Therefore, in order to prove that when

$\lambda_{1j} \leq m_j < \lambda_{1j} + \lambda_{2j}$ then $f(S) > f(P_1)$ we need to show that $\frac{d\Delta_1 f}{dm_j} < 0$.

$$\Delta_1 f = f(S) - f(P_1) = (\frac{m_j}{\lambda_{1j} + \lambda_{2j}})^{(\alpha_j)^{-1}} - \frac{\lambda_{1j} + \lambda_{2j}(\frac{m_j - \lambda_{1j}}{\lambda_{2j}})^{(\alpha_j)^{-1}}}{\lambda_{1j} + \lambda_{2j}}$$

$$\frac{d(\Delta_1 f)}{dm_j} = \frac{(\frac{m_j}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}}}{\alpha_j m_j} - \frac{\lambda_{2j}}{\lambda_{1j} + \lambda_{2j}} \frac{(\frac{m_j - \lambda_{1j}}{\lambda_{2j}})^{(\alpha_j)^{-1}}}{\alpha_j (m_j - \lambda_{1j})}$$

Now we have:

$m_j < \lambda_{1j} + \lambda_{2j} \rightarrow m_j\lambda_{1j} < (\lambda_{1j} + \lambda_{2j})\lambda_{1j} \rightarrow m_j\lambda_{1j} + m_j\lambda_{2j} < (\lambda_{1j} + \lambda_{2j})\lambda_{1j} + m_j\lambda_{2j} \rightarrow m_j\lambda_{1j} + m_j\lambda_{2j} - \lambda_{1j}\lambda_{2j} - \lambda_{1j}^2 < m_j\lambda_{2j} \rightarrow \frac{\lambda_{1j}+\lambda_{2j}}{\lambda_{2j}} < \frac{m_j}{m_j - \lambda_{1j}} \rightarrow (\frac{\lambda_{1j}+\lambda_{2j}}{\lambda_{2j}})^{\alpha_j - 1} < (\frac{m_j}{m_j-\lambda_{1j}})^{\alpha_j-1} \rightarrow \frac{m_j(\lambda_{1j}+\lambda_{2j})^{\alpha_j-1}}{\lambda_{2j}^{\alpha_j}} < \frac{m_j^{\alpha_j}}{\lambda_{2j}(m_j-\lambda_{1j})^{\alpha_j-1}} \rightarrow (\frac{\lambda_{1j}+\lambda_{2j}}{\lambda_{2j}})^{\alpha_j}(\frac{m_j}{\lambda_{1j}+\lambda_{2j}}) < (\frac{m_j}{m_j-\lambda_{1j}})^{\alpha_j}(\frac{m_j-\lambda_{1j}}{\lambda_{2j}}) \rightarrow$
$(\frac{\lambda_{1j}+\lambda_{2j}}{\lambda_{2j}})(\frac{m_j}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}} < (\frac{m_j}{m_j-\lambda_{1j}})(\frac{m_j-\lambda_{1j}}{\lambda_{2j}})^{(\alpha_j)^{-1}} \rightarrow \frac{(\frac{m_j}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}}}{m_j} < \frac{\lambda_{2j}}{\lambda_{1j}+\lambda_{2j}}\frac{(\frac{m_j-\lambda_{1j}}{\lambda_{2j}})^{(\alpha_j)^{-1}}}{(m_j-\lambda_{1j})} \rightarrow$
$\frac{(\frac{m_j}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}}}{\alpha_j m_j} <$
$\frac{\lambda_{2j}}{\lambda_{1j}+\lambda_{2j}}\frac{(\frac{m_j-\lambda_{1j}}{\lambda_{2j}})^{(\alpha_j)^{-1}}}{\alpha_j(m_j-\lambda_{1j})} \Rightarrow \frac{d(\Delta_1 f)}{dm_j} < 0$

The same process can be done in order to show that when $\lambda_{2j} \leq m_j < \lambda_{1j} + \lambda_{2j}$ then $f(S) > f(P_2)$. Define $\Delta_2 f = f(S) - f(P_2)$. If we set $m_j = \lambda_{2j}$ then $f(P_2) = \frac{\lambda_{2j}}{\lambda_{1j}+\lambda_{2j}}$ and $f(S) = (\frac{\lambda_{2j}}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}}$. Since $\alpha_j > 1$ and $0 < \frac{\lambda_{2j}}{\lambda_{1j}+\lambda_{2j}} < 1$ we can conclude that when $m_j = \lambda_{2j} \rightarrow f(S) > f(P_2)$. In addition, if we set $m_j = \lambda_{1j} + \lambda_{2j}$then $f(S) = f(P_2) = 1$. Therefore, in order to prove that when $\lambda_{2j} \leq m_j < \lambda_{1j} + \lambda_{2j}$ then $f(S) > f(P_2)$ we need to show that $\frac{d\Delta_2 f}{dm_j} < 0$.

Therefore, if $\alpha_j > 1$ then point S is the only optimal solution to the provider problem.

$$\Delta_2 f = f(S) - f(P_2) = (\frac{m_j}{\lambda_{1j} + \lambda_{2j}})^{(\alpha_j)^{-1}} - \frac{\lambda_{1j}(\frac{m_j - \lambda_{2j}}{\lambda_{1j}})^{(\alpha_j)^{-1}} + \lambda_{2j}}{\lambda_{1j} + \lambda_{2j}}$$

$$\frac{d(\Delta_2 f)}{dm_j} = \frac{(\frac{m_j}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}}}{\alpha_j m_j} - \frac{\lambda_{1j}}{\lambda_{1j} + \lambda_{2j}} \frac{(\frac{m_j - \lambda_{2j}}{\lambda_{1j}})^{(\alpha_j)^{-1}}}{\alpha_j (m_j - \lambda_{2j})}$$

We have:

$m_j < \lambda_{1j} + \lambda_{2j} \xrightarrow{\times\lambda_{2j}} m_j\lambda_{2j} < (\lambda_{1j} + \lambda_{2j})\lambda_{2j} \xrightarrow{+m_j\lambda_{1j}} m_j\lambda_{2j} + m_j\lambda_{1j} < (\lambda_{1j} + \lambda_{2j})\lambda_{2j} + m_j\lambda_{1j} \rightarrow m_j\lambda_{2j} + m_j\lambda_{1j} - \lambda_{1j}\lambda_{2j} - \lambda_{2j}^2 < m_j\lambda_{1j} \rightarrow \frac{\lambda_{1j}+\lambda_{2j}}{\lambda_{1j}} < \frac{m_j}{m_j-\lambda_{2j}} \rightarrow (\frac{\lambda_{1j}+\lambda_{2j}}{\lambda_{1j}})^{\alpha_j-1} < (\frac{m_j}{m_j-\lambda_{2j}})^{\alpha_j-1} \rightarrow \frac{m_j(\lambda_{1j}+\lambda_{2j})^{\alpha_j-1}}{\lambda_{1j}^{\alpha_j}} < \frac{m_j^{\alpha_j}}{\lambda_{1j}(m_j-\lambda_{2j})^{\alpha_j-1}} \rightarrow (\frac{\lambda_{1j}+\lambda_{2j}}{\lambda_{1j}})^{\alpha_j}(\frac{m_j}{\lambda_{1j}+\lambda_{2j}}) < (\frac{m_j}{m_j-\lambda_{2j}})^{\alpha_j}(\frac{m_j-\lambda_{2j}}{\lambda_{1j}})$
$\rightarrow (\frac{\lambda_{1j}+\lambda_{2j}}{\lambda_{1j}})(\frac{m_j}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}} < (\frac{m_j}{m_j-\lambda_{2j}})(\frac{m_j-\lambda_{2j}}{\lambda_{1j}})^{(\alpha_j)^{-1}} \rightarrow \frac{(\frac{m_j}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}}}{m_j} < \frac{\lambda_{1j}}{\lambda_{1j}+\lambda_{2j}}\frac{(\frac{m_j-\lambda_{2j}}{\lambda_{1j}})^{(\alpha_j)^{-1}}}{(m_j-\lambda_{2j})} \rightarrow$
$\frac{(\frac{m_j}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}}}{\alpha_j m_j} < \frac{\lambda_{1j}}{\lambda_{1j}+\lambda_{2j}}\frac{(\frac{m_j-\lambda_{2j}}{\lambda_{1j}})^{(\alpha_j)^{-1}}}{k_j(m_j-\lambda_{2j})} \Rightarrow \frac{d(\Delta_2 f)}{dm_j} < 0$

Case 2: $\alpha_j < 1$**,** $\lambda_{ij} \leq m_j\ \forall i$**.**

Let's prove that if a provider has enough capacity to serve both types of patients independently then solutions P_1 and P_2 are the only optimal solutions. We prove that when $\lambda_{2j} < \lambda_{1j} \leq m_j < \lambda_{1j} + \lambda_{2j}$ then solution P_1 is the optimal solution to the problem. Let's begin with comparing $f(P_1)$ and $f(P_2)$:

If $m_j = \lambda_{1j} \rightarrow \begin{cases} f(P_1) = \frac{\lambda_{1j}}{\lambda_{1j}+\lambda_{2j}} \\ f(P_2) = \frac{\lambda_{1j}(\frac{\lambda_{1j}-\lambda_{2j}}{\lambda_{1j}})^{\frac{1}{\alpha_j}}+\lambda_{2j}}{\lambda_{1j}+\lambda_{2j}} \end{cases}$

$$\Delta_1 f = f(P_1) - f(P_2) = \frac{\lambda_{1j} - \lambda_{1j}(\frac{\lambda_{1j}-\lambda_{2j}}{\lambda_{1j}})^{(\alpha_j)^{-1}} - \lambda_{2j}}{\lambda_{1j} + \lambda_{2j}}$$

Since $\lambda_{2j} < \lambda_{1j}$ let's define $\lambda_{1j} = \lambda_{2j} + a$. $\Delta_1 f$ can be rewritten as follows:

$$\Delta_1 f = \frac{(\lambda_{2j} + a) - \lambda_{1j}(\frac{a}{\lambda_{1j}})^{(\alpha_j)^{-1}} - \lambda_{2j}}{\lambda_{1j} + \lambda_{2j}} = \frac{a - \lambda_{1j}(\frac{a}{\lambda_{1j}})^{(\alpha_j)^{-1}}}{\lambda_{1j} + \lambda_{2j}} = \frac{\lambda_{1j}(\frac{a}{\lambda_{1j}} - (\frac{a}{\lambda_{1j}})^{(\alpha_j)^{-1}})}{\lambda_{1j} + \lambda_{2j}}$$

Since $0 < \frac{a}{\lambda_{1j}} < 1$ and $0 < \alpha_j < 1$ when $m_j = \lambda_{1j}$, $\Delta_1 f = f(P_1) - f(P_2) > 0$. In addition, when $m_j = \lambda_{1j} + \lambda_{2j}$, $f(P_1) = f(P_2) = 1$ and consequently $\Delta_1 f = 0$

Therefore, showing that $\lambda_{2j} < \lambda_{1j} < m_j < \lambda_{1j} + \lambda_{2j}$ implies $\frac{d\Delta_1 f}{dm_j} < 0$ is sufficient to prove that $f(P_1) > f(P_2)$.

$$\frac{d\Delta_1 f}{dm_j} = \frac{1}{\alpha_j(\lambda_{1j} + \lambda_{2j})}((\frac{m_j - \lambda_{1j}}{\lambda_{2j}})^{(\alpha_j)^{-1}-1} - (\frac{m_j - \lambda_{2j}}{\lambda_{1j}})^{(\alpha_j)^{-1}-1})$$

Let's define $H = (\frac{m_j-\lambda_{1j}}{\lambda_{2j}})^{(\alpha_j)^{-1}-1} - (\frac{m_j-\lambda_{2j}}{\lambda_{1j}})^{(\alpha_j)^{-1}-1}$. Now we show that whenever $0 < \alpha_j < 1$ and $\lambda_{2j} < \lambda_{1j} < m_j < \lambda_{1j} + \lambda_{2j}$ then $H < 0$ and therefore $\frac{d\Delta_1 f}{dm_j} < 0$. We have:

$m_j < \lambda_{1j} + \lambda_{2j} \rightarrow m_j(\lambda_{1j} - \lambda_{2j}) < (\lambda_{1j} + \lambda_{2j})(\lambda_{1j} - \lambda_{2j}) \rightarrow m_j\lambda_{1j} - \lambda_{1j}^2 < m_j\lambda_{2j} - \lambda_{2j}^2 \rightarrow \frac{m_j-\lambda_{1j}}{\lambda_{2j}} < \frac{m_j-\lambda_{2j}}{\lambda_{1j}} \rightarrow (\frac{m_j-\lambda_{1j}}{\lambda_{2j}})^{(\alpha_j)^{-1}-1} < (\frac{m_j-\lambda_{2j}}{\lambda_{1j}})^{(\alpha_j)^{-1}-1} \rightarrow H < 0 \rightarrow \frac{d\Delta G}{dm_j} < 0$.

Now let's compare $f(P_1)$ and $f(S)$. Define $\Delta_2 f = f(P_1) - f(S)$.

If $m_j = \lambda_{1j} \rightarrow \begin{cases} f(P_1) = \frac{\lambda_{1j}}{\lambda_{1j}+\lambda_{2j}} \\ f(S) = (\frac{\lambda_{1j}}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}} \end{cases}$

Since $0 < \frac{\lambda_{1j}}{\lambda_{1j}+\lambda_{2j}} < 1$ and $0 < \alpha_j < 1 \Rightarrow f(P_1) > f(S)$ when $m_j = \lambda_{1j}$. In addition, when $m_j = \lambda_{1j} + \lambda_{2j}$, $f(P_1) = f(S) = 1$.

Therefore, if we prove that when $\lambda_{1j} < m_j < \lambda_{1j} + \lambda_{2j}$ then $\frac{d\Delta_2 f}{dm_j} < 0$then we actually have proved that $f(P_1) > f(S)$.

$$\frac{d\Delta_2 f}{dm_j} = \frac{1}{\alpha_j(\lambda_{1j}+\lambda_{2j})}((\frac{m_j-\lambda_{1j}}{\lambda_{2j}})^{(\alpha_j)^{-1}-1} - (\frac{m_j}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}-1})$$

Let's define $G = (\frac{m_j-\lambda_{1j}}{\lambda_{2j}})^{(\alpha_j)^{-1}-1} - (\frac{m_j-\lambda_{2j}}{\lambda_{1j}})^{(\alpha_j)^{-1}-1}$. Now we show that whenever $0 < \alpha_j < 1$ and $\lambda_{2j} < \lambda_{1j} < m_j < \lambda_{1j} + \lambda_{2j}$ then $G < 0$ and therefore $\frac{d\Delta_2 f}{dm_j} < 0$. We have:

$m_j < \lambda_{1j} + \lambda_{2j} \rightarrow m_j\lambda_{1j} < \lambda_{1j}(\lambda_{1j}+\lambda_{2j}) \rightarrow m_j\lambda_{1j} - \lambda_{1j}^2 - \lambda_{1j}\lambda_{2j} < 0 \rightarrow m_j\lambda_{1j} - \lambda_{1j}^2 - \lambda_{1j}\lambda_{2j} + m_j\lambda_{2j} < m_j\lambda_{2j} \rightarrow \frac{m_j-\lambda_{1j}}{\lambda_{2j}} < \frac{m_j}{\lambda_{1j}+\lambda_{2j}} \rightarrow (\frac{m_j-\lambda_{1j}}{\lambda_{2j}})^{(\alpha_j)^{-1}-1} < (\frac{m_j}{\lambda_{1j}+\lambda_{2j}})^{(\alpha_j)^{-1}-1} \rightarrow G < 0 \rightarrow \frac{d\Delta_2 f}{dm_j} < 0$.

Now, let's compare $f(P_1)$ and $f(\bar{P}_2)$.For solution $\bar{P}_2$ we have $x_{2j} = (\frac{m_j}{\lambda_{2_j}})^{(\alpha_j)^{-1}}$. It can be seen that since $\lambda_{2j} < \lambda_{1j} < m_j < \lambda_{1j} + \lambda_{2j}$, x_{2j} in solution $\bar{P}_2$ is greater than 1. Therefore, if $\lambda_{2j} < \lambda_{1j} < m_j¡\lambda_{1j} + \lambda_{2j}$ then solution $\bar{P}_2$ is not a feasible solution. The same logic is also true for solution $\bar{P}_1$. In fact, for solution $\bar{P}_1$ we have $x_{1j} = (\frac{m_j}{\lambda_{1_j}})^{(\alpha_j)^{-1}}$. Again $\lambda_{2j} < \lambda_{1j} < m_j < \lambda_{1j} + \lambda_{2j}$ and consequently x_{1j} becomes greater than 1 in solution $\bar{P}_1$. Therefore, if $\lambda_{2j} < \lambda_{1j} < m_j¡\lambda_{1j} + \lambda_{2j}$ then solution $\bar{P}_1$ is not a feasible solution too. We proved that if $0 < \alpha_j < 1$ and $\lambda_{2j} < \lambda_{1j} < m_j < \lambda_{1j} + \lambda_{2j}$ then solution P_1 is the optimal solution.

The same reasoning process can applied for other scheduling policies presented in Table 2.1.

□

Results for Section 2.3.1 (optimal referral policy)

We begin by presenting two lemmas which are useful in the proof of Theorems 2-5 and then present the proofs of these theorems.

Lemma 2. *Consider the referrer problem described in Equations 2.15-2.19. When both providers are LOC, $m_1 + m_2$ is an upper bound on the referrer objective function.*

Proof of Lemma 2. The following equations show the referrer objective functions for each case feasible under the capacity and arrival rate assumptions:

$$G(P_1, P_1) = \lambda_{11} + \lambda_{12} + (\lambda_{21})(\frac{m_1-\lambda_{11}}{\lambda_{21}})^{(\alpha_1)^{-1}} + (\lambda_{22})(\frac{m_2-\lambda_{12}}{\lambda_{22}})^{(\alpha_2)^{-1}}$$
$$G(P_1, P_2) = \lambda_{11} + \lambda_{22} + (\lambda_{21})(\frac{m_1-\lambda_{11}}{\lambda_{21}})^{(\alpha_1)^{-1}} + (\lambda_{12})(\frac{m_2-\lambda_{22}}{\lambda_{12}})^{(\alpha_2)^{-1}}$$
$$G(P_2, P_1) = \lambda_{21} + \lambda_{12} + \frac{(m_1-\lambda_{21})^{(\alpha_1)^{-1}}}{(\lambda_{11})^{(\alpha_1)^{-1}-1}} + \frac{(m_2-\lambda_{12})^{(\alpha_2)^{-1}}}{(\lambda_{22})^{(\alpha_2)^{-1}-1}}$$

Let's focus on the case (P_1, P_1). Since both $x_{21} = (\frac{m_1-\lambda_{11}}{\lambda_{21}})^{(\alpha_1)^{-1}}$ and $x_{22} = (\frac{m_2-\lambda_{12}}{\lambda_{22}})^{(\alpha_2)^{-1}}$ are between zero and one and $(\alpha_1)^{-1}$ and $(\alpha_2)^{-1}$ are greater than 1, the maximum values for x_{21} and x_{22} can be achieved if $\alpha_1 = \alpha_2 = 1$ which results in $G(P_1, P_1) = m_1 + m_2$. The same logic can be applied to prove that the maximum achievable values for $G(P_1, P_2)$ and $G(P_2, P_1)$ is $m_1 + m_2$. $\square$

Lemma 3. *Consider the referrer problem described in Equations 2.15-2.19. When both providers are LOC, the referrer objective function is strictly convex.*

Proof of Lemma 3. We need to prove that the Hessian at feasible points in all the three cases (P_1, P_1), (P_1, P_2) and (P_2, P_1) is positive semidefinite.

Case 1: (P_1, P_1)**.**

Since both providers best solutions is solution P_1 and they both have enough capacity to visit each type of patients independently therefore based on conditions in Table 1 we have $\lambda_{2j} \leq \lambda_{1j} \leq m_j, j = 1, 2$.

$$G(P_1, P_1) = \lambda_{11} + \lambda_{12} + (\lambda_{21})(\frac{m_1 - \lambda_{11}}{\lambda_{21}})^{(\alpha_1)^{-1}} + (\lambda_{22})(\frac{m_2 - \lambda_{12}}{\lambda_{22}})^{(\alpha_2)^{-1}}$$

Since $\lambda_{11} + \lambda_{12} = \lambda_1$, $\lambda_{12} + \lambda_{22} = \lambda_2$ and $0 < \alpha_i < 1, \forall i$ the function can be rewritten as follows:

$$G(P_1, P_1) = \lambda_1 + \frac{(m_1 - \lambda_{11})^{(\alpha_1)^{-1}}}{(\lambda_{21})^{(\alpha_1)^{-1}-1}} + \frac{(m_2 - \lambda_1 + \lambda_{11})^{(\alpha_2)^{-1}}}{(\lambda_2 - \lambda_{21})^{(\alpha_2)^{-1}-1}}$$

Let's define $k_1 = (\alpha_1)^{-1}$ and $k_2 = (\alpha_2)^{-1}$. The Hessian matrix for case (P_1, P_1) is $H(P_1, P_1) = \begin{bmatrix} A_1 & B_1 \\ B_1 & C_1 \end{bmatrix}$ where:

$$\begin{aligned}
A_1 = \frac{d^2G(P_1, P_1)}{d\lambda_{11}^2} &= \frac{k_1(k_1-1)(m_1-\lambda_{11})^{k_1-2}}{(\lambda_{21})^{k_1-1}} + \frac{k_2(k_2-1)(m_2+\lambda_{11}-\lambda_1)^{k_2-2}}{(\lambda_2-\lambda_{21})^{k_2-1}} \\
B_1 = \frac{d^2G(P_1, P_1)}{d\lambda_{11}d\lambda_{21}} &= \frac{k_1(k_1-1)(m_1-\lambda_{11})^{k_1-1}}{(\lambda_{21})^{k_1}} + \frac{k_2(k_2-1)(m_2+\lambda_{11}-\lambda_1)^{k_2-1}}{(\lambda_2-\lambda_{21})^{k_2}} \\
C_1 = \frac{d^2G(P_1, P_1)}{d\lambda_{21}^2} &= \frac{k_1(k_1-1)(m_1-\lambda_{11})^{k_1}}{(\lambda_{21})^{k_1+1}} + \frac{k_2(k_2-1)(m_2+\lambda_{11}-\lambda_1)^{k_2}}{(\lambda_2-\lambda_{21})^{k_2+1}}
\end{aligned}$$

Since $k_1 > 1$, $k_2 > 1$ $\lambda_{11} \leq m_1$, $\lambda_{12} \leq m_2 \rightarrow \lambda_1 - m_2 \leq \lambda_{11} \rightarrow 0 \leq m_2 - \lambda_1 + \lambda_{11}$ and $\lambda_{21} \leq \lambda_2$ all A_1, B_1 and C_1 are positive. Therefore, since $\lambda_{i1} \geq 0, i = 1, 2$ it can be concluded that Hessian is positive semidefinite at all feasible points.

Case 2: (P_1, P_2).

For case (P_1, P_2) the assumptions are $\lambda_{21} \leq \lambda_{11} \leq m_1$ and $\lambda_{12} \leq \lambda_{22} \leq m_2$. We have:

$$G(P_1,P_2) = \lambda_{11} + \lambda_{22} + (\lambda_{21})(\frac{m_1 - \lambda_{11}}{\lambda_{21}})^{k_1} + (\lambda_{12})(\frac{m_2 - \lambda_{22}}{\lambda_{12}})^{k_2}$$

The function can be rewritten as follows:

$$G(P_1,P_2) = \lambda_2 + \lambda_{11} - \lambda_{21} + \frac{(m_1 - \lambda_{11})^{k_1}}{(\lambda_{21})^{k_1-1}} + \frac{(m_2 - \lambda_2 + \lambda_{21})^{k_2}}{(\lambda_1 - \lambda_{11})^{k_2-1}}$$

Where $k_1 = (\alpha_1)^{-1}$ and $k_2 = (\alpha_2)^{-1}$. The Hessian matrix for case (P_1, P_2) is $H(P_1, P_2) = \begin{bmatrix} A_2 & B_2 \\ B_2 & C_2 \end{bmatrix}$ where:

$$\begin{aligned}
A_2 &= \frac{d^2G(P_1,P_2)}{d\lambda_{11}^2} = \frac{k_1(k_1-1)(m_1-\lambda_{11})^{k_1-2}}{(\lambda_{21})^{k_1-1}} + \frac{k_2(k_2-1)(m_2+\lambda_{21}-\lambda_2)^{k_2}}{(\lambda_1-\lambda_{11})^{k_2+1}} \\
B_2 &= \frac{d^2G(P_1,P_2)}{d\lambda_{11}d\lambda_{21}} = \frac{k_1(k_1-1)(m_1-\lambda_{11})^{k_1-1}}{(\lambda_{21})^{k_1}} + \frac{k_2(k_2-1)(m_2+\lambda_{21}-\lambda_2)^{k_2-1}}{(\lambda_1-\lambda_{11})^{k_2}} \\
C_2 &= \frac{d^2G(P_1,P_2)}{d\lambda_{21}^2} = \frac{k_1(k_1-1)(m_1-\lambda_{11})^{k_1}}{(\lambda_{21})^{k_1+1}} + \frac{k_2(k_2-1)(m_2+\lambda_{21}-\lambda_2)^{k_2-2}}{(\lambda_1-\lambda_{11})^{k_2-1}}
\end{aligned}$$

Again, since $k_1 > 1$, $k_2 > 1$ $\lambda_{11} \leq m_1$, $\lambda_{22} \leq m_2 \rightarrow \lambda_2 - m_2 \leq \lambda_{21} \rightarrow 0 \leq m_2 + \lambda_{21} - \lambda_2$ and $\lambda_{11} \leq \lambda_1$ all A_2, B_2 and C_2 are positive. Therefore, as $\lambda_{i1} \geq 0, i = 1,2$ it can be concluded that Hessian is positive semidefinite at all feasible points.

Case 3: (P_2, P_1).

For case (P_2, P_1) the assumptions are $\lambda_{11} \leq \lambda_{21} \leq m_1$ and $\lambda_{22} \leq \lambda_{12} \leq m_2$.

$$G(P_2,P_1) = \lambda_{21} + \lambda_{12} + \frac{(m_1 - \lambda_{21})^{(\alpha_1)^{-1}}}{(\lambda_{11})^{(\alpha_1)^{-1}-1}} + \frac{(m_2 - \lambda_{12})^{(\alpha_2)^{-1}}}{(\lambda_{22})^{(\alpha_2)^{-1}-1}}$$

The function can be rewritten as follows:

$$G(P_2,P_1) = \lambda_1 + \lambda_{21} - \lambda_{11} + \frac{(m_1 - \lambda_{21})^{k_1}}{(\lambda_{11})^{k_1-1}} + \frac{(m_2 - \lambda_1 + \lambda_{11})^{k_2}}{(\lambda_2 - \lambda_{21})^{k_2-1}}$$

Where $k_1 = (\alpha_1)^{-1}$ and $k_2 = (\alpha_2)^{-1}$. The Hessian matrix for case (P_2, P_1) is $H(P_2, P_1) = \begin{bmatrix} A_3 & B_3 \\ B_3 & C_3 \end{bmatrix}$ where:

$$A_3 = \frac{d^2G(P_2, P_1)}{d\lambda_{11}^2} = \frac{k_1(k_1-1)(m_1-\lambda_{21})^{k_1}}{(\lambda_{11})^{k_1+1}} + \frac{k_2(k_2-1)(m_2+\lambda_{11}-\lambda_1)^{k_2-2}}{(\lambda_2-\lambda_{21})^{k_2-1}}$$
$$B_3 = \frac{d^2G(P_2, P_1)}{d\lambda_{11}d\lambda_{21}} = \frac{k_1(k_1-1)(m_1-\lambda_{21})^{k_1-1}}{(\lambda_{11})^{k_1}} + \frac{k_2(k_2-1)(m_2+\lambda_{11}-\lambda_1)^{k_2-1}}{(\lambda_2-\lambda_{21})^{k_2}}$$
$$C_3 = \frac{d^2G(P_2, P_1)}{d\lambda_{21}^2} = \frac{k_1(k_1-1)(m_1-\lambda_{21})^{k_1-2}}{(\lambda_{11})^{k_1-1}} + \frac{k_2(k_2-1)(m_2+\lambda_{11}-\lambda_1)^{k_2}}{(\lambda_2-\lambda_{21})^{k_2+1}}$$

Again, since $k_1 > 1$, $k_2 > 1$ $\lambda_{21} \leq m_1$, $\lambda_{12} \leq m_2 \rightarrow \lambda_1 - m_2 \leq \lambda_{11} \rightarrow 0 \leq m_2 + \lambda_{11} - \lambda_1$ and $\lambda_{21} \leq \lambda_2$ all A_3, B_3 and C_3 are positive. Therefore, as $\lambda_{i1} \geq 0, i = 1, 2$ it can be concluded that Hessian is positive semidefinite at all feasible points.

We proved that the referrer function is always convex no matter what case the optimal policy of the referrer results in.

□

Corollary 1. *The candidates for optimal policies to the referrer problem described in Equations 2.15-2.19 are the boundary points that with the exception of Policies 7 and 8 result in $G(\Lambda) = m_1 + m_2$. Table A.1 shows these candidates when both providers are LOC:*

Index	Policy $(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$	Resulted TP (x_{11}, x_{21}) (x_{12}, x_{22})
1	$(m_1+m_2-\lambda_2, \lambda_2-m_2)$ $(\lambda_1+\lambda_2-m_1-m_2, m_2)$	$(1,1)$ $(0,1)$
2	$(m_1, \lambda_1+\lambda_2-m_1-m_2)$ $(\lambda_1-m_1, m_1+m_2-\lambda_1)$	$(1,0)$ $(1,1)$
3	(m_1, λ_2-m_2) (λ_1-m_1, m_2)	$(1,0)$ $(0,1)$
4	$(\lambda_1-m_2, m_1+m_2-\lambda_1)$ $(m_2, \lambda_1+\lambda_2-m_1-m_2)$	$(1,1)$ $(1,0)$
5	$(\lambda_1+\lambda_2-m_1-m_2, m_1)$ $(m_1+m_2-\lambda_2, \lambda_2-m_1)$	$(0,1)$ $(1,1)$
6	(λ_1-m_2, m_1) (m_2, λ_2-m_1)	$(0,1)$ $(1,0)$
7	$(\lambda_1, \lambda_2-m_2)$ $(0, m_2)$	$(1, (\frac{m_1-\lambda_1}{\lambda_2-m_2})^{(\alpha_1)^{-1}})$ $(0,1)$
8	$(\lambda_1-m_2, \lambda_2-m_2)$ (m_2, m_2)	$(1, (\frac{m_1+m_2-\lambda_1}{\lambda_2-m_2})^{(\alpha_1)^{-1}})$ $(1,0) or (0,1)$

Table A.1: Candidate Policies (Both providers are LOC - Regular referral system)

Proof of Corollary 1. This result follows directly from Lemmas 2 and 3. □

Proof of Theorem 2. From Lemma 2 it can be concluded that in a referral system where both providers are LOC if the referrer objective value under a policy is $m_1 + m_2$ then that policy is optimal. Tables A.2 and A.3 show optimal referral policies extracted from Table A.1 for the capacity scenario (Low, Low) and (Mid, Low) respectively:

Optimality Conditions	Optimal Policy
$2\lambda_1 \leq 2m_2 + m_1$	1-6
$\lambda_1 + \lambda_2 \leq 2m_2 + m_1 < 2\lambda_1$	1,3,5,6
$2m_2 + m_1 < \lambda_1 + \lambda_2$ *and* $2m_1 + m_2 \leq \lambda_1 + \lambda_2$	3,6
$2m_2 + m_1 < \lambda_1 + \lambda_2$ *and* $2m_1 + m_2 > \lambda_1 + \lambda_2$	2,3,5,6

Table A.2: Optimal policies (Capacity scenario (Low, Low), Both provider are LOC)

Optimality Conditions	Optimal Policy
$2\lambda_1 \leq 2m_2 + m_1$	1-4
$\lambda_1 + \lambda_2 \leq 2m_2 + m_1 < 2\lambda_1$	1,3,4
$2m_2 + m_1 < \lambda_1 + \lambda_2$	2,3

Table A.3: Optimal policies (Capacity scenario (Mid, Low), Both provider are LOC)

As can be seen in the tables, there are more than one optimal policy under each optimality condition. However, Policy 3 is the only optimal policies in both scenarios. The policy is:

providers 1: $(\lambda_{11}, \lambda_{21}) = (m_1, \lambda_2 - m_2)$

providers 2: $(\lambda_{12}, \lambda_{22}) = (\lambda_1 - m_1, m_2)$

For the first provider we have: $m_1 + m_2 > \lambda_2 \Rightarrow m_1 > \lambda_2 - m_2 \Rightarrow \lambda_{11} > \lambda_{21}$ and $\lambda_{11} = m_1$. Therefore, based on results provided in Table 2.1 the optimal solution for the first provider is solution P_1 and we have $(x_{11}, x_{21}) = (1, 0)$. Similarly for the second provider we have $\lambda_1 < m_1 + m_2 \Rightarrow \lambda_1 - m_1 < m_2 \Rightarrow \lambda_{12} < \lambda_{22}$ and $\lambda_{22} = m_2$. Therefore, the best solution for the second provider is P_2 and $(x_{12}, x_{22}) = (0, 1)$. Now for the referrer we have:

$$G(\Lambda) = \lambda_{11}x_{11} + \lambda_{21}x_{21} + \lambda_{12}x_{12} + \lambda_{22}x_{22} = m_1(1) + (\lambda_2 - m_2)(0) + (\lambda_1 - m_1)(0) + m_2(1) = m_1 + m_2$$

The policy is optimal since it is feasible in both capacity scenarios and results in the highest achievable value for the referrer. □

Proof of Theorem 3. We use Lemma 2 to show that policies mentioned in Table 2.3 are optimal in capacity scenario (High, Low) when $2m_2 + m_1 \geq \lambda_1 + \lambda_2$.

For the first policy we have:

1. $\lambda_{11} + \lambda_{21} = m_1 \Rightarrow (x_{11}, x_{21}) = (1, 1)$

2. Since $\lambda_1 + \lambda_2 \leq 2m_2 + m_1 \Rightarrow \lambda_1 + \lambda_2 - m_1 - m_2 \leq m_2 \Rightarrow \lambda_{12} \leq \lambda_{22}$. Therefore, the second provider's best solution is P_2 and $(x_{12}, x_{22}) = (0, 1)$.

Having target probabilities, it can easily be verified that $G(\Lambda) = m_1 + m_2$. The same process can be done to show that the second policy is also optimal. □

Proof of Theorem 4. Lemma 3 states that when providers are both LOC, $G(\Lambda)$ is strictly convex so that the optimal referral policy Λ is a boundary point. Define $\mathcal{L}$ as the set of boundary referral points for capacity scenario (High, Low) when $2m_2 + m_1 < \lambda_1 + \lambda_2$. $\mathcal{L}$ is listed exhaustively along with the objective functions in Tables A.4-A.6. From Theorem 1 and our assumptions in Section 2.3 we know that there are three potential sets of optimal scheduling policies $\mathcal{S} = (P_1, P_1), (P_1, P_2), (P_2, P_1)$. We consider each of these cases below identifying the optimal referral policies leading to the scheduling policy ($\mathcal{C}^*_S$). In each case we identify $\Lambda \in \mathcal{L}$ which lead to the respective scheduling policy. This mapping generates the set of potential coordinated policies $\mathcal{C}$. The next step is to identify $\mathcal{C}^*$ which eliminates dominated coordinated policies for each scheduling policy. Finally, we compare objective value of policies in $\mathcal{C}^*_{P_1,P_1}, \mathcal{C}^*_{P_1,P_2}$, and $\mathcal{C}^*_{P_2,P_1}$ which results in two potential policies which may be optimal depending on Equation 2.20. Let's begin with the first scheduling policy pair and define $k_1 = (\alpha_1)^{-1}$ and $k_2 = (\alpha_2)^{-1}$.

Case 1: (P_1, P_1)**.** Policies $\{R1, R2, ...\}$ can under specific parameters result in (P_1, P_1):

Index	Referral Policy $(\lambda_{11},\lambda_{21})$ $(\lambda_{12},\lambda_{22})$	$G(\Lambda)$
R1	$(\lambda_1-m_2,\lambda_2-m_2)$ (m_2,m_2)	$\lambda_1+\frac{(m_1+m_2-\lambda_1)^{k_1}}{(\lambda_2-m_2)^{k_1-1}}$
R2	$(\lambda_1-\frac{m_2}{2},\lambda_2-\frac{m_2}{2})$ $(\frac{m_2}{2},\frac{m_2}{2})$	$\lambda_1+\frac{m_2}{2}+$ $\frac{(m_1+\frac{m_2}{2}-\lambda_1)^{k_1}}{(\lambda_2-\frac{m_2}{2})^{k_1-1}}$
R3	$(\frac{\lambda_1+\lambda_2-m_2}{2},\frac{\lambda_1+\lambda_2-m_2}{2})$ $(\frac{\lambda_1-\lambda_2+m_2}{2},\frac{\lambda_2-\lambda_1+m_2}{2})$	$\frac{m_2+\lambda_2+\lambda_1}{2}+$ $\frac{(2m_1+m_2-\lambda_1-\lambda_2)^{k_1}}{2(\lambda_1+\lambda_2-m_2)^{k_1-1}}$
R4	$(\lambda_1-m_2,\lambda_2)$ $(m_2,0)$	$\lambda_1+\frac{(m_1+m_2-\lambda_1)^{k_1}}{(\lambda_2)^{k_1-1}}$
R5	$(\lambda_1-m_2,\lambda_1-m_2)$ $(m_2,\lambda_2+m_2-\lambda_1)$	$\lambda_1+\frac{(m_1+m_2-\lambda_1)^{k_1}}{(\lambda_1-m_2)^{k_1-1}}$

Table A.4: Boundary referral policies resulting in case (P_1, P_1)

First, let's compare $G(R1)$ with $G(R4)$:

$\lambda_2 > \lambda_2 - m_2 \rightarrow \frac{1}{\lambda_2} < \frac{1}{\lambda_2-m_2} \rightarrow \frac{1}{(\lambda_2)^{k_1-1}} < \frac{1}{(\lambda_2-m_2)^{k_1-1}} \xrightarrow{\times(m_1+m_2-\lambda_1)^{k_1}} \frac{(m_1+m_2-\lambda_1)^{k_1}}{(\lambda_2)^{k_1-1}} < \frac{(m_1+m_2-\lambda_1)^{k_1}}{(\lambda_2-m_2)^{k_1-1}} \xrightarrow{+\lambda_1} \lambda_1+\frac{(m_1+m_2-\lambda_1)^{k_1}}{(\lambda_2)^{k_1-1}} < \lambda_1+\frac{(m_1+m_2-\lambda_1)^{k_1}}{(\lambda_2-m_2)^{k_1-1}} \Longrightarrow G(R4) < G(R1)$.

Now we compare $G(R1)$ with $G(R5)$:

$\lambda_1 > \lambda_2 \rightarrow \lambda_1-m_2 > \lambda_2-m_2 \rightarrow \frac{1}{\lambda_1-m_2} < \frac{1}{\lambda_2-m_2} \rightarrow \frac{1}{(\lambda_1-m_2)^{k_1-1}} < \frac{1}{(\lambda_2-m_2)^{k_1-1}} \xrightarrow{\times(m_1+m_2-\lambda_1)^{k_1}} \frac{(m_1+m_2-\lambda_1)^{k_1}}{(\lambda_1-m_2)^{k_1-1}} < \frac{(m_1+m_2-\lambda_1)^{k_1}}{(\lambda_2-m_2)^{k_1-1}} \xrightarrow{+\lambda_1} \lambda_1+\frac{(m_1+m_2-\lambda_1)^{k_1}}{(\lambda_1-m_2)^{k_1-1}} < \lambda_1+\frac{(m_1+m_2-\lambda_1)^{k_1}}{(\lambda_2-m_2)^{k_1-1}} \rightarrow G(R5) < G(R1)$.

So far we have shown that Policy R1 results in higher $G(\Lambda)$ in comparison with policies R4 and R5. Therefore, $\mathcal{C}^*_{(P1,P1)}$ definitely does not include policies R4 and R5.

Now, let's compare $G(R2)$ with $G(R3)$. Define:

$$\Delta G = G(R2) - G(R3) = \frac{\lambda_1-\lambda_2}{2} + \frac{(m_1+\frac{m_2}{2}-\lambda_1)^{k_1}}{(\lambda_2-\frac{m_2}{2})^{k_1-1}} - \frac{(m_1-\frac{\lambda_1+\lambda_2-m_2}{2})^{k_1}}{(\frac{\lambda_1+\lambda_2-m_2}{2})^{k_1-1}}$$

We have:

$$\frac{d\Delta G}{dm_1} = k_1\left(\left(\frac{m_1+\frac{m_2}{2}-\lambda_1}{\lambda_2-\frac{m_2}{2}}\right)^{k_1-1} - \left(\frac{m_1-\frac{\lambda_1+\lambda_2-m_2}{2}}{\frac{\lambda_1+\lambda_2-m_2}{2}}\right)^{k_1-1}\right)$$

First we show that $\frac{d\Delta G}{dm_1} < 0$. We have:

$m_1+m_2 < \lambda_1+\lambda_2 \Rightarrow 2m_1+2m_2 < 2\lambda_1+2\lambda_2 \xrightarrow{\times(\lambda_1-\lambda_2)} 2m_1(\lambda_1-\lambda_2)+2m_2(\lambda_1-\lambda_2) < 2(\lambda_1+\lambda_2)(\lambda_1-\lambda_2) \Rightarrow 2m_1\lambda_1-2\lambda_1^2+2\lambda_1m_2 < 2m_1\lambda_2+2\lambda_2m_2-2\lambda_2^2 \Rightarrow (2m_1+m_2-2\lambda_1)(\lambda_1+\lambda_2-m_2) < (2m_1+m_2-\lambda_1-\lambda_2)(2\lambda_2-m_2) \Rightarrow (m_1+\frac{m_2}{2}-\lambda_1)(\frac{\lambda_1+\lambda_2-m_2}{2}) < (m_1-\frac{\lambda_1+\lambda_2-m_2}{2})(\lambda_2-$

$\frac{m_2}{2}) \Rightarrow (\frac{m_1+\frac{m_2}{2}-\lambda_1}{\lambda_2-\frac{m_2}{2}}) < (\frac{m_1-\frac{\lambda_1+\lambda_2-m_2}{2}}{\frac{\lambda_1+\lambda_2-m_2}{2}}) \xrightarrow{k_1>1} (\frac{m_1+\frac{m_2}{2}-\lambda_1}{\lambda_2-\frac{m_2}{2}})^{k_1-1} < (\frac{m_1-\frac{\lambda_1+\lambda_2-m_2}{2}}{\frac{\lambda_1+\lambda_2-m_2}{2}})^{k_1-1} \Rightarrow \frac{d\Delta G}{dm_1} < 0.$

Therefore, as m_1 increases ΔG decreases. So, if we show that ΔG is positive when m_1 is equal to its upper limit then we can conclude that $\Delta G > 0 \Rightarrow G(R2) > G(R3)$. We have $2m_2 + m_1 < \lambda_1 + \lambda_2 \Rightarrow m_1 < \lambda_1 + \lambda_2 - 2m_2$. Now, since $\lambda_1 + \lambda_2 - m_2 > \lambda_1 + \lambda_2 - 2m_2$ if we show that $\Delta G_{m_1=\lambda_1+\lambda_2-m_2} \geq 0$ it implies that $\Delta G_{m_1=\lambda_1+\lambda_2-2m_2} > 0$.

$$\Delta G_{m_1=\lambda_1+\lambda_2-m_2} = \frac{\lambda_1-\lambda_2}{2} + \frac{(\lambda_1+\lambda_2+\frac{m_2}{2}-\lambda_1)^{k_1}}{(\lambda_2-\frac{m_2}{2})^{k_1-1}} - \frac{(\lambda_1+\lambda_2-\frac{\lambda_1+\lambda_2-m_2}{2})^{k_1}}{(\frac{\lambda_1+\lambda_2-m_2}{2})^{k_1-1}} = 0$$

Therefore, $\Delta G_{m_1=\lambda_1+\lambda_2-2m_2} > 0 \Rightarrow \Delta G > 0 \Rightarrow G(R2) > G(R3)$.

So far we have shown that amongst policies presented in Table A.4 only Policies 1 and 2 are eligible to be in $\mathcal{C}^*_{(P1,P1)}$. Through applying the same process we extract $\mathcal{C}^*$ policies for the next two cases in the followings.

Case 2: (P_1, P_2)**.**

The following table shows referral policies in $\mathcal{C}_{(P_1,P_2)}$ resulting in case (P_1, P_2):

Index	Referral Policy $(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$	$G(\Lambda)$
R1	$(\lambda_1 - m_2, \lambda_2 - m_2)$ (m_2, m_2)	$\lambda_1 + \frac{(m_1+m_2-\lambda_1)^{k_1}}{(\lambda_2-m_2)^{k_1-1}}$
R2	$(\lambda_1 - \frac{m_2}{2}, \lambda_2 - \frac{m_2}{2})$ $(\frac{m_2}{2}, \frac{m_2}{2})$	$\lambda_1 + \frac{m_2}{2} + \frac{(m_1+\frac{m_2}{2}-\lambda_1)^{k_1}}{(\lambda_2-\frac{m_2}{2})^{k_1-1}}$
R6	$(\lambda_1, \lambda_2 - m_2)$ $(0, m_2)$	$\lambda_1 + m_2 + \frac{(m_1-\lambda_1)^{k_1}}{(\lambda_2-m_2)^{k_1-1}}$

Table A.5: Boundary referral policies resulting in case (P_1, P_2)

It can be seen that the first two policies (i.e. R1 and R2) are the same as the policies in $\mathcal{C}^*_{(P1,P1)}$. Applying the same process we used to compare $G(R2)$ with $G(R3)$ in case $(P1, P1)$, it can be shown that $G(R6) > G(R2)$. Therefore, Policy R2 in $\mathcal{C}^*_{(P1,P1)}$ is not anymore a candidate policy and we can eliminate it from $\mathcal{C}^*_{(P1,P1)}$. So far, we have $\mathcal{C}^*_{(P1,P1)} = \{R1\}$ and $\mathcal{C}^*_{(P1,P2)} = \{R1, R6\}$ where Policy R1 is the common policy between the two sets.

Case 3: (P_2, P_1)**:** We now show that any policies in $C^*_{(P2,P1)}$ are dominated by other feasible policies. The following table shows referral policies in $\mathcal{C}_{(P_2,P_1)}$ resulting in case (P_2, P_1):

Index	Referral Policy $(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$	$G(\Lambda)$
R3	$(\frac{\lambda_1+\lambda_2-m_2}{2}, \frac{\lambda_1+\lambda_2-m_2}{2})$ $(\frac{\lambda_1-\lambda_2+m_2}{2}, \frac{\lambda_2-\lambda_1+m_2}{2})$	$\frac{m_2+\lambda_2+\lambda_1}{2} + \frac{(2m_1+m_2-\lambda_1-\lambda_2)^{k_1}}{2(\lambda_1+\lambda_2-m_2)^{k_1-1}}$
R5	$(\lambda_1 - m_2, \lambda_1 - m_2)$ $(m_2, \lambda_2 + m_2 - \lambda_1)$	$\lambda_1 + \frac{(m_1+m_2-\lambda_1)^{k_1}}{(\lambda_1-m_2)^{k_1-1}}$
R7	$(\lambda_1 - m_2, \lambda_2)$ $(m_2, 0)$	$\lambda_2 + m_2 + \frac{(m_1-\lambda_2)^{k_1}}{(\lambda_1-m_2)^{k_1-1}}$

Table A.6: Boundary solutions that result in case (P_2, P_1)

In the analysis of cases (P_1, P_1) and (P_1, P_2) we showed that $G(R6) > G(R2) > G(R3)$ and $G(R1) > G(R5)$. Therefore, R3 and R5 in Table A.6 cannot be optimal. Using the same process used to compare R2 and R3 in case $(P1, P1)$ it can be shown that $G(R6) > G(R7)$. Therefore, $C^*_{(P2,P1)}$ can be ignored.

Conclusion: From the above cases, R1 and R6 (policies 8 and 7 in Table A.1) are the only potential optimal policies. Comparing the objective values for these two policies leads directly to the condition EC1. It can be verified that there exist parameters leading to both policies being optimal. To conclude we have shown that in a centralized referral system where both providers are LOC if the capacity scenario is (High, Low) and $2m_2+m_1 < \lambda_1+\lambda_2$ then depending on the resulted objective value it is best for the referrer to apply one of the policies presented in the following table:

Index	Referral Policy $(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$	$G(\Lambda)$
R1	$(\lambda_1 - m_2, \lambda_2 - m_2)$ (m_2, m_2)	$\lambda_1 + \frac{(m_1+m_2-\lambda_1)^{k_1}}{(\lambda_2-m_2)^{k_1-1}}$
R6	$(\lambda_1, \lambda_2 - m_2)$ $(0, m_2)$	$\lambda_1 + m_2 + \frac{(m_1-\lambda_1)^{k_1}}{(\lambda_2-m_2)^{k_1-1}}$

Table A.7: Optimal Policies (Capacity scenario (High, Low), $2m_2 + m_1 < \lambda_1 + \lambda_2$)

□

Proof of Theorem 5. We use Lemma 2 to show that when capacity scenario is (Mid, Mid), policies mentioned in Table 2.5 are optimal.

Since the capacity scenario is (Mid, Mid) (i.e. $\lambda_2 < m_2 < m_1 < \lambda_1$) therefore $\lambda_{11} = \lambda_1 - m_2 > 0$ and $\lambda_{12} = \lambda_1 - m_1 > 0$. The following shows two possible different situations for the value of $2m_2 + m_1$:

1. $2m_2 + m_1 \geq \lambda_1 + \lambda_2$

2. $2m_2 + m_1 < \lambda_1 + \lambda_2$

From the assumptions we made in Section 2.3 we have $\lambda_1 < m_1 + m_2$ (i). In addition, in the capacity scenario (Mid, Mid) $\lambda_2 < m_2$ (ii). Adding (i) and (ii) results in $\lambda_1 + \lambda_2 < 2m_2 + m_1$ which implies that the second situation is not feasible. Therefore, the only feasible situation is $2m_2 + m_1 \geq \lambda_1 + \lambda_2$. For the first referral policy we have:

1. $\lambda_{11} + \lambda_{21} = m_1 \Rightarrow (x_{11}, x_{21}) = (1, 1)$

2. $\lambda_1 + \lambda_2 \leq 2m_2 + m_1 \Rightarrow \lambda_1 + \lambda_2 - m_1 - m_2 \leq m_2 \Rightarrow \lambda_{22} \leq \lambda_{12}$ which implies that the second provider best solution is P_1 and $(x_{12}, x_{22}) = (1, 0)$.

For the second policy we have:

1. $\lambda_{12} + \lambda_{22} = m_2 \Rightarrow (x_{12}, x_{22}) = (1, 1)$

2. Since $m_1 > m_2 \Rightarrow 2m_1 + m_2 > 2m_2 + m_1 \Rightarrow 2m_1 + m_2 > 2m_2 + m_1 > \lambda_1 + \lambda_2$. Now, for the second policy we have: $2m_1 + m_2 > \lambda_1 + \lambda_2 \Rightarrow m_1 > \lambda_1 + \lambda_2 - m_1 - m_2 \Rightarrow \lambda_{11} > \lambda_{21}$ which implies that the best solution for the first provider is P_1 and $(x_{11}, x_{21}) = (1, 0)$

Having target probabilities and referral rates, it is now easy to verify that under both policies we have $G(\Lambda) = m_1 + m_2$. □

Results for Section 2.3.2 (optimal referral policy)

In this section we present the proofs of Theorems 6 and 7.

Lemma 4. *With the exception of capacity scenario (Low,Low), in all other capacity scenarios we always have* $2m_1 + m_2 \geq \lambda_1 + \lambda_2$

Proof of Lemma 4. Based on our assumptions in Section 2.3 in all capacity scenarios except (Low,Low) we have $\lambda_1 < m_1 + m_2$ (i) and $\lambda_2 < m_1$ (ii). Adding (i) and (ii) together results in $2m_1 + m_2 \geq \lambda_1 + \lambda_2$. □

Lemma 5. *Given* $2m_1 + m_2 < \lambda_1 + \lambda_2$*, no feasible policy satisfies* $\lambda_{11} + \lambda_{12} = \lambda_1 + \lambda_2 - m_2$.

Proof of Lemma 5. First, based on Lemma 4 the only capacity scenario where it is possible to have $2m_1 + m_2 < \lambda_1 + \lambda_2$ is capacity scenario (Low,Low). Since we assumed that each provider has enough capacity to serve both types of patients allocated to her independently for provider 1 we have $\lambda_{11} \leq m_1$ and $\lambda_{21} \leq m_1$ and consequently $\lambda_{11} + \lambda_{21} \leq 2m_1$ (i). Since $2m_1 + m_2 < \lambda_1 + \lambda_2 \Rightarrow 2m_1 < \lambda_1 + \lambda_2 - m_2$ (ii). From (i) and (ii) we have $\lambda_{11} + \lambda_{21} < \lambda_1 + \lambda_2 - m_2$.

Therefore, in capacity scenario (Low, Low) if $2m_1+m_2 < \lambda_1+\lambda_2$ then there is no feasible policy that that satisfies $\lambda_{11} + \lambda_{12} = \lambda_1 + \lambda_2 - m_2$. □

Proof of Theorem 6-a. Let's define $k_1 = (\alpha_1)^{-1}$ and $k_2 = (\alpha_2)^{-1}$ where $0 < k_1 < 1$ and $k_2 > 1$. Since the first provider is HOC and applies the shared policy we have:

$$G(\Lambda) = \lambda_{11}x_{11} + \lambda_{12}x_{12} + \lambda_{21}x_{21} + \lambda_{22}x_{22} = (\lambda_{11} + \lambda_{21})(\frac{m_1}{\lambda_{11} + \lambda_{21}})^{k_1} + \lambda_{21}x_{21} + \lambda_{22}x_{22}$$

Since the optimal policy for the second provider is either P_1 or P_2 we need to show that in both cases the optimal policy for the referrer satisfies $\lambda_{11} + \lambda_{21} = \lambda_1 + \lambda_2 - m_2$. Let's define $G(S, P_1)$ and $G(S, P_2)$ as the referrer objective functions when the best solutions for the second provider are P_1 and P_2 respectively. We have:

$$G(S, P_1) = (\lambda_{11} + \lambda_{21})(\frac{m_1}{\lambda_{11} + \lambda_{21}})^{k_1} + \lambda_{21} + \lambda_{22}(\frac{m_2 - \lambda_{12}}{\lambda_{22}})^{k_2}$$
$$G(S, P_2) = (\lambda_{11} + \lambda_{21})(\frac{m_1}{\lambda_{11} + \lambda_{21}})^{k_1} + \lambda_{12}(\frac{m_2 - \lambda_{22}}{\lambda_{12}})^{k_2} + \lambda_{22}$$

Let's assume that the optimal policy satisfies $\lambda_{11} + \lambda_{21} = \lambda_1 + \lambda_2 - x$ and consequently $\lambda_{12} + \lambda_{22} = x$. Based on our assumptions in Section 2.3, for each provider j we have $m_j \leq \lambda_{1j} + \lambda_{2j}$ which implies $m_2 \leq x$ and $m_1 \leq \lambda_1 + \lambda_2 - x \Rightarrow x \leq \lambda_1 + \lambda_2 - m_1$. Therefore $m_2 \leq x \leq \lambda_1 + \lambda_2 - m_1$. We need to show that for any optimal policy, x is equal to m_2 no matter what the second provider solution is. Let's begin with $G(S, P_1)$. If $x = m_2$ then $G(S, P_1) = (\lambda_1 + \lambda_2 - m_2)^{1-k_1} m_1^{k_1} + m_2$. Now we need to show that this is the maximum achievable value for $G(S, P_1)$ and as x increases and moves toward its upper limit $G(S, P_1)$ decreases.

Since $k_2 > 1$ we can rewrite $G(S, P_1)$ as follow:

$$G(S, P_1) = (\lambda_1 + \lambda_2 - x)^{1-k_1} m_1^{k_1} + \lambda_{12} + \frac{(m_2 - \lambda_{12})^{k_2}}{(\lambda_{22})^{k_2-1}}$$

Where $\lambda_{i2} \leq m_2 \leq \sum_i \lambda_{i2}$. Let's split the function into two terms where the first term is $(\lambda_1 + \lambda_2 - x)^{1-k_1} m_1^{k_1}$ and the second term is $\lambda_{12} + \frac{(m_2-\lambda_{12})^{k_2}}{(\lambda_{22})^{k_2-1}}$. It is obvious that as x

increases the first term decreases. Now, let's focus on the second term. Since $\lambda_{12} + \lambda_{22} = x$ if x increases both λ_{12} and λ_{22} can increase. It can be seen that if λ_{22} increases the second term and consequently $G(S, P_1)$ decreases. Therefore, in the best policy, λ_{22} should get the minimum feasible value. Therefore, if x is increased, it is best to increase λ_{12} and not λ_{22}. Now, let's assume that we increase x by ϵ. From $m_2 \leq x \leq \lambda_1 + \lambda_2 - m_1$ we have $x = m_2 + \epsilon$. As $\lambda_{12} \leq m_2$ the maximum value that can be allocated to λ_{12} is m_2 and consequently $\lambda_{22} = \epsilon$. Now from the fact that if λ_{12} increases $G(S, P_1)$ decreases we can conclude that if x increases $G(S, P_1)$ decreases and consequently the optimal policy is to set $x = m_2$. Finally, for the optimal referral policy (i.e. Λ) we have $\lambda_{11} + \lambda_{21} = \lambda_1 + \lambda_2 - m_2$, $\lambda_{12} = m_2$ and $\lambda_{22} = 0$.

The same process can be done for $G(S, P_2)$. However the difference is that as λ_{12} increases $G(S, P_2)$ decreases. Therefore, for the optimal policy we have $\lambda_{11} + \lambda_{21} = \lambda_1 + \lambda_2 - m_2$, $\lambda_{12} = 0$ and $\lambda_{22} = m_2$.

From Lemma 4 and 5 as well as our explanations above we can conclude that in a centralized system where the first provider is HOC , in all capacity scenarios except when both the capacity scenario is (Low, Low) and $2m_1 + m_2 < \lambda_1 + \lambda_2$, all the feasible policies that satisfy $\lambda_{11} + \lambda_{21} = \lambda_1 + \lambda_2 - m_2$ are optimal. □

Proof of Theorem 6-b. Based on Lemma 5 in capacity scenario (Low, Low) if $2m_1 + m_2 < \lambda_1 + \lambda_2$ then it is not possible to achieve the highest achievable objective value for the referrer which is $\lambda_{11} + \lambda_{12} = \lambda_1 + \lambda_2 - m_2$. The process of finding optimal policies for this situation is similar to the one we used to prove Theorem 4. The second provider's policy can be either P_1 or P_2. Therefore, depending on the second provider's policy there can be two cases (S, P_1) and (S, P_2). First, we simplify referrer objective function in for case by finding the optimal value of one of the streams to each provider. Having the simplified (Λ), we then prove that in each case we need to focus on the boundary referral points. Let's define $\mathcal{L}$ as the set of boundary referral points for capacity scenario (Low, Low) when $2m_1 + m_2 < \lambda_1 + \lambda_2$. For each case, we find $\mathcal{C} \in \mathcal{L}$ which defines the boundary referral points that result in that case. Based on the optimality conditions extracted, for each case, we then find $\mathcal{C}^*$ which eliminates dominated coordinated policies for that scheduling policy. Finally, we compare objective value of policies in $\mathcal{C}^*$ which results in two potential policies which may be optimal depending on Equation 2.22.

Let's begin with $G(S, P_1)$ and define $k_1 = (\alpha_1)^{-1}$ and $k_2 = (\alpha_2)^{-1}$ where $0 < k_1 < 1$ and $k_2 > 1$.

$$G(S, P_1) = (\lambda_{11} + \lambda_{21})(\tfrac{m_1}{\lambda_{11}+\lambda_{21}})^{k_1} + \lambda_{12} + \lambda_{22}(\tfrac{m_2-\lambda_{12}}{\lambda_{22}})^{k_2} = (\lambda_{11} + \lambda_{21})^{1-k_1} m_1^{k_1} + \lambda_1 - \lambda_{11} + \tfrac{(m_2-\lambda_1+\lambda_{11})^{k_2}}{(\lambda_2-\lambda_{21})^{k_2-1}}$$

It can be seen that as λ_{21} increases $G(S, P_1)$ also increases. Since $\lambda_{21} < m_1 < \lambda_2$, in the optimal policy λ_{21} is equal to its upper limit which is m_1 and consequently $\lambda^*_{22} = \lambda_2 - m_1$. Therefore, $G(S, P_1)$ can be rewritten as follows:

$$G(S, P_1) = (\lambda_{11} + m_1)^{1-k_1} m_1^{k_1} + \lambda_1 - \lambda_{11} + \tfrac{(m_2-\lambda_1+\lambda_{11})^{k_2}}{(\lambda_2-m_1)^{k_2-1}} = (\lambda_1 + m_1 - \lambda_{12})^{1-k_1} m_1^{k_1} + \lambda_{12} + \tfrac{(m_2-\lambda_{12})^{k_2}}{(\lambda_2-m_1)^{k_2-1}}$$

Since $\lambda_{11} \le m_1 \Rightarrow \lambda_1 - m_1 \le \lambda_{12}$. So, for the $G(S, P_1)$ we have:

$$\begin{cases} G(S, P_1) = (\lambda_1 + m_1 - \lambda_{12})^{1-k_1} m_1^{k_1} + \lambda_{12} + \frac{(m_2-\lambda_{12})^{k_2}}{(\lambda_2-m_1)^{k_2-1}} \\ \lambda_1 - m_1 \le \lambda_{12} \le m_2 \end{cases}$$

Now, it all depends on the value of λ_{12}. The same process can be done for $G(S, P_2)$ and for the optimal policy in case (S, P_2) we have $\lambda^*_{12} = \lambda_1 - m_1$ and:

$$\begin{cases} G(S, P_2) = (\lambda_2 + m_1 - \lambda_{22})^{1-k_1} m_1^{k_1} + \lambda_{22} + \frac{(m_2-\lambda_{22})^{k_2}}{(\lambda_1-m_1)^{k_2-1}} \\ \lambda_1 - m_1 \le \lambda_{22} \le m_2 \end{cases}$$

Now, we prove that in both cases we need to focus on the boundary points. To do so, we begin with $G(S, P_1)$ and the same logic can be applied for the second case. We have:

$$\frac{dG(S, P_1)}{d\lambda_{12}} = -k_2 \left(\frac{m_2 - \lambda_{12}}{\lambda_2 - m_1}\right)^{k_2-1} - (1 - k_1)\left(\frac{m_1}{\lambda_1 + m_1 - \lambda_{12}}\right)^{k_1} + 1$$

If we show that only three following situation for $\frac{dG(S,P_1)}{d\lambda_{12}}$ are possible then we can conclude that in order to maximize $G(S, P_1)$ we only need to focus on the boundary points.

1. Strictly positive
2. Strictly negative
3. First negative then positive

To do so, we prove that it is not possible for $\frac{dG(S,P_1)}{d\lambda_{12}}$ to move from positive to negative. If we set $k_1 = k_2 = 1$ then $\frac{dG(S,P_1)}{d\lambda_{12}} = 0$. In addition, as k_1 decreases $\frac{dG(S,P_1)}{d\lambda_{12}}$ decreases as well. Therefore, if we keep $k_2 = 1$ and reduce $0 < k_1 < 1$ then $\frac{dG(S,P_1)}{d\lambda_{12}} < 0$. Let's define $B = k_2(\frac{m_2-\lambda_{12}}{\lambda_2-m_1})^{k_2-1}$ where $k_2 > 1$ and $0 < \frac{m_2-\lambda_{12}}{\lambda_2-m_1} < 1$. We have:

$$\frac{dB}{dk_2} = \left(\frac{m_2 - \lambda_{12}}{\lambda_2 - m_1}\right)^{k_2-1} \left(k_2 ln\left(\frac{m_2 - \lambda_{12}}{\lambda_2 - m_1}\right) + 1\right) \tag{A.1}$$

It can be verified that the maximum value of B can be achieved if $k_2 = \frac{-1}{ln(\frac{m_2-\lambda_{12}}{\lambda_2-m_1})}$. Note that since $0 < \frac{m_2-\lambda_{12}}{\lambda_2-m_1} < 1$, $ln(\frac{m_2-\lambda_{12}}{\lambda_2-m_1}) < 0$ and $\frac{-1}{ln(\frac{m_2-\lambda_{12}}{\lambda_2-m_1})} > 0$. Two cases can be considered:

1. $\frac{-1}{ln(\frac{m_2-\lambda_{12}}{\lambda_2-m_1})} < 1 < k_2$: In this case $\frac{dB}{dk_2} > 0$. Therefore, as k_2 increases B increases as well and $\frac{dG(S,P_1)}{d\lambda_{12}}$ decreases.

2. $\frac{-1}{ln(\frac{m_2-\lambda_{12}}{\lambda_2-m_1})} > 1$.

   (a) If $1 < k_2 < \frac{-1}{ln(\frac{m_2-\lambda_{12}}{\lambda_2-m_1})}$ then $\frac{dB}{dk_2} < 0$. Therefore, as k_2 increases B decreases and $\frac{dG(S,P_1)}{d\lambda_{12}}$ increases

   (b) If $k_2 > \frac{-1}{ln(\frac{m_2-\lambda_{12}}{\lambda_2-m_1})}$ then then $\frac{dB}{dk_2} > 0$. Therefore, as k_2 increases B increases as well and $\frac{dG(S,P_1)}{d\lambda_{12}}$ decreases

Previously we show that if $k_2 = 1$ and $0 < k_1 < 1$ then $\frac{dG(S,P_1)}{d\lambda_{12}} < 0$. Therefore, for case 2-a we have $\frac{-1}{ln(\frac{m_2-\lambda_{12}}{\lambda_2-m_1})} < 1 < k_2$ and therefore $\frac{dG(S,P_1)}{d\lambda_{12}}$ may remain negative or move from negative to positive. For case 1 and 2-b since as k_2 increases $\frac{dG(S,P_1)}{d\lambda_{12}}$ decreases we can conclude that when $1 < k_2 < \frac{-1}{ln(\frac{m_2-\lambda_{12}}{\lambda_2-m_1})}$ and $0 < k_1 < 1$ then $\frac{dG(S,P_1)}{d\lambda_{12}}$ remains negative. Therefore, we show that it is not possible for $\frac{dG(S,P_1)}{d\lambda_{12}}$ value to move from positive to negative and therefore in order to maximize $G(\Lambda)$ we always need to focus on the boundary points. The following table shows boundary referral points (i.e. $\mathcal{C}$) resulting in each case:

Case	Index	Boundary Referral Points $(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$
(S, P_1)	1	$(\lambda_1 - m_2, \lambda_2 - m_2)$ (m_2, m_2)
	2	$(\lambda_1 - m_2, m_1)$ $(m_2, \lambda_2 - m_1)$
	3	(m_1, m_1) $(\lambda_1 - m_1, \lambda_2 - m_1)$
	4	$(m_1, m_1 + \lambda_2 - \lambda_1)$ $(\lambda_1 - m_1, \lambda_1 - m_1)$
(S, P_2)	4	$(m_1, m_1 + \lambda_2 - \lambda_1)$ $(\lambda_1 - m_1, \lambda_1 - m_1)$
	5	$(m_1, \lambda_2 - m_2)$ $(\lambda_1 - m_1, m_2)$
	6	$(\lambda_1 - m_2, \lambda_2 - m_2)$ (m_2, m_2)

Table A.8: Boundary referral points resulting in each case (First provider is HOC)

Since we proved that in case (S, P_1) we have $\lambda_{22}^* = \lambda_2 - m_1$ and in case (S, P_2) $\lambda_{12}^* =$

$\lambda_1 - m_1$ therefore policies 1 and 4 in case (S, P_1) and policy 6 in case (S, P_2) can be ignored. The following table shows candidate referral policies (i.e. $\mathcal{C}^*$) resulting in each case.

Case	Index	Referral Policy $(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$	$G(\Lambda)$
S, P_1	R1	$(\lambda_1 - m_2, m_1)$ $(m_2, \lambda_2 - m_1)$	$(\lambda_1 + m_1 - m_2)^{1-k_1} m_1^{k_1} + m_2$
	R2	(m_1, m_1) $(\lambda_1 - m_1, \lambda_2 - m_1)$	$(2m_1)^{1-k_1} m_1^{k_1} +$ $\lambda_1 - m_1 + \frac{(m_1+m_2-\lambda_1)^{k_2}}{(\lambda_2-m_1)^{k_2-1}}$
S, P_2	R3	$(m_1, m_1 + \lambda_2 - \lambda_1)$ $(\lambda_1 - m_1, \lambda_1 - m_1)$	$(2m_1 + \lambda_2 - \lambda_1)^{1-k_1} m_1^{k_1} +$ $\lambda_1 - m_1 + \frac{(m_1+m_2-\lambda_1)^{k_2}}{(\lambda_1-m_1)^{k_2-1}}$
	R4	$(m_1, \lambda_2 - m_2)$ $(\lambda_1 - m_1, m_2)$	$(\lambda_2 + m_1 - m_2)^{1-k_1} m_1^{k_1} + m_2$

Table A.9: Candidate policies (First provider is HOC, capacity scenario (Low, Low), $2m_1 + m_2 < \lambda_1 + \lambda_2$)

Now we show that $G(R1) > G(R4)$ and $G(R2) > G(R3)$ and therefore in a centralized system where the first provider is HOC if the capacity scenario is (Low, Low) and $2m_1+m_2 < \lambda_1 + \lambda_2$ then the optimal policy for the referrer always result in case (S, P_1). First, let's begin with the comparison of $G(R1)$ and $G(R4)$:

$\lambda_1 > \lambda_2 \xrightarrow{+(m_1-m_2)} \lambda_1 + m_1 - m_2 > \lambda_2 + m_1 - m_2 \xrightarrow{0<k_1<1} (\lambda_1 + m_1 - m_2)^{1-k_1} > (\lambda_2 + m_1 - m_2)^{1-k_1} \xrightarrow{\times m_1^{k_1}} (\lambda_1 + m_1 - m_2)^{1-k_1} m_1^{k_1} > (\lambda_2 + m_1 - m_2)^{1-k_1} m_1^{k_1} \xrightarrow{+m_2} (\lambda_1 + m_1 - m_2)^{1-k_1} m_1^{k_1} + m_2 > (\lambda_2 + m_1 - m_2)^{1-k_1} m_1^{k_1} + m_2 \Rightarrow G(R1) > G(R4)$

For $G(R2)$ and $G(R3)$ we have:

$\lambda_2 - \lambda_1 < 0 \xrightarrow{+2m_1} \lambda_2 - \lambda_1 + 2m_1 < 2m_1 \xrightarrow{0<k_1<1} (\lambda_2 - \lambda_1 + 2m_1)^{1-k_1} < (2m_1)^{1-k_1} \xrightarrow{\times m_1^{k_1}} (\lambda_2 - \lambda_1 + 2m_1)^{1-k_1} m_1^{k_1} < (2m_1)^{1-k_1} m_1^{k_1} \xrightarrow{+\lambda_1 - m_1} \xrightarrow{\times m_1^{k_1}} (\lambda_2 - \lambda_1 + 2m_1)^{1-k_1} m_1^{k_1} + \lambda_1 - m_1 < (2m_1)^{1-k_1} m_1^{k_1} + \lambda_1 - m_1$ (i)

In addition:

$\lambda_1 > \lambda_2 \xrightarrow{-m_1} \lambda_1 - m_1 > \lambda_2 - m_1 \xrightarrow{k_2>1} (\lambda_1 - m_1)^{k_2-1} > (\lambda_2 - m_1)^{k_2-1} \Rightarrow \frac{1}{(\lambda_1-m_1)^{k_2-1}} < \frac{1}{(\lambda_2-m_1)^{k_2-1}}$
$\xrightarrow{\times(m_1+m_2-\lambda_1)^{k_2}} \frac{(m_1+m_2-\lambda_1)^{k_2}}{(\lambda_1-m_1)^{k_2-1}} < \frac{(m_1+m_2-\lambda_1)^{k_2}}{(\lambda_2-m_1)^{k_2-1}}$ (ii)

If we add (i) and (ii) we have:

$(\lambda_2 - \lambda_1 + 2m_1)^{1-k_1} m_1^{k_1} + \lambda_1 - m_1 + \frac{(m_1+m_2-\lambda_1)^{k_2}}{(\lambda_1-m_1)^{k_2-1}} < (2m_1)^{1-k_1} m_1^{k_1} + \lambda_1 - m_1 + \frac{(m_1+m_2-\lambda_1)^{k_2}}{(\lambda_2-m_1)^{k_2-1}} \Rightarrow G(R3) < G(R2)$

Therefore, depending on the resulted value for $G(\Lambda)$, the optimum policies for the capacity scenario (Low, Low) when the first provider is HOC and $2m_1 + m_2 < \lambda_1 + \lambda_2$ can be either of the policies in the following table:

Index	Optimal Policy $(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$	TP x_{i1} (x_{12}, x_{22})	Optimality Condition (Eq. 2.22)
1	$(\lambda_1 - m_2, m_1)$ $(m_2, \lambda_2 - m_1)$	$(\frac{m_1}{\lambda_1+m_1-m_2})^{(\alpha_1)^{-1}}$ $(1, 0)$	True
2	(m_1, m_1) $(\lambda_1 - m_1, \lambda_2 - m_1)$	$(\frac{1}{2})^{(\alpha_1)^{-1}}$ $(1, (\frac{m_2+m_1-\lambda_1}{\lambda_2-m_1})^{(\alpha_2)^{-1}})$	False

Table A.10: Optimal policies (First provider is HOC - Capacity scenario is (Low, Low) and $2m_1 + m_2 < \lambda_1 + \lambda_2$)

$\square$

Lemma 6. *In a centralized system where the second provider is HOC, if* $2m_2 + m_1 < \lambda_1 + \lambda_2$ *then it is not possible to have a policy that satisfies* $\lambda_{12} + \lambda_{22} = \lambda_1 + \lambda_2 - m_1$.

Proof of Lemma 6. Based on our assumptions in Section 2.3 we have $\lambda_{12} \leq m_2$ and $\lambda_{22} \leq m_2$ and consequently $\lambda_{12} + \lambda_{22} \leq 2m_2$. Since $2m_2 + m_1 < \lambda_1 + \lambda_2 \Rightarrow 2m_2 < \lambda_1 + \lambda_2 - m_1 \Rightarrow \lambda_{12} + \lambda_{22} < \lambda_1 + \lambda_2 - m_1$. $\square$

Proof of Theorem 7-a. The same process applied in Theorem 6-a can be applied here. Let's define $k_1 = (\alpha_1)^{-1}$ and $k_2 = (\alpha_2)^{-1}$ where $k_1 > 1$ and $0 < k_2 < 1$. Since the second provider is HOC we have:

$$G(\Lambda) = \lambda_{11}x_{11} + \lambda_{12}x_{12} + \lambda_{21}x_{21} + \lambda_{22}x_{22} = \lambda_{11}x_{11} + \lambda_{21}x_{21} + (\lambda_{12} + \lambda_{22})(\frac{m_2}{\lambda_{12} + \lambda_{22}})^{k_2}$$

Since the optimal policy for the first provider is either P_1 or P_2 we need to show that in both cases the optimal policy for the referrer satisfies $\lambda_{11} + \lambda_{21} = m_1$. Let's define $G(P_1, S)$ and $G(P_2, S)$ as the referrer functions where the best solutions for the first provider are P_1 and P_2 respectively. We have:

$$G(P_1, S) = \lambda_{11} + \lambda_{21}(\frac{m_1 - \lambda_{11}}{\lambda_{21}})^{k_1} + (\lambda_{12} + \lambda_{22})(\frac{m_2}{\lambda_{12} + \lambda_{22}})^{k_2}$$
$$G(P_2, S) = \lambda_{11}(\frac{m_1 - \lambda_{21}}{\lambda_{11}})^{k_1} + \lambda_{21} + (\lambda_{12} + \lambda_{22})(\frac{m_2}{\lambda_{12} + \lambda_{22}})^{k_2}$$

Let's assume that the optimal policy satisfies $\lambda_{11} + \lambda_{21} = x$ and consequently $\lambda_{12} + \lambda_{22} = \lambda_1 + \lambda_2 - x$. Based on our assumptions in Section 2.3, for any provider like j we have

$m_j \leq \lambda_{1j} + \lambda_{2j}$ which implies that $m_1 \leq x$ and $m_2 \leq \lambda_1 + \lambda_2 - x \Rightarrow x \leq \lambda_1 + \lambda_2 - m_2$. Therefore $m_1 \leq x \leq \lambda_1 + \lambda_2 - m_2$. We need to show that for any optimal policy, x is equal to m_1 no matter what the first provider solution is. Let's begin with $G(P_1, S)$.

If $x = m_1$ then $G(P_1, S) = m_1 + (\lambda_1 + \lambda_2 - m_1)(\frac{m_2}{\lambda_1+\lambda_2-m_1})^{k_2}$. Now we need to show that this is the maximum achievable value for $G(P_1, S)$ and as x increases and moves toward its upper limit $G(P_1, S)$ decreases.

Since $k_1 > 1$ and $0 < k_2 < 1$ we can rewrite $G(P_1, S)$ as follow:

$$G(P_1, S) = \lambda_{11} + \frac{(m_1 - \lambda_{11})^{k_1}}{(\lambda_{21})^{k_1-1}} + (\lambda_1 + \lambda_2 - x)^{1-k_2} m_2^{k_2}$$

Where $\lambda_{i1} \leq m_1 \leq \sum_i \lambda_{i1}$. Let's split the function into two terms where the first term is $(\lambda_1 + \lambda_2 - x)^{1-k_2} m_2^{k_2}$ and the second term is $\lambda_{11} + \frac{(m_1-\lambda_{11})^{k_1}}{(\lambda_{21})^{k_1-1}}$. It can easily be verified that as x increases the first term decreases. Now, let's focus on the second term. Since $\lambda_{11} + \lambda_{21} = x$ if x increases both λ_{11} and λ_{21} can increase. It can be seen that if λ_{21} increases the second term and consequently $G(P_1, S)$ decreases. Therefore, in the best policy, λ_{21} should get the minimum feasible value. Therefore, if x is increased, it is best to increase λ_{11} and not λ_{22}. Now, let's assume that we increase x by ϵ. From $m_1 \leq x \leq \lambda_1 + \lambda_2 - m_2$ we have $x = m_1 + \epsilon$. As $\lambda_{11} \leq m_1$ the maximum value that can be allocated to λ_{11} is m_1 and consequently $\lambda_{21} = \epsilon$. Now from the fact that if λ_{11} increases $G(P_1, S)$ decreases we can conclude that if x increases $G(P_1, S)$ decreases and consequently the optimal policy is to set $x = m_1$. Finally, for the optimal policy we have $\lambda_{12} + \lambda_{22} = \lambda_1 + \lambda_2 - m_1$, $\lambda_{11} = m_1$ and $\lambda_{21} = 0$.

The same process can be done for $G(P_2, S)$. However the difference is that as λ_{11} increases $G(P_2, S)$ decreases. Therefore, for the optimal policy $(\lambda_{11}, \lambda_{21}, \lambda_{12}, \lambda_{22})$ we have $\lambda_{12} + \lambda_{22} = \lambda_1 + \lambda_2 - m_1$, $\lambda_{11} = 0$ and $\lambda_{21} = m_1$.

From what is discussed above as well as Lemma 5 we can conclude that in a centralized system where the second provider is HOC and in all the capacity scenarios if $2m_2 + m_1 > \lambda_1 + \lambda_2$ then all the feasible policies that satisfy $\lambda_{11} + \lambda_{21} = m_1$ are optimal. □

Proof of Theorem 7-b. Based on Lemma 6 if $2m_2 + m_1 < \lambda_1 + \lambda_2$ then it is not possible to achieve the highest objective value for the referrer which is discussed in the proof of Theorem 7-a. Note that for the capacity scenario (Mid, Mid) we have $m_2 > \lambda_2$ and $m_1 + m_2 > \lambda_1$. If we add these two inequalities together it results in $2m_2 + m_1 > \lambda_1 + \lambda_2$. This implies that in the capacity scenario (Mid, Mid) the referrer is always able to achieve the highest objective value.

For this part of the theorem we use the same process applied in Theorem 6-b. For each capacity scenario we will take the following steps:

1. First we simplify the referrer objective function for each case through finding optimal values for specific streams to providers.
2. Then, we show that to find the optimal referral solutions for each case (P_1, S) and (P_2, S) we need to focus on the boundary referral points.
3. Boundary referral points resulting in each case (i.e. $\mathcal{C}$) are extracted.
4. For each case, policies in $\mathcal{C}$ are compared with each other to find the ones that result in higher $G(\Lambda)$ (i.e. $\mathcal{C}^*$).
5. Finally, policies in $\mathcal{C}^*$ in both cases are compared to each other to find the optimal referral policies for referrer.

Let's define $k_1 = (\alpha_1)^{-1}$ and $k_2 = (\alpha_2)^{-1}$ where $k_1 > 1$ and $0 < k_2 < 1$. The followings show the referrer objective function for each case:

$$G(P_1, S) = \lambda_{11} + \lambda_{21}(\tfrac{m_1-\lambda_{11}}{\lambda_{21}})^{k_1} + (\lambda_{12} + \lambda_{22})(\tfrac{m_2}{\lambda_{12}+\lambda_{22}})^{k_2} = \lambda_{11} + \tfrac{(m_1-\lambda_{11})^{k_1}}{(\lambda_{21})^{k_1-1}} + (\lambda_1 + \lambda_2 - \lambda_{11} - \lambda_{21})^{1-k_2} m_2^{k_2}$$
$$G(P_2, S) = \lambda_{11}(\tfrac{m_1-\lambda_{21}}{\lambda_{11}})^{k_1} + \lambda_{21} + (\lambda_{12} + \lambda_{22})(\tfrac{m_2}{\lambda_{12}+\lambda_{22}})^{k_2} = \lambda_{21} + \tfrac{(m_1-\lambda_{21})^{k_1}}{(\lambda_{11})^{k_1-1}} + (\lambda_1 + \lambda_2 - \lambda_{11} - \lambda_{21})^{1-k_2} m_2^{k_2}$$

For the case (P_1, S) it can be seen that as λ_{21} increases $G(P_1, S)$ decreases. Therefore, in the optimal solution λ_{21}^* should be equal to its lower limit which is $\lambda_2 - m_2$ ($\lambda_{22} \leq m_2 \Rightarrow \lambda_2 - m_2 \leq \lambda_{21}$). The same logic is true for λ_{11} in case (P_2, S). Therefore, $\lambda_{11}^* = \lambda_1 - m_2$ in the optimal solution for the case (P_2, S). Therefore, we have:

$$\begin{cases} G(P_1, S) = (\lambda_1 + m_2 - \lambda_{11})^{1-k_2} m_2^{k_2} + \lambda_{11} + \frac{(m_1-\lambda_{11})^{k_1}}{(\lambda_2-m_2)^{k_1-1}} \\ \lambda_1 - m_2 \leq \lambda_{11} \leq m_1 \end{cases}$$
$$\begin{cases} G(P_2, S) = \lambda_{21} + \frac{(m_1-\lambda_{21})^{k_1}}{(\lambda_1-m_2)^{k_1-1}} + (\lambda_2 + m_2 - \lambda_{21})^{1-k_2} m_2^{k_2} \\ \lambda_1 - m_2 \leq \lambda_{21} \leq m_1 \end{cases}$$

Now we show that to find optimal solutions in both cases we need to focus on the boundary referral points. We begin with $G(P_1, S)$ and the same logic can be applied for the other case.

$$\frac{dG(P_1, S)}{d\lambda_{11}} = -k_1(\frac{m_1 - \lambda_{11}}{\lambda_2 - m_2})^{k_1-1} - (1 - k_2)(\frac{m_2}{\lambda_1 + m_2 - \lambda_{11}})^{k_2} + 1$$

If we show that the only three situations for $\frac{dG(P_1,S)}{d\lambda_{11}}$ value are as follows then we can conclude that in order to maximize $G(P_1, S)$ we only need to focus on the boundary points.

1. Strictly positive

2. Strictly negative

3. First negative then positive

To do so, we prove that it is not possible for $\frac{dG(P_1,S)}{d\lambda_{11}}$ value to move from positive to negative. If we set $k_1 = k_2 = 1$ then $\frac{dG(P_1,S)}{d\lambda_{11}} = 0$. In addition, as k_2 decrease $\frac{dG(P_1,S)}{d\lambda_{11}}$ also decreases. Therefore, if we keep $k_1 = 1$ and reduce the value of k_2 (i.e. $0 < k_2 < 1$) then $\frac{dG(P_1,S)}{d\lambda_{11}} < 0$. Let's define $B = k_1(\frac{m_1-\lambda_{11}}{\lambda_2-m_2})^{k_1-1}$ where $k_1 > 1$ and $0 < \frac{m_1-\lambda_{11}}{\lambda_2-m_2} < 1$.

We have:

$$\frac{dB}{dk_1} = (\frac{m_1-\lambda_{11}}{\lambda_2-m_2})^{k_1-1}(k_1 ln(\frac{m_1-\lambda_{11}}{\lambda_2-m_2}) + 1) \tag{A.2}$$

It an be verified that the maximum value of B can be achieved if $k_1 = \frac{-1}{ln(\frac{m_1-\lambda_{11}}{\lambda_2-m_2})}$. Two cases can be considered:

1. $\frac{-1}{ln(\frac{m_1-\lambda_{11}}{\lambda_2-m_2})} < 1 < k_1$: In this case $\frac{dB}{dk_1} > 0$. Therefore, as k_1 increases B increases as well and $\frac{dG(P_1,S)}{d\lambda_{11}}$ decreases.

2. $\frac{-1}{ln(\frac{m_1-\lambda_{11}}{\lambda_2-m_2})} > 1$.

   (a) If $1 < k_1 < \frac{-1}{ln(\frac{m_1-\lambda_{11}}{\lambda_2-m_2})}$ then $\frac{dB}{dk_1} < 0$. Therefore, as k_1 increases B decreases and $\frac{dG(P_1,S)}{d\lambda_{11}}$ increases

   (b) If $k_1 > \frac{-1}{ln(\frac{m_1-\lambda_{11}}{\lambda_2-m_2})}$ then then $\frac{dB}{dk_1} > 0$. Therefore, as k_1 increases B increases as well and $\frac{dG(P_1,S)}{d\lambda_{11}}$ decreases

Previously we show that if $k_1 = 1$ and $0 < k_2 < 1$ then $\frac{dG(P_1,S)}{d\lambda_{11}} < 0$. Therefore, for case 2-a where $1 < k_1 < \frac{-1}{ln(\frac{m_1-\lambda_{11}}{\lambda_2-m_2})}$, $\frac{dG(P_1,S)}{d\lambda_{11}}$ may remain negative or move from negative to positive. For cases 1 and 2-b since as k_1 increases $\frac{dG(P_1,S)}{d\lambda_{11}}$ decreases we can conclude that when $1 < k_1$ and $0 < k_2 < 1$, $\frac{dG(P_1,S)}{d\lambda_{11}}$ remains negative. Therefore, we show that it is not possible for $\frac{dG(P_1,S)}{d\lambda_{11}}$ value to move from positive to negative and therefore we always need to focus on the boundary points. Now we focus on extracting boundary referral points that result in each case (i.e. $\mathcal{C}$).

We begin begin with the capacity scenario (Low, Low). The following table shows boundary referral points resulting in each case when the capacity scenario is (Low, Low) and $2m_2 + m_1 < \lambda_1 + \lambda_2$:

Case	Index	Referral Policy $(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$
(P_1, S)	R1	$(\lambda_1 - m_2, \lambda_2 - m_2)$ (m_2, m_2)
	R2	$(\frac{\lambda_1+\lambda_2-m_2}{2}, \frac{\lambda_1+\lambda_2-m_2}{2})$ $(\frac{\lambda_1+m_2-\lambda_2}{2}, \frac{\lambda_2+m_2-\lambda_1}{2})$
	R3	$(\lambda_1 - m_2, \lambda_1 - m_2)$ $(m_2, \lambda_2 + m_2 - \lambda_1)$
	R4	$(m_1, \lambda_1 + \lambda_2 - m_1 - m_2)$ $(\lambda_1 - m_1, m_1 + m_2 - \lambda_1)$
	R5	$(m_1, \lambda_2 - m_2)$ $(\lambda_1 - m_1, m_2)$
	R6	(m_1, m_1) $(\lambda_1 - m_1, \lambda_2 - m_1)$
(P_2, S)	R2	$(\frac{\lambda_1+\lambda_2-m_2}{2}, \frac{\lambda_1+\lambda_2-m_2}{2})$ $(\frac{\lambda_1+m_2-\lambda_2}{2}, \frac{\lambda_2+m_2-\lambda_1}{2})$
	R3	$(\lambda_1 - m_2, \lambda_1 - m_2)$ $(m_2, \lambda_2 + m_2 - \lambda_1)$
	R7	$(\lambda_1 + \lambda_2 - m_1 - m_2, m_1)$ $(m_1 + m_2 - \lambda_2, \lambda_2 - m_1)$
	R8	$(\lambda_1 - m_2, m_1)$ $(m_2, \lambda_2 - m_1)$
	R6	(m_1, m_1) $(\lambda_1 - m_1, \lambda_2 - m_1)$

Table A.11: Boundary referral points (Second provider is HOC, capacity scenario (Low, Low), $2m_2+m_1 < \lambda_1 + \lambda_2$)

Based on optimality conditions we extracted in the first step, policies R2, R3, R4 and R6 in case (P_1, S) and policies R2, R7 and R6 in case (P_2, S) cannot be ignored. Therefore, policies R1 and R5 in case (P_1, S) and policies R3 and R8 in case (P_2, S) are the candidate solutions for the referrer (i.e. these policies are in $\mathcal{C}^*$). Now, we show that $G(R1)_{(P_1,S)} > G(R3)_{(P_2,S)}$ and $G(R5)_{(P_1,S)} > G(R8)_{(P_2,S)}$ and therefore in a centralized system where the second provider is HOC if the capacity scenario is (Low, Low) and $2m_2 + m_1 < \lambda_1 + \lambda_2$ then the optimal policy for the referrer always results in case (P_1, S). Let's compare $G(R1)_{(P_1,S)}$ with $G(R3)_{(P_2,S)}$:

$$G(R1)_{(P_1,S)} = \lambda_1 - m_2 + \frac{(m_1 + m_2 - \lambda_1)^{k_1}}{(\lambda_2 - m_2)^{k_1-1}} + (2m_2)^{1-k_2} m_2^{k_2}$$
$$G(R3)_{(P_2,S)} = \lambda_1 - m_2 + \frac{(m_1 + m_2 - \lambda_1)^{k_1}}{(\lambda_1 - m_2)^{k_1-1}} + (\lambda_2 + 2m_2 - \lambda_1)^{1-k_2} m_2^{k_2}$$

Since $\lambda_1 > \lambda_2$ we have $\frac{(m_1+m_2-\lambda_1)^{k_1}}{(\lambda_1-m_2)^{k_1-1}} < \frac{(m_1+m_2-\lambda_1)^{k_1}}{(\lambda_2-m_2)^{k_1-1}}$ (i) and $(\lambda_2 + 2m_2 - \lambda_1)^{1-k_2} m_2^{k_2} < (2m_2)^{1-k_2}$ (ii). From (i) and (ii) we can conclude $G(R1)_{(P_1,S)} > G(R3)_{(P_2,S)}$. For $G(R5)_{(P_1,S)}$ and $G(R8)_{(P_2,S)}$ we have:

$$G(R5)_{(P_1,S)} = m_1 + (\lambda_1 + m_2 - m_1)^{1-k_2} m_2^{k_2}$$
$$G(R8)_{(P_2,S)} = m_1 + (\lambda_2 + m_2 - m_1)^{1-k_2} m_2^{k_2}$$

Again, since $\lambda_1 > \lambda_2 \Rightarrow (\lambda_1 + m_2 - m_1)^{1-k_2} m_2^{k_2} > (\lambda_2 + m_2 - m_1)^{1-k_2} m_2^{k_2} \Rightarrow G(R5)_{(P_1,S)} > G(R8)_{(P_2,S)}$.

Doing the same process for the capacity scenario (Mid, Low) we have:

Case	Index	Referral Policy $(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$
(P_1, S)	R1	(λ_2, λ_2) $(\lambda_1 - \lambda_2, 0)$
	R2	$(\lambda_1 - m_2, \lambda_2 - m_2)$ (m_2, m_2)
	R3	$(\frac{\lambda_1+\lambda_2-m_2}{2}, \frac{\lambda_1+\lambda_2-m_2}{2})$ $(\frac{\lambda_1+m_2-\lambda_2}{2}, \frac{\lambda_2+m_2-\lambda_1}{2})$
	R4	$(\lambda_1 - m_2, \lambda_1 - m_2)$ $(m_2, \lambda_2 + m_2 - \lambda_1)$
	R5	$(m_1, \lambda_1 + \lambda_2 - m_1 - m_2)$ $(\lambda_1 - m_1, m_1 + m_2 - \lambda_1)$
	R6	$(m_1, \lambda_2 - m_2)$ $(\lambda_1 - m_2, m_2)$
	R7	$(\lambda_1 - m_2, \lambda_2)$ $(m_2, 0)$
(P_2, S)	R8	(λ_2, λ_2) $(\lambda_1 - \lambda_2, 0)$
	R3	$(\frac{\lambda_1+\lambda_2-m_2}{2}, \frac{\lambda_1+\lambda_2-m_2}{2})$ $(\frac{\lambda_1+m_2-\lambda_2}{2}, \frac{\lambda_2+m_2-\lambda_1}{2})$
	R4	$(\lambda_1 - m_2, \lambda_1 - m_2)$ $(m_2, \lambda_2 + m_2 - \lambda_1)$
	R7	$(\lambda_1 - m_2, \lambda_2)$ $(m_2, 0)$

Table A.12: Boundary referral points (Second provider is HOC, capacity scenario: (Mid, Low), $2m_2+m_1 < \lambda_1 + \lambda_2$)

Based on the logic we used before, the only policies that are potential to be optimal (i.e. policies in $\mathcal{C}^*$) are policies R2 and R6 in the case (P_1, S) and policies R4 and R7 in the case (P_2, S). Now we show that $G(R2)_{(P_1,S)} > G(R4)_{(P_2,S)}$ and $G(R6)_{(P_1,S)} > G(R7)_{(P_2,S)}$ and therefore like the capacity scenario (Low, Low), the optimal policy for the referrer always results in the case (P_1, S). We have:

$$G(R2)_{(P_1,S)} = \lambda_1 - m_2 + \frac{(m_1 + m_2 - \lambda_1)^{k_1}}{(\lambda_2 - m_2)^{k_1 - 1}} + (2m_2)^{1-k_2} m_2^{k_2}$$

$$G(R4)_{(P_2,S)} = \lambda_1 - m_2 + \frac{(m_1 + m_2 - \lambda_1)^{k_1}}{(\lambda_1 - m_2)^{k_1 - 1}} + (\lambda_2 + 2m_2 - \lambda_1)^{1-k_2} m_2^{k_2}$$

Since $\lambda_1 > \lambda_2$ we have $\frac{(m_1+m_2-\lambda_1)^{k_1}}{(\lambda_1-m_2)^{k_1-1}} < \frac{(m_1+m_2-\lambda_1)^{k_1}}{(\lambda_2-m_2)^{k_1-1}}$ (i) and $(\lambda_2 + 2m_2 - \lambda_1)^{1-k_2} m_2^{k_2} <$

$(2m_2)^{1-k_2}$ (ii). From (i) and (ii) it can be concluded that $G(R2)_{(P_1,S)} > G(R4)_{(P_2,S)}$. For $G(R6)_{(P_1,S)}$ and $G(R7)_{(P_2,S)}$ we have:

$$G(R6)_{(P_1,S)} = m_1 + (\lambda_1 + m_2 - m_1)^{1-k_2} m_2^{k_2}$$
$$G(R7)_{(P_2,S)} = \lambda_2 + \frac{(m_1 - \lambda_2)^{k_1}}{(\lambda_1 - m_2)^{k_1-1}} + (m_2)^{1-k_2} m_2$$

Now we show that $G(R6)_{(P_1,S)} > G(R7)_{(P_2,S)}$:

$\lambda_1 > m_1 \Rightarrow \lambda_1 - m_1 > 0 \xrightarrow{+m_2} \lambda_1 + m_2 - m_1 > m_2 \xrightarrow{0<k_2<1} (\lambda_1 + m_2 - m_1)^{1-k_2} > (m_2)^{1-k_2} \xrightarrow{\times m_2^{k_2}} m_2^{k_2}(\lambda_1 + m_2 - m_1)^{1-k_2} > m_2^{k_2}(m_2)^{1-k_2}$ (i)

$\lambda_1 + \lambda_2 > m_1 + m_2 \Rightarrow \lambda_1 - m_2 > m_1 - \lambda_2 \Rightarrow 1 > (\frac{m_1-\lambda_2}{\lambda_1-m_2}) \xrightarrow{k_1>1} 1 > (\frac{m_1-\lambda_2}{\lambda_1-m_2})^{k_1-1} \Rightarrow (m_1 - \lambda_2) > \frac{(m_1-\lambda_2)^{k_1}}{(\lambda_1-m_2)^{k_1-1}} \Rightarrow m_1 > \lambda_2 + \frac{(m_1-\lambda_2)^{k_1}}{(\lambda_1-m_2)^{k_1-1}}$ (ii)

If we add (i) and (ii) together it results in $m_1 + (\lambda_1 + m_2 - m_1)^{1-k_2} m_2^{k_2} > \lambda_2 + \frac{(m_1-\lambda_2)^{k_1}}{(\lambda_1-m_2)^{k_1-1}} + (m_2)^{1-k_2} m_2 \Rightarrow G(R6)_{(P_1,S)} > G(R7)_{(P_2,S)}$.

Finally for the capacity scenario (High, Low) we have:

Case	Index	Referral Policy $(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$
(P_1, S)	R1	(λ_2, λ_2) $(\lambda_1 - \lambda_2, 0)$
	R2	$(\lambda_1 - m_2, \lambda_2 - m_2)$ (m_2, m_2)
	R3	$(\frac{\lambda_1+\lambda_2-m_2}{2}, \frac{\lambda_1+\lambda_2-m_2}{2})$ $(\frac{\lambda_1+m_2-\lambda_2}{2}, \frac{\lambda_2+m_2-\lambda_1}{2})$
	R4	$(\lambda_1 - m_2, \lambda_1 - m_2)$ $(m_2, \lambda_2 + m_2 - \lambda_1)$
	R5	$(\lambda_1, \lambda_2 - m_2)$ $(0, m_2)$
	R7	$(\lambda_1 - m_2, \lambda_2)$ $(m_2, 0)$
(P_2, S)	R1	(λ_2, λ_2) $(\lambda_1 - \lambda_2, 0)$
	R3	$(\frac{\lambda_1+\lambda_2-m_2}{2}, \frac{\lambda_1+\lambda_2-m_2}{2})$ $(\frac{\lambda_1+m_2-\lambda_2}{2}, \frac{\lambda_2+m_2-\lambda_1}{2})$
	R4	$(\lambda_1 - m_2, \lambda_1 - m_2)$ $(m_2, \lambda_2 + m_2 - \lambda_1)$
	R7	$(\lambda_1 - m_2, \lambda_2)$ $(m_2, 0)$

Table A.13: Boundary points (Second provider is HOC, Capacity scenario(High, Low), $2m_2+m_1 < \lambda_1+\lambda_2$)

The only policies that remain potential to be optimal for the referrer are policies 2 and 5 in case (P_1, S) and policies 4 and 7 in case (P_2, S). In the previous capacity scenarios we proved that $G(R2)_{(P_1,S)} > G(R4)_{(P_2,S)}$. Now we show that $G(R5)_{(P_1,S)} > G(R7)_{(P_2,S)}$ and like the previous capacity scenario, the optimal policy for the referrer always results in case (P_1, S).

$$G(R5)_{(P_1,S)} = \lambda_1 + \frac{(m_1 - \lambda_1)^{k_1}}{(\lambda_2 - m_2)^{k_1-1}} + m_2$$
$$G(R7)_{(P_2,S)} = \lambda_2 + \frac{(m_1 - \lambda_2)^{k_1}}{(\lambda_1 - m_2)^{k_1-1}} + m_2$$

Let's define:

$$\Delta G = G(R5)_{(P_1,S)} - G(R7)_{(P_2,S)} = \lambda_1 - \lambda_2 + \frac{(m_1 - \lambda_1)^{k_1}}{(\lambda_2 - m_2)^{k_1-1}} - \frac{(m_1 - \lambda_2)^{k_1}}{(\lambda_1 - m_2)^{k_1-1}}$$

Where $m_2 < \lambda_2 < \lambda_1 < m_1 < m_1 + m_2 < \lambda_1 + \lambda_2$ and $2m_2 + m_1 < \lambda_1 + \lambda_2$. Therefore, $m_1 \in [\lambda_1, \lambda_1 + \lambda_2 - 2m_2]$

If we set $m_1 = \lambda_1$ then $\Delta G = \lambda_1 - \lambda_2 - \frac{(\lambda_1-\lambda_2)^{k_1}}{(\lambda_1-m_2)^{k_1-1}}$. First we prove that if we set m_1 equal to its lower limit then $\Delta G > 0$.

$m_2 < \lambda_2 \Rightarrow -m_2 > -\lambda_2 \xrightarrow{+\lambda_1} \lambda_1 - m_2 > \lambda_1 - \lambda_2 \Rightarrow 1 > \frac{\lambda_1-\lambda_2}{\lambda_1-m_2} \xrightarrow{k_1>1} 1 > (\frac{\lambda_1-\lambda_2}{\lambda_1-m_2})^{k_1-1} \Rightarrow \lambda_1 - \lambda_2 > \frac{(\lambda_1-\lambda_2)^{k_1}}{(\lambda_1-m_2)^{k_1-1}} \Rightarrow \lambda_1 - \lambda_2 - \frac{(\lambda_1-\lambda_2)^{k_1}}{(\lambda_1-m_2)^{k_1-1}} > 0 \Rightarrow \Delta G_{m_1=\lambda_1} > 0$

Now, let's take the derivative of ΔG with respect to m_1:

$$\frac{d\Delta G}{dm_1} = (k_1 - 1)(\frac{m_1 - \lambda_1}{\lambda_2 - m_2})^{k_1-2} - (k_1 - 1)(\frac{m_1 - \lambda_2}{\lambda_1 - m_2})^{k_1-2}$$

Let's consider two scenarios where in the first scenario $1 < k_1 < 2$ and in the second scenario $k_1 > 2$. For the first scenario since $1 < k_2 < 2$ we can rewrite $\frac{d\Delta G}{dm_1}$ as follows:

$$\frac{d\Delta G}{dm_1} = (k_1 - 1)(\frac{\lambda_2 - m_2}{m_1 - \lambda_1})^{2-k_1} - (k_1 - 1)(\frac{\lambda_1 - m_2}{m_1 - \lambda_2})^{2-k_1}$$

Now we have:

$\lambda_1 + \lambda_2 > m_1 + m_2 \xrightarrow{\times(m_1-m_2)} (\lambda_1 + \lambda_2)(m_1 - m_2) > m_1^2 - m_2^2 \Rightarrow \lambda_2 m_1 - \lambda_2 m_2 + \lambda_1 m_1 - \lambda_1 m_2 > m_1^2 - m_2^2 \xrightarrow{+\lambda_1\lambda_2} \lambda_2 m_1 - \lambda_2 m_2 + \lambda_1 m_1 - \lambda_1 m_2 + \lambda_1\lambda_2 > m_1^2 - m_2^2 + \lambda_1\lambda_2 \Rightarrow \lambda_2\lambda_1 - \lambda_1 m_2 - \lambda_2 m_2 + m_2^2 > m_1^2 - m_1\lambda_2 - \lambda_1 m_1 + \lambda_1\lambda_2 \Rightarrow \frac{\lambda_2-m_2}{m_1-\lambda_1} > \frac{m_1-\lambda_2}{\lambda_1-m_2} \xrightarrow{1<k_1<2} (\frac{\lambda_2-m_2}{m_1-\lambda_1})^{2-k_1} > (\frac{m_1-\lambda_2}{\lambda_1-m_2})^{2-k_1} \xrightarrow{\times(k_1-1)} (k_1 - 1)(\frac{\lambda_2-m_2}{m_1-\lambda_1})^{2-k_1} > (k_1 - 1)(\frac{m_1-\lambda_2}{\lambda_1-m_2})^{2-k_1} \Rightarrow \frac{d\Delta G}{dm_1} > 0$.

Therefore, if $1 < k_2 < 2$ as m_1 increases ΔG also increases. Since we also showed that $\Delta G_{m_1=lower-limit} > 0$ therefore as a conclusion we can say that if $1 < k_2 < 2$ then $\Delta G > 0$. Now we prove that even if $k_2 > 1$ then $\Delta G > 0$ and therefore we can conclude that $G(R5)_{(P_1,S)}$ is always greater than $G(R7)_{(P_2,S)}$.

For $k_1 > 2$ we have:

$$\frac{d\Delta G}{dm_1} = (k_1 - 1)(\frac{m_1 - \lambda_1}{\lambda_2 - m_2})^{k_1-2} - (k_1 - 1)(\frac{m_1 - \lambda_2}{\lambda_1 - m_2})^{k_1-2}$$

$m_1 + m_2 < \lambda_1 + \lambda_2 \xrightarrow{\times(m_1-m_2)} m_1^2 - m_2^2 < (\lambda_1 + \lambda_2)(m_1 - m_2) \xrightarrow{+\lambda_1\lambda_2} m_1^2 - m_2^2 + \lambda_1\lambda_2 < (\lambda_1 + \lambda_2)(m_1 - m_2) + \lambda_1\lambda_2 \Rightarrow \frac{m_1-\lambda_1}{\lambda_2-m_2} < \frac{m_1-\lambda_2}{\lambda_1-m_2} \xrightarrow{k_1>2} (\frac{m_1-\lambda_1}{\lambda_2-m_2})^{k_1-2} < (\frac{m_1-\lambda_2}{\lambda_1-m_2})^{k_1-2} \xrightarrow{\times(k_1-1)} (k_1 - 1)(\frac{m_1-\lambda_1}{\lambda_2-m_2})^{k_1-2} < (k_1 - 1)(\frac{m_1-\lambda_2}{\lambda_1-m_2})^{k_1-2} \Rightarrow \frac{d\Delta G}{dm_1} < 0$. Therefore, if $k_1 > 2$ then as m_1 increases ΔG decreases. Now if we show that $\Delta G > 0$ when m_1 is equal to its upper limit then we can conclude that ΔG is greater than zero for all $m_1 \in [\lambda_1, \lambda_1 + \lambda_2 - 2m_2]$. Since the capacity scenario is (High, Low) we have $m_2 < \lambda_2 < \lambda_1 < m_1 < m_1 + m_2 < \lambda_1 + \lambda_2$. We know that $\lambda_1 + \lambda_2 > \lambda_1 + \lambda_2 - 2m_2$. Since $\frac{d\Delta G}{dm_1} < 0$ if we prove that $\Delta G_{m_1=\lambda_1+\lambda_2} > 0$ then we can conclude that $\Delta G_{m_1=\lambda_1+\lambda_2-2m_2} > 0$. Let's assume that the higher limit for m_1

is $\lambda_1 + \lambda_2$ and prove that $\Delta G_{m_1=\lambda_1+\lambda_2} > 0$.

$$\Delta G_{m_1=\lambda_1+\lambda_2} = \lambda_1 - \lambda_2 + \frac{\lambda_2^{k_1}}{(\lambda_2-m_2)^{k_1-1}} - \frac{\lambda_1^{k_1}}{(\lambda_1-m_2)^{k_1-1}} = \lambda_1(1-(\frac{\lambda_1}{\lambda_1-m_2})^{k_1-1}) - \lambda_2(1-(\frac{\lambda_2}{\lambda_2-m_2})^{k_1-1})$$

Let's define $A = \lambda_1(1-(\frac{\lambda_1}{\lambda_1-m_2})^{k_1-1})$ and $B = \lambda_2(1-(\frac{\lambda_2}{\lambda_2-m_2})^{k_1-1})$. Therefore, $\Delta G_{m_1=\lambda_1+\lambda_2} = A - B$. We have:

$\lambda_2 < \lambda_1 \xrightarrow{\times m_2} \lambda_2 m_2 < \lambda_1 m_2 \Rightarrow -\lambda_1 m_2 < -\lambda_2 m_2 \xrightarrow{+\lambda_1\lambda_2} \lambda_1\lambda_2 - \lambda_1 m_2 < \lambda_1\lambda_2 - \lambda_2 m_2 \Rightarrow \frac{\lambda_1}{\lambda_1-m_2} < \frac{\lambda_2}{\lambda_2-m_2} \xrightarrow{k_1-1} (\frac{\lambda_1}{\lambda_1-m_2})^{k_1-1} < (\frac{\lambda_2}{\lambda_2-m_2})^{k_1-1} \Rightarrow -(\frac{\lambda_2}{\lambda_2-m_2})^{k_1-1} < -(\frac{\lambda_1}{\lambda_1-m_2})^{k_1-1} \xrightarrow{+1} 1-(\frac{\lambda_2}{\lambda_2-m_2})^{k_1-1} < 1-(\frac{\lambda_1}{\lambda_1-m_2})^{k_1-1}$ (i)

In addition, we have $\lambda_2 < \lambda_1$ (ii).

Multiplying (i) and (ii) results in $\lambda_2(1-(\frac{\lambda_2}{\lambda_2-m_2})^{k_1-1}) < \lambda_1(1-(\frac{\lambda_1}{\lambda_1-m_2})^{k_1-1}) \Rightarrow \Delta G_{m_1=\lambda_1+\lambda_2} > 0$. Therefore, we have proved that $\Delta G > 0$ and consequently $G(R5)_{(P_1,S)} > G(R7)_{(P_2,S)}$.

We proved that the optimal policy and target probabilities for different capacity scenarios when the second provider is HOC and $2m_2 + m_1 < \lambda_1 + \lambda_2$ are as follows:

Capacity Scenario	Index	Optimal Policy $(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$	TP (x_{11}, x_{21}) x_{i2}	Optimality Condition
(Low, Low) (Mid, Low)	1	$(\lambda_1 - m_2, \lambda_2 - m_2)$ (m_2, m_2)	$(1, (\frac{m_1+m_2-\lambda_1}{\lambda_2-m_2})^{(\alpha_1)^{-1}})$ $(\frac{1}{2})^{(\alpha_2)^{-1}}$	Eq. 2.24 True
	2	$(m_1, \lambda_2 - m_2)$ $(\lambda_1 - m_1, m_2)$	$(1, 0)$ $(\frac{m_2}{\lambda_1+m_2-m_1})^{(\alpha_2)^{-1}}$	Eq. 2.24 False
(High, Low)	1	$(\lambda_1 - m_2, \lambda_2 - m_2)$ (m_2, m_2)	$(1, (\frac{m_1+m_2-\lambda_1}{\lambda_2-m_2})^{(\alpha_1)^{-1}})$ $(\frac{1}{2})^{(\alpha_2)^{-1}}$	Eq. 2.25 True
	3	$(\lambda_1, \lambda_2 - m_2)$ $(0, m_2)$	$(1, (\frac{m_1-\lambda_1}{\lambda_2-m_2})^{(\alpha_1)^{-1}})$ 1	Eq. 2.25 False

Table A.14: Optimal policies (Second provider is HOC and $2m_2 + m_1 < \lambda_1 + \lambda_2$)

□

A.2.1 Fair-Allocation Model Analysis

In this section we extract optimal referral policies for a Fair-allocation referral network. A summary the optimal policies can be found at the end of the section.

Lemma 7. *Consider a referral system where both providers are LOC and fairness is an important factor in the system. The referrer objective function in this system is convex.*

Proof of Lemma 7. The following shows $G(\Lambda)$ where $k_1 = (\alpha_1)^{-1}$ and $k_2 = (\alpha_2)^{-1}$ and $k_1, k_2 > 1$.

$$G(\Lambda) = (\lambda_{11} + \lambda_{21})(\frac{m_1}{\lambda_{11} + \lambda_{21}})^{k_1} + (\lambda_{12} + \lambda_{22})(\frac{m_2}{\lambda_{12} + \lambda_{22}})^{k_2} =$$
$$\frac{m_1^{k_1}}{(\lambda_{11} + \lambda_{21})^{k_1 - 1}} + \frac{m_2^{k_2}}{(\lambda_{12} + \lambda_{22})^{k_2 - 1}} = \frac{m_1^{k_1}}{(\lambda_{11} + \lambda_{21})^{k_1 - 1}} + \frac{m_2^{k_2}}{(\lambda_1 + \lambda_2 - \lambda_{11} - \lambda_{21})^{k_2 - 1}}$$

And:

$$\frac{d^2F}{d\lambda_{11}^2} = \frac{d^2F}{d\lambda_{21}^2} = \frac{d^2F}{d\lambda_{11}\lambda_{21}} = k_1(k_1 - 1)\frac{m_1^{k_1}}{(\lambda_{11} + \lambda_{21})^{k_1 + 1}} + k_2(k_2 - 1)\frac{m_2^{k_2}}{(\lambda_{12} + \lambda_{22})^{k_2 + 1}}$$

Since $\frac{d^2F}{d\lambda_{11}^2} = \frac{d^2F}{d\lambda_{21}^2} = \frac{d^2F}{d\lambda_{11}\lambda_{21}}$ and all of them are positive we can conclude that the Hessian matrix is positive and therefore $G(\Lambda)$ is convex. $\square$

Lemma 7 implies that in order to find the maximum value for the $G(\Lambda)$ we should focus on boundary points.

Optimal Referral Policies Capacity Scenario (Low, Low)

The following table shows boundary points when the capacity scenario is (Low, Low) and $\lambda_1 + \lambda_2 > 2m_1 + m_2$:

Index	Policy $(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$	$G(\Lambda)$
1	$(m_1, \lambda_2 - m_2)$ $(\lambda_1 - m_1, m_2)$	$\frac{m_1^{k_1}}{(m_1+\lambda_2-m_2)^{k_1-1}} + \frac{m_2^{k_2}}{(\lambda_1+m_2-m_1)^{k_2-1}}$
2	(m_1, m_1) $(\lambda_1 - m_1, \lambda_2 - m_1)$	$\frac{m_1^{k_1}}{(2m_1)^{k_1-1}} + \frac{m_2^{k_2}}{(\lambda_1+\lambda_2-2m_1)^{k_2-1}}$
3	$(\lambda_1 - m_2, \lambda_2 - m_2)$ (m_2, m_2)	$\frac{m_1^{k_1}}{(\lambda_1+\lambda_2-2m_2)^{k_1-1}} + \frac{m_2^{k_2}}{(2m_2)^{k_2-1}}$
4	$(\lambda_1 - m_2, m_1)$ $(m_2, \lambda_1 - m_1)$	$\frac{m_1^{k_1}}{(\lambda_1+m_1-m_2)^{k_1-1}} + \frac{m_2^{k_2}}{(\lambda_1+m_2-m_1)^{k_2-1}}$

Table A.15: Boundary points (Capacity scenario (Low, Low), $\lambda_1 + \lambda_2 > 2m_1 + m_2$)

Now based on KKT conditions for the $G(\Lambda)$ we show that the optimal policy for the referrer is either Policy 2 or Policy 3. Let's first compare $G(policy-1)$ and $G(policy-4)$:

$$\Delta G = m_1^{k_1}\left(\frac{1}{(m_1+\lambda_2-m_2)^{k_1-1}} - \frac{1}{(\lambda_1+m_1-m_2)^{k_1-1}}\right)$$

Since $\lambda_2 < \lambda_1 \xrightarrow{+(m_1-m_2)} m_1 - m_2 + \lambda_2 < m_1 - m_2 + \lambda_1 \Rightarrow \frac{1}{m_1-m_2+\lambda_2} > \frac{1}{m_1-m_2+\lambda_1} \Rightarrow \frac{1}{(m_1-m_2+\lambda_2)^{k_1-1}} > \frac{1}{(m_1-m_2+\lambda_1)^{k_1-1}} \Rightarrow \Delta G > 0 \Rightarrow G(policy-1) > G(policy-4)$. Now we need to prove that Policy 1 cannot be optimal. In other words, we need to show that $max(G(policy-2), G(policy-3)) > G(policy-1)$.

The following shows the referrer problem when the capacity scenario is (Low, Low):

$$\begin{aligned}
&Max \frac{m_1^{k_1}}{(\lambda_{11}+\lambda_{21})^{k_1-1}} + \frac{m_2^{k_2}}{(\lambda_1+\lambda_2-\lambda_{11}-\lambda_{21})^{k_2-1}} \\
&St: \\
&\lambda_{11} - m_1 \leq 0 \\
&\lambda_{21} - m_1 \leq 0 \\
&m_1 - \lambda_{11} - \lambda_{21} \leq 0 \\
&\lambda_1 - m_2 - \lambda_{11} \leq 0 \\
&\lambda_2 - m_2 - \lambda_{21} \leq 0 \\
&\lambda_{11} + \lambda_{21} - \lambda_1 - \lambda_2 + m_2 \leq 0
\end{aligned}$$

It should be mentioned that we ignored two conditions $\lambda_{11} \leq \lambda_1$ and $\lambda_{21} \leq \lambda_2$ as in the capacity scenario (Low, Low) m_1 and m_2 are both lower than λ_1 and λ_2.

KKT conditions for the problem are as follows:

1. $\frac{dG}{d\lambda_{11}} = (k_2-1)(\frac{m_2}{\lambda_1+\lambda_2-\lambda_{11}-\lambda_{21}})^{k_2} - (k_1-1)(\frac{m_1}{\lambda_{11}+\lambda_{21}})^{k_1} = u_1 - u_3 - u_4 + u_6$
2. $\frac{dG}{d\lambda_{21}} = (k_2-1)(\frac{m_2}{\lambda_1+\lambda_2-\lambda_{11}-\lambda_{21}})^{k_2} - (k_1-1)(\frac{m_1}{\lambda_{11}+\lambda_{21}})^{k_1} = u_2 - u_3 - u_5 + u_6$
3. $u_1(\lambda_{11} - m_1) = 0$
4. $u_2(\lambda_{21} - m_1) = 0$
5. $u_3(m_1 - \lambda_{11} - \lambda_{21}) = 0$
6. $u_4(\lambda_1 - m_2 - \lambda_{11}) = 0$
7. $u_5(\lambda_2 - m_2 - \lambda_{21}) = 0$

8. $u_6(\lambda_{11}+\lambda_{21}-\lambda_1-\lambda_2+m_2)=0$

From the first and second conditions we have $u_1-u_4=u_2-u_5$.

Where $u_i\geq 0, i=1,...,6$ are Lagrangian multipliers. Since the objective function is convex and the problem is a maximization problem as mentioned earlier we need to focus on the boundary points.

Let's focus on the Policy 1 we have:

$$\begin{cases}\lambda_{11}=m_1\\ \lambda_{21}=\lambda_2-m_2\end{cases}\Rightarrow\begin{cases}u_1\begin{cases}=0\\ \neq 0\end{cases}\\ u_2=0\\ u_3=0\\ u_4=0\\ u_5\begin{cases}=0\\ \neq 0\end{cases}\\ u_6=0\end{cases}$$

Since $u_1-u_4=u_2-u_5$ and $u_2=u_4=0$ we have $u_1=-u_5$. Since $u_i\geq 0$ therefore the only possible situation is $u_1=u_5=0$. This results in $u_1=u_2=...=u_6$. Therefore, from the first and second conditions we have:

$(k_2-1)(\frac{m_2}{\lambda_1+\lambda_2-\lambda_{11}-\lambda_{21}})^{k_2}-(k_1-1)(\frac{m_1}{\lambda_{11}+\lambda_{21}})^{k_1}=0\Rightarrow(k_2-1)(\frac{m_2}{\lambda_1+\lambda_2-\lambda_{11}-\lambda_{21}})^{k_2}=(k_1-1)(\frac{m_1}{\lambda_{11}+\lambda_{21}})^{k_1}$ (i)

Therefore, Policy 1 can be considered as a potential solution if equation (i) holds. Now, let's assume that $(k_2-1)(\frac{m_2}{\lambda_1+\lambda_2-\lambda_{11}-\lambda_{21}})^{k_2}=(k_1-1)(\frac{m_1}{\lambda_{11}+\lambda_{21}})^{k_1}$ and see if under this condition Policy 1 can be optimal:

$(k_2-1)(\frac{m_2}{\lambda_1+\lambda_2-\lambda_{11}-\lambda_{21}})^{k_2}=(k_1-1)(\frac{m_1}{\lambda_{11}+\lambda_{21}})^{k_1}\Rightarrow(\frac{m_2}{\lambda_1+\lambda_2-\lambda_{11}-\lambda_{21}})^{k_2}=\frac{k_1-1}{k_2-1}(\frac{m_1}{\lambda_{11}+\lambda_{21}})^{k_1}\Rightarrow G_m(\Lambda)=(\lambda_{11}+\lambda_{21}+(\frac{k_1-1}{k_2-1})(\lambda_{12}+\lambda_{22}))(\frac{m_1}{\lambda_{11}+\lambda_{21}})^{k_1}$. Where $G_m(\Lambda)$ indicates the fact that $G(\Lambda)$ is calculated under the the condition $(k_2-1)(\frac{m_2}{\lambda_1+\lambda_2-\lambda_{11}-\lambda_{21}})^{k_2}=(k_1-1)(\frac{m_1}{\lambda_{11}+\lambda_{21}})^{k_1}$.

Therefore:

$$G_m(policy-1)=(m_1+\lambda_2-m_2+(\frac{k_1-1}{k_2-1})(\lambda_1+m_2-m_1))(\frac{m_1}{m_1+\lambda_2-m_2})^{k_1}$$
$$G_m(policy-2)=(2m_1+(\frac{k_1-1}{k_2-1})(\lambda_1+\lambda_2-2m_1))(\frac{m_1}{2m_1})^{k_1}$$
$$G_m(policy-3)=(\lambda_1+\lambda_2-2m_2+(\frac{k_1-1}{k_2-1})(2m_2))(\frac{m_1}{\lambda_1+\lambda_2-2m_2})^{k_1}$$

Let's consider two scenarios where in the first scenario $k_1\geq k_2$ and in the second scenario $k_1<k_2$. In the following we show that if $k_1\geq k_2$ then $G_m(policy-3)>G_m(policy-1)>G_m(policy-2)$. Let's first compare $G_m(policy-1)$ and $G_m(policy-2)$:

1. $\lambda_2 < m_1 + m_2 \Rightarrow m_1 + \lambda_2 - m_2 < 2m_1 \Rightarrow \frac{1}{m_1+\lambda_2-m_2} > \frac{1}{2m_1} \Rightarrow \frac{m_1}{m_1+\lambda_2-m_2} > \frac{m_1}{2m_1} \Rightarrow (\frac{m_1}{m_1+\lambda_2-m_2})^{k_1} > (\frac{m_1}{2m_1})^{k_1}$

2. $k_1 > k_2 \Rightarrow k_1 - 1 > k_2 - 1 \Rightarrow \frac{k_1-1}{k_2-1} > 1 \Rightarrow (\frac{k_1-1}{k_2-1})(m_1 + m_2 - \lambda_2) > (m_1 + m_2 - \lambda_2) \Rightarrow (\frac{k_1-1}{k_2-1})((\lambda_1 + m_2 - m_1) - (\lambda_1 + \lambda_2 - 2m_1)) > (2m_1) - (m_1 + \lambda_2 - m_2) \Rightarrow (m_1 + \lambda_2 - m_2 + (\frac{k_1-1}{k_2-1})(\lambda_1 + m_2 - m_1)) > (2m_1 + (\frac{k_1-1}{k_2-1})(\lambda_1 + \lambda_2 - 2m_1))$

If we multiply 1 and 2 we have:

$(m_1+\lambda_2-m_2+(\frac{k_1-1}{k_2-1})(\lambda_1+m_2-m_1))(\frac{m_1}{m_1+\lambda_2-m_2})^{k_1} > (2m_1+(\frac{k_1-1}{k_2-1})(\lambda_1+\lambda_2-2m_1))(\frac{m_1}{2m_1})^{k_1} \Rightarrow G_m(policy-1) > G_m(policy-2)$.

Now, let's compare $G_m(policy-1)$ and $G_m(policy-3)$:

1. $\lambda_1 < m_1 + m_2 \Rightarrow \lambda_1 + \lambda_2 - 2m_2 < m_1 + \lambda_2 - m_2 \Rightarrow \frac{1}{\lambda_1+\lambda_2-2m_2} > \frac{1}{m_1+\lambda_2-m_2} \Rightarrow \frac{m_1}{\lambda_1+\lambda_2-2m_2} > \frac{m_1}{m_1+\lambda_2-m_2} \Rightarrow (\frac{m_1}{\lambda_1+\lambda_2-2m_2})^{k_1} > (\frac{m_1}{m_1+\lambda_2-m_2})^{k_1}$

2. $k_1 > k_2 \Rightarrow k_1 - 1 > k_2 - 1 \Rightarrow \frac{k_1-1}{k_2-1} > 1 \Rightarrow (\frac{k_1-1}{k_2-1})(m_1 + m_2 - \lambda_1) > (m_1 + m_2 - \lambda_1) \Rightarrow (\frac{k_1-1}{k_2-1})((2m_2) - (\lambda_1 + m_2 - m_1)) > (m_1 + \lambda_2 - m_2) - (\lambda_1 + \lambda_2 - 2m_2) \Rightarrow (\lambda_1 + \lambda_2 - 2m_2 + (\frac{k_1-1}{k_2-1})(2m_2)) > (m_1 + \lambda_2 - m_2 + (\frac{k_1-1}{k_2-1})(\lambda_1 + m_2 - m_1)) \Rightarrow G_m(policy-3) > G_m(policy-1)$.

So far we have shown that if $k_1 \geq k_2$ then $G_m(policy-3) > G_m(policy-1) > G_m(policy-2)$ and therefore Policy 1 cannot be optimal under this condition. Let's now focus on the second scenario where $k_1 < k_2$.

Let's define:

$\Delta G = G_m(policy-3) - G_m(policy-1) = (\lambda_1 + \lambda_2 - 2m_2 + (\frac{k_1-1}{k_2-1})(2m_2))(\frac{m_1}{\lambda_1+\lambda_2-2m_2})^{k_1} - (m_1 + \lambda_2 - m_2 + (\frac{k_1-1}{k_2-1})(\lambda_1 + m_2 - m_1))(\frac{m_1}{m_1+\lambda_2-m_2})^{k_1}$

We have:

$$\frac{d\Delta G}{dk_2} = \frac{(\lambda_1 + m_2 - m_1)(\frac{m_1}{m_1+\lambda_2-m_2})^{k_1} - (2m_2)(\frac{m_1}{\lambda_1+\lambda_2-2m_2})^{k_1}}{(k_2-1)^2}(k_1 - 1)$$

Now we show that $\frac{d\Delta G}{dk_2} < 0$ and consequently as k_2 increases ΔG decreases.

1. $\lambda_1 < m_1 + m_2 \Rightarrow \lambda_1 + m_2 - m_1 < 2m_2$

2. $\lambda_1 < m_1 + m_2 \Rightarrow \lambda_1 + \lambda_2 - 2m_2 < m_1 + \lambda_2 - m_2 \Rightarrow \frac{1}{\lambda_1+\lambda_2-2m_2} > \frac{1}{m_1+\lambda_2-m_2} \xrightarrow{\times m_1} \frac{m_1}{\lambda_1+\lambda_2-2m_2} > \frac{m_1}{m_1+\lambda_2-m_2} \Rightarrow (\frac{m_1}{\lambda_1+\lambda_2-2m_2})^{k_1} > (\frac{m_1}{m_1+\lambda_2-m_2})^{k_1}$

From (1) and (2) we have:

$$(2m_2)(\frac{m_1}{\lambda_1 + \lambda_2 - 2m_2})^{k_1} > (\lambda_1 + m_2 - m_1)(\frac{m_1}{m_1 + \lambda_2 - m_2})^{k_1} \Rightarrow \frac{d\Delta G}{dk_2} < 0$$

Since $k_2 > 1$ if we show that ΔG is positive when $k_2 \to \infty$ then we can conclude that $\Delta G > 0 \Rightarrow G_m(policy-3) > G_m(policy-1)$ and therefore Policy 1 cannot be optimal at all. If $k_2 \to \infty$ then we have:

$$\Delta G_{k_2\to\infty} = (\lambda_1+\lambda_2-2m_2)(\frac{m_1}{\lambda_1+\lambda_2-2m_2})^{k_1} - (m_1+\lambda_2-m_2)(\frac{m_1}{m_1+\lambda_2-m_2})^{k_1}$$

We now show that $\Delta G_{k_2\to\infty} > 0 \Rightarrow \Delta G > 0 \Rightarrow G_m(policy-3) > G_m(policy-1)$.

$\lambda_1 < m_1+m_2 \Rightarrow \lambda_1+\lambda_2-2m_2 < m_1+\lambda_2-m_2 \Rightarrow \frac{\lambda_1+\lambda_2-2m_2}{m_1+\lambda_2-m_2} < 1$

Since $\frac{\lambda_1+\lambda_2-2m_2}{m_1+\lambda_2-m_2} < 1$ and $k_1 > 1$ we have $\frac{\lambda_1+\lambda_2-2m_2}{m_1+\lambda_2-m_2} > (\frac{\lambda_1+\lambda_2-2m_2}{m_1+\lambda_2-m_2})^{k_1} \Rightarrow \frac{\lambda_1+\lambda_2-2m_2}{m_1+\lambda_2-m_2} > (\frac{\frac{m_1}{m_1+\lambda_2-m_2}}{\frac{m_1}{\lambda_1+\lambda_2-2m_2}})^{k_1} \Rightarrow (\lambda_1+\lambda_2-2m_2)(\frac{m_1}{\lambda_1+\lambda_2-2m_2})^{k_1} > (m_1+\lambda_2-m_2)(\frac{m_1}{m_1+\lambda_2-m_2})^{k_1} \Rightarrow \Delta G_{k_2\to\infty} > 0$

The following table shows boundary points when the capacity scenario is (Low, Low) and $2m_2+m_1 < \lambda_1+\lambda_2 < 2m_1+m_2$:

Index	Policy $(\lambda_{11},\lambda_{21})$ $(\lambda_{12},\lambda_{22})$	$G(\Lambda)$
1	(m_1,λ_2-m_2) (λ_1-m_1,m_2)	$\frac{m_1^{k_1}}{(m_1+\lambda_2-m_2)^{k_1-1}}+\frac{m_2^{k_2}}{(\lambda_1+m_2-m_1)^{k_2-1}}$
2	$(m_1,\lambda_1+\lambda_2-m_1-m_2)$ $(\lambda_1-m_1,m_1+m_2-\lambda_1)$	$m_2+\frac{m_1^{k_1}}{(\lambda_1+\lambda_2-m_2)^{k_1-1}}$
3	$(\lambda_1+\lambda_2-m_1-m_2,m_1)$ $(m_1+m_2-\lambda_2,\lambda_2-m_1)$	$m_2+\frac{m_1^{k_1}}{(\lambda_1+\lambda_2-m_2)^{k_1-1}}$
4	$(\lambda_1-m_2,\lambda_2-m_2)$ (m_2,m_2)	$\frac{m_1^{k_1}}{(\lambda_1+\lambda_2-2m_2)^{k_1-1}}+\frac{m_2^{k_2}}{(2m_2)^{k_2-1}}$
5	(λ_1-m_2,m_1) (m_2,λ_1-m_1)	$\frac{m_1^{k_1}}{(\lambda_1+m_1-m_2)^{k_1-1}}+\frac{m_2^{k_2}}{(\lambda_1+m_2-m_1)^{k_2-1}}$

Table A.16: Boundary points (Capacity scenario (Low, Low), $2m_2+m_1 < \lambda_1+\lambda_2 < 2m_1+m_2$)

Previously we proved that policies 1 and 5 cannot be optimal. In addition, $G(policy-2) = G(policy-3)$. Therefore, depending on which policy results in the highest objective value for the referrer each one of the policies 2, 3 and 4 can be optimal.

The following table shows boundary points when the capacity scenario is (Low, Low) and $\lambda_1+\lambda_2 < 2m_2+m_1$:

Index	Policy $(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$	$G(\Lambda)$
1	$(m_1, \lambda_2 - m_2)$ $(\lambda_1 - m_1, m_2)$	$\frac{m_1^{k_1}}{(m_1+\lambda_2-m_2)^{k_1-1}} + \frac{m_2^{k_2}}{(\lambda_1+m_2-m_1)^{k_2-1}}$
2	$(m_1, \lambda_1 + \lambda_2 - m_1 - m_2)$ $(\lambda_1 - m_1, m_1 + m_2 - \lambda_1)$	$m_2 + \frac{m_1^{k_1}}{(\lambda_1+\lambda_2-m_2)^{k_1-1}}$
3	$(m_1 + m_2 - \lambda_2, \lambda_2 - m_2)$ $(\lambda_1 + \lambda_2 - m_1 - m_2, m_2)$	$m_1 + \frac{m_2^{k_2}}{(\lambda_1+\lambda_2-m_1)^{k_2-1}}$
4	$(\lambda_1 - m_2, m_1 + m_2 - \lambda_1)$ $(m_2, \lambda_1 + \lambda_2 - m_1 - m_2)$	$m_1 + \frac{m_2^{k_2}}{(\lambda_1+\lambda_2-m_1)^{k_2-1}}$
5	$(\lambda_1 - m_2, m_1)$ $(m_2, \lambda_1 - m_1)$	$\frac{m_1^{k_1}}{(\lambda_1+m_1-m_2)^{k_1-1}} + \frac{m_2^{k_2}}{(\lambda_1+m_2-m_1)^{k_2-1}}$
6	$(\lambda_1 + \lambda_2 - m_1 - m_2, m_1)$ $(m_1 + m_2 - \lambda_2, \lambda_2 - m_1)$	$m_2 + \frac{m_1^{k_1}}{(\lambda_1+\lambda_2-m_2)^{k_1-1}}$

Table A.17: Boundary points (Capacity scenario (Low, Low), $\lambda_1 + \lambda_2 < 2m_2 + m_1$)

Again policies 1 and 5 can be ignored. In addition, $G(policy-2) = G(policy-6)$ and $G(policy-3) = G(policy-4)$. Therefore, depending on which policy results in the highest objective value each one of the policies 2, 3, 4 and 6 can be optimal.

Optimal Referral Policies Capacity Scenario (Mid, Low)

Now let's focus on the capacity scenario (Mid, Low). In comparison with the referrer problem for the capacity scenario (Low, Low), the only constraint that needs to be changed is the second constraint. The new condition is $\lambda_{21} \leq \lambda_2$.

Therefore, KKT conditions for the problem are:

1. $\frac{dG}{d\lambda_{11}} = (k_2-1)(\frac{m_2}{\lambda_1+\lambda_2-\lambda_{11}-\lambda_{21}})^{k_2} - (k_1-1)(\frac{m_1}{\lambda_{11}+\lambda_{21}})^{k_1} = u_1 - u_3 - u_4 + u_6$

2. $\frac{dG}{d\lambda_{21}} = (k_2-1)(\frac{m_2}{\lambda_1+\lambda_2-\lambda_{11}-\lambda_{21}})^{k_2} - (k_1-1)(\frac{m_1}{\lambda_{11}+\lambda_{21}})^{k_1} = u_2 - u_3 - u_5 + u_6$

3. $u_1(\lambda_{11} - m_1) = 0$

4. $u_2(\lambda_{21} - \lambda_2) = 0$

5. $u_3(m_1 - \lambda_{11} - \lambda_{21}) = 0$

6. $u_4(\lambda_1 - m_2 - \lambda_{11}) = 0$

7. $u_5(\lambda_2 - m_2 - \lambda_{21}) = 0$

8. $u_6(\lambda_{11} + \lambda_{21} - \lambda_1 - \lambda_2 + m_2) = 0$

The following table shows boundary points when the capacity scenario is (Mid, Low) and $\lambda_1 + \lambda_2 > 2m_2 + m_1$:

Index	Policy $(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$	$G(\Lambda)$
1	$(m_1, \lambda_2 - m_2)$ $(\lambda_1 - m_1, m_2)$	$\frac{m_1^{k_1}}{(m_1+\lambda_2-m_2)^{k_1-1}} + \frac{m_2^{k_2}}{(\lambda_1+m_2-m_1)^{k_2-1}}$
2	$(m_1, \lambda_1 + \lambda_2 - m_1 - m_2)$ $(\lambda_1 - m_1, m_1 + m_2 - \lambda_1)$	$m_2 + \frac{m_1^{k_1}}{(\lambda_1+\lambda_2-m_2)^{k_1-1}}$
3	$(\lambda_1 - m_2, \lambda_2)$ $(m_2, 0)$	$m_2 + \frac{m_1^{k_1}}{(\lambda_1+\lambda_2-m_2)^{k_1-1}}$
4	$(\lambda_1 - m_2, \lambda_2 - m_2)$ (m_2, m_2)	$\frac{m_1^{k_1}}{(\lambda_1+\lambda_2-2m_2)^{k_1-1}} + \frac{m_2^{k_2}}{(2m_2)^{k_2-1}}$

Table A.18: Boundary points (Capacity scenario (Mid, Low), $\lambda_1 + \lambda_2 > 2m_2 + m_1$)

It can be seen that the first and second KKT conditions are remained unchanged and therefore the results for the capacity scenario (Low, Low) also apply here. As a result, Policy 1 can be eliminated from the potential optimal solutions for the referrer. In addition, $G(policy - 2) = G(policy - 3)$. Therefore, depending on which policy results in the highest objective value for the referrer each one of the policies 2, 3 and 4 can be optimal.

The following table shows boundary points when the capacity scenario is (Mid, Low) and $\lambda_1 + \lambda_2 < 2m_2 + m_1$:

Index	Policy $(\lambda_{11},\lambda_{21})$ $(\lambda_{12},\lambda_{22})$	$G(\Lambda)$
1	(m_1,λ_2-m_2) (λ_1-m_1,m_2)	$\frac{m_1^{k_1}}{(m_1+\lambda_2-m_2)^{k_1-1}}+\frac{m_2^{k_2}}{(\lambda_1+m_2-m_1)^{k_2-1}}$
2	$(m_1,\lambda_1+\lambda_2-m_1-m_2)$ $(\lambda_1-m_1,m_1+m_2-\lambda_1)$	$m_2+\frac{m_1^{k_1}}{(\lambda_1+\lambda_2-m_2)^{k_1-1}}$
3	$(\lambda_1-m_2,\lambda_2)$ $(m_2,0)$	$m_2+\frac{m_1^{k_1}}{(\lambda_1+\lambda_2-m_2)^{k_1-1}}$
4	$(m_1+m_2-\lambda_2,\lambda_2-m_2)$ $(\lambda_1+\lambda_2-m_1-m_2,m_2)$	$m_1+\frac{m_2^{k_2}}{(\lambda_1+\lambda_2-m_1)^{k_2-1}}$
5	$(\lambda_1-m_2,m_1+m_2-\lambda_1)$ $(m_2,\lambda_1+\lambda_2-m_1-m_2)$	$m_1+\frac{m_2^{k_2}}{(\lambda_1+\lambda_2-m_1)^{k_2-1}}$

Table A.19: Boundary points (Capacity scenario (Mid, Low), $\lambda_1+\lambda_2<2m_2+m_1$)

Again Policy 1 can be ignored and $G(policy-2)=G(policy-3)$ and $G(policy-4)=G(policy-5)$. The optimal policies can be determined based on the objective value for the referrer. Therefore, any of the policies 2, 3, 4 and 5 are potential to be optimal.

Optimal Referral Policies Capacity Scenario (High, Low)

The referrer problem and KKT conditions for the capacity scenario (High, Low) are as follows:

$$
\begin{aligned}
&Max\frac{m_1^{k_1}}{(\lambda_{11}+\lambda_{21})^{k_1-1}}+\frac{m_2^{k_2}}{(\lambda_1+\lambda_2-\lambda_{11}-\lambda_{21})^{k_2-1}}\\
&St:\\
&\lambda_{11}-\lambda_1\leq 0\\
&\lambda_{21}-\lambda_2\leq 0\\
&m_1-\lambda_{11}-\lambda_{21}\leq 0\\
&\lambda_1-m_2-\lambda_{11}\leq 0\\
&\lambda_2-m_2-\lambda_{21}\leq 0\\
&\lambda_{11}+\lambda_{21}-\lambda_1-\lambda_2+m_2\leq 0
\end{aligned}
$$

It should be mentioned that we ignored two conditions $\lambda_{11}\leq\lambda_1$ and $\lambda_{21}\leq\lambda_2$ as in the capacity scenario (Low, Low) m_1 and m_2 are both lower than λ_1 and λ_2.

KKT conditions for the problem are as follows:

1. $\frac{dG}{d\lambda_{11}} = (k_2-1)(\frac{m_2}{\lambda_1+\lambda_2-\lambda_{11}-\lambda_{21}})^{k_2} - (k_1-1)(\frac{m_1}{\lambda_{11}+\lambda_{21}})^{k_1} = u_1 - u_3 - u_4 + u_6$
2. $\frac{dG}{d\lambda_{21}} = (k_2-1)(\frac{m_2}{\lambda_1+\lambda_2-\lambda_{11}-\lambda_{21}})^{k_2} - (k_1-1)(\frac{m_1}{\lambda_{11}+\lambda_{21}})^{k_1} = u_2 - u_3 - u_5 + u_6$
3. $u_1(\lambda_{11}-\lambda_1) = 0$
4. $u_2(\lambda_{21}-\lambda_2) = 0$
5. $u_3(m_1-\lambda_{11}-\lambda_{21}) = 0$
6. $u_4(\lambda_1-m_2-\lambda_{11}) = 0$
7. $u_5(\lambda_2-m_2-\lambda_{21}) = 0$
8. $u_6(\lambda_{11}+\lambda_{21}-\lambda_1-\lambda_2+m_2) = 0$

The following table shows boundary points when the capacity scenario is (High, Low) and $\lambda_1+\lambda_2 > 2m_2+m_1$:

Index	Policy $(\lambda_{11},\lambda_{21})$ $(\lambda_{12},\lambda_{22})$	$G(\Lambda)$
1	$(\lambda_1, \lambda_2-m_2)$ $(0, m_2)$	$m_2 + \frac{m_1^{k_1}}{(\lambda_1+\lambda_2-m_2)^{k_1-1}}$
2	$(\lambda_1-m_2, \lambda_2)$ $(m_2, 0)$	$m_2 + \frac{m_1^{k_1}}{(\lambda_1+\lambda_2-m_2)^{k_1-1}}$
3	$(\lambda_1-m_2, \lambda_2-m_2)$ (m_2, m_2)	$\frac{m_1^{k_1}}{(\lambda_1+\lambda_2-2m_2)^{k_1-1}} + \frac{m_2^{k_2}}{(2m_2)^{k_2-1}}$

Table A.20: Boundary points (Capacity scenario (High, Low), $\lambda_1+\lambda_2 > 2m_2+m_1$)

$G(policy-1) = G(policy-2)$ an all the three policies above has the potential to be optimal. The policy that results in the highest objective value for the referrer is the optimal policy.

The following table shows boundary points when the capacity scenario is (High, Low) and $\lambda_1+\lambda_2 < 2m_2+m_1$:

Index	Policy $(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$	$G(\Lambda)$
1	$(\lambda_1, \lambda_2 - m_2)$ $(0, m_2)$	$m_2 + \frac{m_1^{k_1}}{(\lambda_1+\lambda_2-m_2)^{k_1-1}}$
2	$(\lambda_1 - m_2, \lambda_2)$ $(m_2, 0)$	$m_2 + \frac{m_1^{k_1}}{(\lambda_1+\lambda_2-m_2)^{k_1-1}}$
3	$(m_1 + m_2 - \lambda_2, \lambda_2 - m_2)$ $(\lambda_1 + \lambda_2 - m_1 - m_2, m_2)$	$m_1 + \frac{m_2^{k_2}}{(\lambda_1+\lambda_2-m_1)^{k_2-1}}$
4	$(\lambda_1 - m_2, m_1 + m_2 - \lambda_1)$ $(m_2, \lambda_1 + \lambda_2 - m_1 - m_2)$	$m_1 + \frac{m_2^{k_2}}{(\lambda_1+\lambda_2-m_1)^{k_2-1}}$

Table A.21: Boundary points (Capacity scenario (High, Low), $\lambda_1 + \lambda_2 < 2m_2 + m_1$)

Like the previous situation, all the four policies here are potential to be optimal and the optimal policy is determined based on the highest resulted objective value.

Finally for the capacity scenario (Mid, Mid) we have:

$$Max \frac{m_1^{k_1}}{(\lambda_{11}+\lambda_{21})^{k_1-1}} + \frac{m_2^{k_2}}{(\lambda_1+\lambda_2-\lambda_{11}-\lambda_{21})^{k_2-1}}$$

$$St:$$

$$\lambda_{11} - m_1 \leq 0$$

$$\lambda_{21} - m_1 \leq 0$$

$$m_1 - \lambda_{11} - \lambda_{21} \leq 0$$

$$\lambda_1 - \lambda_2 - \lambda_{11} \leq 0$$

$$-\lambda_{21} \leq 0$$

$$\lambda_{11} + \lambda_{21} - \lambda_1 - \lambda_2 + m_2 \leq 0$$

KKT conditions for the problem are as follows:

1. $\frac{dG}{d\lambda_{11}} = (k_2-1)(\frac{m_2}{\lambda_1+\lambda_2-\lambda_{11}-\lambda_{21}})^{k_2} - (k_1-1)(\frac{m_1}{\lambda_{11}+\lambda_{21}})^{k_1} = u_1 - u_3 - u_4 + u_6$
2. $\frac{dG}{d\lambda_{21}} = (k_2-1)(\frac{m_2}{\lambda_1+\lambda_2-\lambda_{11}-\lambda_{21}})^{k_2} - (k_1-1)(\frac{m_1}{\lambda_{11}+\lambda_{21}})^{k_1} = u_2 - u_3 - u_5 + u_6$
3. $u_1(\lambda_{11} - m_1) = 0$
4. $u_2(\lambda_{21} - m_1) = 0$
5. $u_3(m_1 - \lambda_{11} - \lambda_{21}) = 0$

6. $u_4(\lambda_1 - \lambda_2 - \lambda_{11}) = 0$

7. $u_5(-\lambda_{21}) = 0$

8. $u_6(\lambda_{11} + \lambda_{21} - \lambda_1 - \lambda_2 + m_2) = 0$

Optimal Referral Policies Capacity Scenario (Mid, Mid)

The following table shows boundary points when the capacity scenario is (Mid, Mid):

Index	Policy $(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$	$G(\Lambda)$
1	$(m_1, 0)$ $(\lambda_1 - m_1, \lambda_2)$	$m_1 + \frac{m_2^{k_2}}{(\lambda_1+\lambda_2-m_1)^{k_2-1}}$
2	$(m_1, \lambda_1 + \lambda_2 - m_1 - m_2)$ $(\lambda_1 - m_1, m_1 + m_2 - \lambda_1)$	$m_2 + \frac{m_1^{k_1}}{(\lambda_1+\lambda_2-m_2)^{k_1-1}}$
3	$(\lambda_1 - m_2, \lambda_2)$ $(m_2, 0)$	$m_2 + \frac{m_1^{k_1}}{(\lambda_1+\lambda_2-m_2)^{k_1-1}}$
4	$(\lambda_1 - m_2, m_1 + m_2 - \lambda_1)$ $(m_2, \lambda_1 + \lambda_2 - m_1 - m_2)$	$m_1 + \frac{m_2^{k_2}}{(\lambda_1+\lambda_2-m_1)^{k_2-1}}$

Table A.22: Boundary points (Capacity scenario (Mid, Mid))

Each one of the policies in the table above can be optimal and the optimal policy is the one which results in the highest objective value for the referrer.

Now, we focus on the situations where one of the providers is HOC. Let's begin with the situation where the first provider is HOC.

$$G(\Lambda) = (\lambda_{11} + \lambda_{21})^{1-k_1} m_1^{k_1} + \frac{m_2^{k_2}}{(\lambda_1 + \lambda_2 - \lambda_{11} - \lambda_{21})^{k_2-1}}$$

First, let's show that if $\lambda_1+\lambda_2 > 2m_1+m_2$ then the optimal policy is $(\lambda_{11}, \lambda_{21}) = (m_1, m_1)$. It can be seen that as $\lambda_{11}+\lambda_{21}$ increases $G(\Lambda)$ also increases. The referrer problem constraints are as follows:

1. $\lambda_{11} \leq m_1$

2. $\lambda_{21} \leq m_1$

3. $m_1 \leq \lambda_{11} + \lambda_{21}$

4. $\lambda_{12} \leq m_2 \Rightarrow \lambda_1 - m_2 \leq \lambda_{11}$

5. $\lambda_{22} \leq m_2 \Rightarrow \lambda_2 - m_2 \leq \lambda_{21}$

6. $m_2 \leq \lambda_{12} + \lambda_{22} \Rightarrow \lambda_{11} + \lambda_{21} \leq \lambda_1 + \lambda_2 - m_2$

7. $\lambda_{11} \leq \lambda_1$

8. $\lambda_{21} \leq \lambda_2$

From constraints 1, 2, 7 and 8 and the fact that in all the capacity scenarios except (Low, Low) $\lambda_2 \leq m_1$ it can be concluded that the only capacity scenario in which $(\lambda_{11}, \lambda_{21}) = (m_1, m_1)$ can be a feasible solution is the capacity scenario (Low, Low). If we add constraints 1 and 2 we have $\lambda_{11} + \lambda_{21} \leq 2m_1$. Therefore, if $2m_1 \leq \lambda_1 + \lambda_2 - m_2$ then the optimal policy for the referrer is $\lambda_{11} + \lambda_{21} = 2m_1$ which implies $(\lambda_{11}, \lambda_{21}) = (m_1, m_1)$. On the other hand, if $2m_1 > \lambda_1 + \lambda_2 - m_2$ then the optimal policy for the referrer is to set λ_{11} and λ_{12} in such a way that satisfies $\lambda_{11} + \lambda_{21} = \lambda_1 + \lambda_2 - m_2$.

Now, let's focus on the situation where the second provider is HOC. We have:

$$G(\Lambda) = (\lambda_{12} + \lambda_{22})^{1-k_2} m_2^{k_2} + \frac{m_1^{k_1}}{(\lambda_1 + \lambda_2 - \lambda_{12} - \lambda_{22})^{k_1 - 1}}$$

The same logic that we used above can be applied here and therefore if $2m_2 \leq \lambda_1 + \lambda_2 - m_1$ then the optimal policy for the referrer is $\lambda_{12} + \lambda_{22} = 2m_2$ which implies $(\lambda_{12}, \lambda_{22}) = (m_2, m_2)$. On the other hand, if $2m_2 \leq \lambda_1 + \lambda_2 - m_1$ then the optimal policy for the referrer is to set λ_{12} and λ_{22} in such a way that satisfies $\lambda_{12} + \lambda_{22} = \lambda_1 + \lambda_2 - m_1$ which implies $\lambda_{11} + \lambda_{21} = m_1$.

Summary

Consider the policies shown in the following table:

Index	Policy $(\lambda_{11}, \lambda_{21})$ $(\lambda_{12}, \lambda_{22})$	TP x_{i1} x_{i2}
1	(m_1, m_1) $(\lambda_1 - m_1, \lambda_2 - m_1)$	$(\frac{1}{2})^{(\alpha_1)^{-1}}$ $(\frac{m_2}{\lambda_1+\lambda_2-2m_1})^{(\alpha_2)^{-1}}$
2	$(\lambda_1 - m_2, \lambda_2 - m_2)$ (m_2, m_2)	$(\frac{m_1}{\lambda_1+\lambda_2-2m_2})^{(\alpha_1)^{-1}}$ $(\frac{1}{2})^{(\alpha_2)^{-1}}$
3	$(m_1, \lambda_1 + \lambda_2 - m_1 - m_2)$ $(\lambda_1 - m_1, m_1 + m_2 - \lambda_1)$	$(\frac{m_1}{\lambda_1+\lambda_2-m_2})^{(\alpha_1)^{-1}}$ 1
4	$(\lambda_1 + \lambda_2 - m_1 - m_2, m_1)$ $(m_1 + m_2 - \lambda_2, \lambda_2 - m_1)$	$(\frac{m_1}{\lambda_1+\lambda_2-m_2})^{(\alpha_1)^{-1}}$ 1
5	$(\lambda_1 - m_2, m_1 + m_2 - \lambda_1)$ $(m_2, \lambda_1 + \lambda_2 - m_1 - m_2)$	1 $(\frac{m_2}{\lambda_1+\lambda_2-m_1})^{(\alpha_2)^{-1}}$
6	$(m_1 + m_2 - \lambda_2, \lambda_2 - m_2)$ $(\lambda_1 + \lambda_2 - m_1 - m_2, m_2)$	1 $(\frac{m_2}{\lambda_1+\lambda_2-m_1})^{(\alpha_2)^{-1}}$
7	$(\lambda_1 - m_2, \lambda_2)$ $(m_2, 0)$	$(\frac{m_1}{\lambda_1+\lambda_2-m_2})^{(\alpha_1)^{-1}}$ 1
8	$(\lambda_1, \lambda_2 - m_2)$ $(0, m_2)$	$(\frac{m_1}{\lambda_1+\lambda_2-m_2})^{(\alpha_1)^{-1}}$ 1
9	$(m_1, 0)$ $(\lambda_1 - m_1, \lambda_2)$	1 $(\frac{m_2}{\lambda_1+\lambda_2-m_1})^{(\alpha_2)^{-1}}$

Table A.23: Potential Optimal Policies (Fair-Allocation Referral System)

Tables A.24-A.26 show optimal referral policies in a fair-allocation referral system for different capacity scenarios and operational competency level.

Both providers are LOC The following table shows optimal policies for the referrer in a referral system where both providers are LOC and fairness is taken into account.

Capacity Scenario	Optimality Conditions	Optimal Policy
(Low, Low)	$\lambda_1 + \lambda_2 > 2m_1 + m_2$	$Policy - 1$ *or* $Policy - 2$
	$2m_2 + m_1 < \lambda_1 + \lambda_2$ *and* $\lambda_1 + \lambda_2 < 2m_1 + m_2$	$\begin{cases} Policy - 3 \\ Policy - 4 \end{cases}$ *or* $Policy - 2$
	$\lambda_1 + \lambda_2 < 2m_2 + m_1$	$\begin{cases} Policy - 3 \\ Policy - 4 \end{cases}$ *or* $\begin{cases} Policy - 5 \\ Policy - 6 \end{cases}$
(Mid, Low)	$\lambda_1 + \lambda_2 > 2m_2 + m_1$	$\begin{cases} Policy - 3 \\ Policy - 7 \end{cases}$ *or* $Policy - 2$
	$\lambda_1 + \lambda_2 < 2m_2 + m_1$	$\begin{cases} Policy - 3 \\ Policy - 7 \end{cases}$ *or* $\begin{cases} Policy - 5 \\ Policy - 6 \end{cases}$
(High, Low)	$\lambda_1 + \lambda_2 > m_1 + 2m_2$	$\begin{cases} Policy - 7 \\ Policy - 8 \end{cases}$ *or* $Policy - 2$
	$\lambda_1 + \lambda_2 < m_1 + 2m_2$	$\begin{cases} Policy - 7 \\ Policy - 8 \end{cases}$ *or* $\begin{cases} Policy - 5 \\ Policy - 6 \end{cases}$
(Mid, Mid)	- - - -	$\begin{cases} Policy - 7 \\ Policy - 3 \end{cases}$ *or* $\begin{cases} Policy - 5 \\ Policy - 9 \end{cases}$

Table A.24: Optimal policies (Fair-allocation system, Both providers are LOC)

First provider is HOC The following table shows optimal policies for the referrer in a referral system where the first provider is HOC.

Capacity Scenario	Optimality Conditions	Optimal Policy
(Low, Low)	$\lambda_1 + \lambda_2 > 2m_1 + m_2$	$Policy - 1$
	$\lambda_1 + \lambda_2 < 2m_1 + m_2$	$\lambda_{11} + \lambda_{21} = \lambda_1 + \lambda_2 - m_2$
(Mid, Low)	- - -	
(High, Low)	- - -	
(Mid, Mid)	- - -	

Table A.25: Optimal policies (Fair-allocation system, First provider is HOC)

Second provider is HOC The following table shows optimal policies for the referrer in a referral system where the second provider is HOC.

Optimality Conditions	Optimal Policy
$\lambda_1 + \lambda_2 > 2m_1 + m_2$	$Policy - 2$
$\lambda_1 + \lambda_2 < 2m_2 + m_1$	$\lambda_{11} + \lambda_{21} = m_1$

Table A.26: Optimal policies (Fair-allocation system, Second provider is HOC)

Appendix B

Third Chapter Multinomial Logistic Regression Model

B.1 Multinomial Regression Model

In this section we present the result of the multinomial regression model used in Section 4.4.2. The dependent variable is patient preference and the independent variables are patient gender (Figure 4.3); patient location (Figure 4.4); patient age (Figure 4.5); and three variables: moving average for the past 10 patients, and the 5th and 95th percentiles, calculated based on wait time information.

```
Optimization terminated successfully.
         Current function value: 0.998359
         Iterations 6
                          MNLogit Regression Results
==============================================================================
Dep. Variable:            Preferences   No. Observations:                 1240
Model:                        MNLogit   Df Residuals:                     1222
Method:                           MLE   Df Model:                           16
Date:                Sat, 19 Mar 2022   Pseudo R-squ.:                 0.03361
Time:                        14:43:46   Log-Likelihood:                -1238.0
converged:                       True   LL-Null:                       -1281.0
Covariance Type:            nonrobust   LLR p-value:                 1.299e-11
==============================================================================================
Preferences=First Available Surgeon       coef    std err          z      P>|z|      [0.025      0.975]
----------------------------------------------------------------------------------------------
F                                      -0.0994      0.170     -0.583      0.560      -0.433       0.235
over_70                                -0.7804      0.178     -4.379      0.000      -1.130      -0.431
Guelph                                 -0.7774      0.394     -1.972      0.049      -1.550      -0.005
Kitchener                               0.6133      0.406      1.509      0.131      -0.183       1.410
Townships                               0.1070      0.428      0.250      0.802      -0.731       0.945
Wait times MA                          -0.0700      1.275     -0.055      0.956      -2.569       2.429
Bottom 5%                              -0.5076      1.095     -0.464      0.643      -2.653       1.638
Top 5%                                  0.4773      0.809      0.590      0.555      -1.109       2.063
const                                   1.0151      0.463      2.190      0.028       0.107       1.923
----------------------------------------------------------------------------------------------
Preferences=Specific Surgeon       coef    std err          z      P>|z|      [0.025      0.975]
----------------------------------------------------------------------------------------------
F                               0.1031      0.162      0.637      0.524      -0.214       0.421
over_70                        -0.3333      0.172     -1.935      0.053      -0.671       0.004
Guelph                         -0.8675      0.375     -2.312      0.021      -1.603      -0.132
Kitchener                       0.5169      0.389      1.330      0.184      -0.245       1.279
Townships                       0.2540      0.406      0.626      0.531      -0.541       1.049
Wait times MA                   1.2052      1.190      1.013      0.311      -1.128       3.538
Bottom 5%                      -1.2396      1.011     -1.226      0.220      -3.221       0.742
Top 5%                          0.0426      0.764      0.056      0.956      -1.455       1.540
const                           0.8958      0.443      2.021      0.043       0.027       1.765
==============================================================================================
```

Figure B.1: Multinomial Logistic Model Statistics

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