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Computing closed forms for the convergent series $\displaystyle\sum_{n \in \mathbb{Z}}\frac{1}{(n^3+Bn^2+Cn+D)^k}$

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by

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Abstract

In this thesis we discuss the various approaches that will be taken to evaluate and find a finite closed form for the sum

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n^3 + Bn^2 + Cn + D)^k}$$

where $B, C, D \in \mathbb{C}$ and k is a positive integer. We begin this thesis by studying the cubic equations and discussing briefly various methods of finding their roots. Cardano's method (1545) for finding the roots of cubic polynomials is explored in detail as this method is used in later parts of the thesis to make calculations while evaluating the sums. Various tools and techniques from Fourier analysis are reviewed for these aid in computing the sums. To obtain finite closed forms for the sums $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$, we make use of different methods and approaches from combinatorics and identities involving well-known trigonometric functions.

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Gagandeep Kaur Virk

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Chapter 1

Introduction

This thesis is an attempt to evaluate the closed forms for the sums $\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$ and $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$. The thesis is divided into four chapters that are further divided into sub-sections. Chapter 1 of the thesis outlines the subject and background required for the research. Chapter 2 discusses the tools and techniques of Fourier analysis that shall be used to compute the closed forms for the sums $\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$ and $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$ in the later parts of the chapter. In Chapter 3, we evaluate the sums by using different approaches, namely, by connecting the sums to the cotangent function. Chapter 4 summarizes and binds together the results derived in the earlier chapters and future directions for these results. We start this chapter by discussing the background required for the subject of research.

1.1 Motivation: The Riemann zeta function

An important function in mathematics is the Riemann zeta function, $\zeta(s)$. The Riemann zeta function, also known as the Euler-Riemann zeta function, is defined by the following absolutely convergent series,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

where $s \in \mathbb{C}$ with $Re(s) > 1$.

This function was first introduced by Euler in the early eighteenth century. In 1859, its interpretation was extended to complex variables by Bernard Riemann in [4]. Euler computed the values of $\zeta(s)$ for positive even integers [21]. The most commonly discussed value of $\zeta(s)$ for positive even integers is that of $\zeta(2)$ which appears in a number of research papers and publications (for example, [2], [6], [22]). Euler proved that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

and more generally that

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = (-1)^{k-1} \frac{B_{2k}(2\pi)^{2k}}{2(2k)!},$$

where B_k denotes the k^{th} Bernoulli number. With Lindemann's work [10] in 1882 showing that π is transcendental, it was effectively proved that the values of $\zeta(2n)$, where $n \in \mathbb{N}$, were transcendental too. The examination of $\zeta(2n+1)$ is somewhat difficult to examine and is still unsolved. Though Apéry [1], in 1979, proved that $\zeta(3)$ is irrational, the irrationality of $\zeta(2n+1)$ for $n > 1$ is still unknown. Due to this contribution by Apéry, $\zeta(3)$ is known as Apéry's constant. Ball and Rivoal [3] proved that $\zeta(2n+1)$ is irrational for infinitely many positive integers. Rivoal [14] also showed that at least one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, $\ldots$, $\zeta(21)$ is irrational. In 2001, Zudilin [25] who improved the results of Rivoal and Ball by showing that at least one of $\zeta(5)$, $\zeta(7)$, $\zeta(9)$, or $\zeta(11)$ is irrational.

The Riemann zeta function is closely related to the prime number theorem. Euler discovered an important formula involving a product over primes, given by,

$$\prod_p (1 - p^{-s})^{-1} = \zeta(s)$$

and hence, the name Euler product was given to this formula. Whilst a number of properties of $\zeta(s)$ have been studied and explored, there are significant fundamental postulations that are still unproved. One such conjecture is the Riemann Hypothesis. The function $\zeta(s)$ has two different types of zeroes. The "trivial" zeroes occur at all negative even integers $s = -2, -4, -6, \ldots$ and the "non-trivial" zeroes occur at certain complex numbers $s = \sigma + it$. The Riemann Hypothesis states that the real part of these non-trivial zeroes all lie on a line called the "critical line" where $Re(s) = \sigma = \frac{1}{2}$. To this day, 250×10^9 non-trivial roots of $\zeta(s)$ have been found and all are on the critical line [23].

With such important conjectures still remaining unproved, it becomes crucial to explore and analyze mathematical expressions involving or related to $\zeta(s)$ in one way or another. In the note [13], Murty and Weatherby have elegantly worked out closed forms for the related sum

$$\sum_{n \in \mathbb{Z}} \frac{1}{n^2 + Bn + C}$$

and more generally for

$$\sum_{n \in \mathbb{Z}} \frac{1}{(n^2 + Bn + C)^k} \tag{1.1}$$

where $B, C \in \mathbb{C}$ and k is a positive integer, by using concepts of Fourier analysis and combinatorics. In [13], the authors use Fourier analysis to show that for $B, C \in \mathbb{C}$ with $-D = B^2 - 4C$ such that $n^2 + Bn + C \neq 0$ for any integer n, we have

$$\sum_{n \in \mathbb{Z}} \frac{1}{n^2 + Bn + C} = \frac{2\pi}{\sqrt{D}} \left(\frac{e^{2\pi\sqrt{D}} - 1}{e^{2\pi\sqrt{D}} - 2\cos(B\pi)e^{\pi\sqrt{D}} + 1} \right)$$

and also that $\sum_{n\in\mathbb{Z}} \frac{1}{(n^2+Bn+C)^k}$ is equal to

$$\frac{2\pi(-4)^{k-1}}{(k-1)!}\left[\frac{1}{\sqrt{D}}\left(\frac{1}{e^{B\pi i+\pi\sqrt{D}}-1}-\frac{1}{e^{B\pi i-\pi\sqrt{D}}-1}\right)\right]^{(k-1)}$$

The authors of [13] also described how methods of combinatorics can be used to find a closed form for (1.1). In [13], it is proved that

Theorem 1.1 *([13], Theorem 3) For $B, C \in \mathbb{C}$ with $-D = B^2 - 4C$ such that $n^2 + Bn + C \neq 0$ for any integer n, and k a positive integer, we have that $\sum_{n\in\mathbb{Z}} \frac{1}{(n^2+Bn+C)^k}$ is equal to*

$$\frac{2\pi\sqrt{D}}{D^k}\sum_{r=0}^{k-1}\binom{2k-2-2r}{k-1-r}\sum_{b_1,\ldots,b_r}\frac{(2\pi)^R D^{\frac{R}{2}} C_0^{b_1}\cdots C_{r-1}^{b_r}}{b_1!\cdots b_r!}\times$$
$$\sum_{s=1}^{R+1}\left(\frac{(s-1)!S(R+1,s)}{(e^{B\pi i+\pi\sqrt{D}}-1)^s}-\frac{(-1)^R(s-1)!S(R+1,s)}{(e^{B\pi i-\pi\sqrt{D}}-1)^s}\right)$$

where the summation in b_i's is over all nonnegative solutions to $b_1+2b_2+\cdots+rb_r = r$ and where $R = b_1+\cdots+b_r$, C_m is the m^{th} Catalan number and $S(R+1, s)$ is a Stirling number of the second kind. The sum lies in $\pi\mathbb{Q}(\sqrt{D}, e^{\pi\sqrt{D}}, e^{B\pi i})[\pi]$ with π^k being the largest power of π present.

Inspired by the necessity to consider and study mathematical expressions related to $\zeta(s)$ in some way or another, we attempt to evaluate the convergent series

$$\sum_{n\in\mathbb{Z}}\frac{1}{n^3+Bn^2+Cn+D} \tag{1.2}$$

and more generally

$$\sum_{n\in\mathbb{Z}}\frac{1}{(n^3+Bn^2+Cn+D)^k} \tag{1.3}$$

for parameters $B, C, D \in \mathbb{C}$ and positive integer k.

Since we are looking at series that involve cubic polynomials, it is important that we discuss cubic functions and methods for finding their roots.

1.2 Cubic equations and Cardano's method

A function of the form,

$$f(x) = ax^3 + bx^2 + cx + d \tag{1.4}$$

with $a \neq 0$, is known as the cubic function. With $f(x) = 0$, it produces a cubic equation

$$ax^3 + bx^2 + cx + d = 0. \tag{1.5}$$

The solutions of (1.5) are the roots of (1.4).

If the coefficients a, b, c and d are all real, (1.5) will have at least one real root and if the coefficients of (1.5) are complex and not all real, there will be at least one complex root. The roots of (1.5) can be found either algebraically by finding the discriminant of (1.5) or trigonometrically by dividing (1.5) by a and substituting $t - \frac{b}{3a}$ for x and reducing it to a depressed cubic of the form,

$$t^3 + pt + q = 0. \tag{1.6}$$

The roots of (1.5) can also be found by factorization if one root, r is known. Equation (1.4) can be factored as

$$ax^3 + bx^2 + cx + d = (x - r)(ax^2 + (b + ar)x + c + br + ar^2). \tag{1.7}$$

Hence, if we know at least one root r of (1.5) (by the guess and check method or rational root test, etc.) we can find the other roots by finding roots of the quadratic factor on the right hand side of (1.7).

Other than the methods discussed above, the solutions to the cubic equation (1.5) can also be derived using a number of different methods such as Cardano's method [5], Vieta's substitution [17] or Lagrange's method [9]. For working out the calculations required for this thesis, we have made use of Cardano's method to derive roots of a cubic equation. Cardano's derivation of the roots is as follows. We have a given cubic equation

$$ax^3 + bx^2 + cx + d = 0$$

with $a \neq 0$.

The previous equation is divided by a on both the sides to get

$$x^3 + Bx^2 + Cx + D = 0$$

where $B = \frac{b}{a}, C = \frac{c}{a}, D = \frac{d}{a}$. Substituting values of x as $x = y - \frac{B}{3}$ and simplifying the previous equation becomes

$$y^3 + \left(C - \frac{1}{3}B^2\right) y + \left(\frac{2}{27}B^3 - \frac{1}{3}BC + D\right) = 0. \tag{1.8}$$

Substituting values of $p = \left(C - \frac{1}{3}B^2\right)$ and $q = \left(\frac{2}{27}B^3 - \frac{1}{3}BC + D\right)$ in (1.8), we obtain a depressed cubic of the form

$$y^3 + py + q = 0 \tag{1.9}$$

We know that the discriminants of a cubic equation $a'x_1^3 + b'x_1^2 + c'x_1 + d' = 0$ are given by

$$\begin{aligned}
\Delta_0 &= b'^2 - 3a'c' \\
\Delta_1 &= 2b'^3 - 9a'b'c' + 27a'^2 d' \\
\Delta &= \frac{-1}{27a'^2}(\Delta_1^2 - 4\Delta_0^3)
\end{aligned}$$

This implies for (1.8), we have

$$\begin{aligned}\Delta_0 &= -3p \\ \Delta_1 &= 27q \\ \Delta &= \frac{-1}{27}(27q^2 - 4(-3p)^3) \\ &= -(27q^2 + 4p^3) \qquad (1.10)\end{aligned}$$

We shall make use of (1.10) in later sections to compute the sums (1.2) and (1.3).

Making the substitution $y = z - \frac{p}{3z}$ and simplifying (1.9) we obtain a quadratic equation in z^3 of the form

$$z^6 + qz^3 - \frac{p^3}{27} = 0.$$

Solutions to this quadratic equation are given by,

$$z^3 = \frac{-q \pm \sqrt{q^2 + \frac{4p^3}{27}}}{2}$$

or say by:

$$z^3 = \frac{-q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}.$$

If we let M be the solution to z^3 with $(+)$ and N be the solution to z^3 with $(-)$,

$$M = \frac{-q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}, \;\; N = \frac{-q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}, \qquad (1.11)$$

and let $\omega = e^{\frac{2\pi i}{3}}$, a cubic root of unity, we obtain six solutions for z:

$$\begin{aligned} z_1 &= \sqrt[3]{M}, & z_2 &= \sqrt[3]{M}\omega, \\ z_3 &= \sqrt[3]{M}\omega^2, & z_4 &= \sqrt[3]{N}, \\ z_5 &= \sqrt[3]{N}\omega, & z_6 &= \sqrt[3]{N}\omega^2 \end{aligned}$$

Taking a closer look at M and N, we find that $\sqrt[3]{M}\sqrt[3]{N}$ is equal to
$\sqrt[3]{\frac{-q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}.\sqrt[3]{\frac{-q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}} = \sqrt[3]{\left(\frac{-q}{2}\right)^2-\left(\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}\right)^2}.$
Simplifying this, we get

$$\sqrt[3]{M}\sqrt[3]{N} = \frac{-p}{3}. \tag{1.12}$$

Recall that $y = z - \frac{p}{3z}$. Putting z_1 into this equation, we obtain

$$y = \sqrt[3]{M} - \frac{p}{3\sqrt[3]{M}}.$$

The previous equation can be further modified by making the identity (1.12) to obtain

$$y = \sqrt[3]{M} + \sqrt[3]{N}.$$

Similarly, putting $z_2, z_3, \ldots, z_6$ into the equation for y and making modifications using (1.12), we find three distinct values for y

$$y_1 = \sqrt[3]{M}\omega + \sqrt[3]{N}\omega^2, \quad y_2 = \sqrt[3]{M}\omega^2 + \sqrt[3]{N}\omega, \quad y_3 = \sqrt[3]{N} + \sqrt[3]{M}$$

Recall our substitution $x = y - \frac{B}{3}$, or $y = x + \frac{B}{3}$. By back substitution of y_1, y_2, y_3, we find that the three roots of the original cubic equation $x^3 + Bx^2 + Cx + D = 0$ are

$$\begin{aligned} x_1 &= \sqrt[3]{M} + \sqrt[3]{N} - \frac{B}{3} \\ x_2 &= \sqrt[3]{M}\omega + \sqrt[3]{N}\omega^2 - \frac{B}{3}, \\ x_3 &= \sqrt[3]{M}\omega^2 + \sqrt[3]{N}\omega - \frac{B}{3}. \end{aligned} \tag{1.13}$$

or more generally

$$\begin{aligned} x_1 &= \sqrt[3]{M} + \sqrt[3]{N} - \frac{b}{3a}, \\ x_2 &= \sqrt[3]{M}\omega + \sqrt[3]{N}\omega^2 - \frac{b}{3a} \\ x_3 &= \sqrt[3]{M}\omega^2 + \sqrt[3]{N}\omega - \frac{b}{3a} \end{aligned} \tag{1.14}$$

where $M = \frac{-q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}$, $N = \frac{-q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}$, $p = \left(C - \frac{1}{3}B^2\right) = \frac{c}{a} - \frac{1}{3}\left(\frac{b}{a}\right)^2$ and $q = \left(\frac{2}{27}B^3 - \frac{1}{3}BC + D\right) = \frac{2}{27}\left(\frac{b}{a}\right)^3 - \frac{1}{3}\frac{bc}{a^2} + \frac{d}{a}$.

It is clearly evident from (1.13) and (1.14), that if coefficients of the cubic equation $x^3 + Bx^2 + Cx + D = 0$ or more generally, the coefficients of the cubic equation $ax^3 + bx^2 + cx + d = 0$ are real, there will be at least one real root to the cubic equation given by x_1 from (1.13) since M and N are real in that case.

In the following sections, we will make use of the roots found using Cardano's method in order to analyze the sums (1.2) and (1.3).

Chapter 2

Evaluating the sum using Fourier Analysis

In this chapter, we shall evaluate the closed forms for the sums of the form $\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$ and $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$ using methods of Fourier analysis. We first discuss briefly the tools and techniques of Fourier analysis that shall be used to examine the sums.

2.1 Fourier Analysis

It is assumed that the reader has preliminary background knowledge of Fourier transforms and series. The reader is referred to [15] for an introduction to the subject. Some of the fundamental facts of Fourier analysis are reviewed herein.

For a function $f(x) \in L^1[0,1]$, the set of 1-periodic, Lebesgue integrable functions with

$$\int_0^1 |f(x)|\, dx < \infty,$$

the *mth* Fourier coefficient is defined by

$$f_m = \int_0^1 f(x) e^{-2\pi imx} dx.$$

Given this definition, the Fourier series associated with $f(x)$ is given by

$$\sum_{m \in \mathbb{Z}} f_m e^{2\pi imx}.$$

Note that if the function f is differentiable, then the Fourier series associated with the function converges to the function f. It is also known that the Fourier series of f at x converges to

$$\frac{f(x_+) + f(x_-)}{2}$$

if the function f is piecewise smooth. A function as such is said to satisfy *Dirichlet's conditions* [16]. Such a function f is said to be piecewise smooth if it can be broken into distinct pieces for which both the function f and its derivative f' are continuous functions and where the only admissible discontinuities are jump discontinuities. It should be noted that it is not easy to precisely deduce the convergence of the Fourier series for a given function f to the function f. For convergence purposes, here we only study piecewise smooth functions $f \in L^1(\mathbb{R})$ which form the set of Lebesgue integrable functions satisfying

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty.$$

The Fourier transform for such functions is defined as

$$\widehat{f}(u) = \int_{-\infty}^{\infty} f(x) e^{-2\pi ixu} dx.$$

If for $f \in L^1(\mathbb{R})$, we have $\widehat{f} \in L^1(\mathbb{R})$, we can relate f to the Fourier transform of $\hat{f}$ giving the following inversion formula.

Lemma 2.1 (Inversion Formula) *For* $f, \widehat{f} \in L^1(\mathbb{R})$*, we have*

$$f(x) = \int_{-\infty}^{\infty} \widehat{f}(u) e^{2\pi i x u} du,$$

implying $\widehat{\widehat{f}}(x) = f(-x)$.

The proof of inversion formula is not obvious and the reader is referred to [15] for a detailed derivation of it.

Related to Fourier transforms is the notion of convolution of two functions. If $f, g \in L^1(\mathbb{R})$, then $f * g \in L^1(\mathbb{R})$ is defined by

$$(f * g)(x) = \int_{-\infty}^{\infty} f(y) g(x - y) dy,$$

and is called the *convolution* of the functions f and g. The Fourier transform translates between convolution and multiplication of functions.

Theorem 2.2 (Convolution Theorem) *If* $f, g \in L^1(\mathbb{R}^1)$ *with Fourier transforms* $\widehat{f}, \widehat{g}$ *respectively, then the Fourier transform of convolution of the two functions is given by the product of their Fourier transforms. That is, given*

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(y) g(x - y) dy,$$

we have

$$\widehat{h}(u) = \widehat{f}(u) \widehat{g}(u).$$

Proof: The proof of this theorem is followed from [20]. By the definition of Fourier transforms,

$$\begin{aligned}
\widehat{h}(u) &= \int_{-\infty}^{\infty} h(z)e^{-2\pi izu}dz \\
&= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(x)g(z-x)dx \right) e^{-2\pi izu}dz \\
&= \int_{-\infty}^{\infty} f(x) \left(\int_{-\infty}^{\infty} g(z-x)e^{-2\pi izu}dz \right) dx
\end{aligned}$$

We substitute $y = z - x$; then, $dy = dz$. Thus,

$$\begin{aligned}
\widehat{h}(u) &= \int_{-\infty}^{\infty} f(x) \left(\int_{-\infty}^{\infty} g(y)e^{-2\pi i(y+x)u}dy \right) dx \\
&= \int_{-\infty}^{\infty} f(x)e^{-2\pi ixu} \left(\int_{-\infty}^{\infty} g(y)e^{-2\pi iyu}dy \right) dx \\
&= \int_{-\infty}^{\infty} f(x)e^{-2\pi ixu}dx \int_{-\infty}^{\infty} g(y)e^{-2\pi iyu}dy \\
&= \widehat{f}(u)\widehat{g}(u).
\end{aligned}$$

□

We now state and prove a version of Poisson's summation formula using the tools that we have discussed so far. The version presented here is fairly simple, particularly useful for our purposes, and has a concise and elementary proof. There are stronger versions of the formula and we refer the reader to [7] for the stronger versions where the assumptions made on the function f are minimal. The proof here follows [12].

Theorem 2.3 (Poisson Summation) *If $f \in L^1(\mathbb{R})$ is continuous and piecewise smooth where the sum*

$$\sum_{m\in\mathbb{Z}} f(m+v)$$

converges absolutely and uniformly in v, and if

$$\sum_{m\in\mathbb{Z}} |\widehat{f}(m)| < \infty$$

then

$$\sum_{m\in\mathbb{Z}} f(m+v) = \sum_{m\in\mathbb{Z}} \widehat{f}(m)e^{2\pi imv}.$$

Proof: We let $F(v) = \sum_{m\in\mathbb{Z}} f(m+v)$. F is a continuous and piecewise smooth periodic function in v with period 1, by the assumption of absolute and uniform convergence. Thus, F can be viewed as a function on $[0,1]$ and the Fourier coefficients can be calculated directly.

$$\begin{aligned} F_m &= \int_0^1 F(v)e^{-2\pi imv}dv \\ &= \sum_{n\in\mathbb{Z}} \int_0^1 f(n+v)e^{-2\pi imv}dv. \end{aligned}$$

Replacing $x = n+v$ we obtain

$$\begin{aligned} F_m &= \sum_{n\in\mathbb{Z}} \int_n^{n+1} f(x)e^{-2\pi imx}dx \\ &= \int_{-\infty}^{\infty} f(x)e^{-2\pi imx}dx = \widehat{f}(m). \end{aligned}$$

Since $\sum_{m\in\mathbb{Z}} |\widehat{f}(m)| < \infty$, the Fourier series for F converges and we have that

$$\sum_{m\in\mathbb{Z}} f(m+v) = F(v) = \sum_{m\in\mathbb{Z}} F_m e^{2\pi imv} = \sum_{m\in\mathbb{Z}} \widehat{f}(m)e^{2\pi imv}.$$

□

The tools and techniques of Fourier analysis discussed here shall be helpful in finding explicit closed forms for the sums $\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$ and $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$.

2.2 Computing a closed form for $\displaystyle\sum_{n\in\mathbb{Z}} \frac{1}{n^3 + Bn^2 + Cn + D}$

We begin this section with some well known lemmas that shall prove to be useful in computing the sum.

Lemma 2.4 *If α is a complex number with $Im(\alpha) > 0$ and we let $f(x) = 2\pi i e^{2\pi i \alpha x} H(x)$ where $H(x)$ is the Heaviside function given by,*

$$H(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

then,

$$\hat{f}(u) = \frac{1}{u - \alpha}.$$

Proof: For $f(x) = 2\pi i e^{2\pi i \alpha x} H(x)$, we have,

$$\begin{aligned}
\hat{f}(u) &= \int_{-\infty}^{\infty} 2\pi i e^{2\pi i \alpha x} H(x) e^{-2\pi i u x} dx \\
&= \lim_{t\to\infty} \int_0^t 2\pi i e^{2\pi i \alpha x} e^{-2\pi i u x} dx \\
&= 2\pi i \lim_{t\to\infty} \int_0^t e^{2\pi i(\alpha - u)x} dx \\
&= 2\pi i \lim_{t\to\infty} \int_0^t e^{2\pi i(Re(\alpha) + iIm(\alpha) - u)x} dx \\
&= 2\pi i \lim_{t\to\infty} \int_0^t e^{2\pi i(Re(\alpha) - u)x} e^{-2\pi Im(\alpha)x} dx.
\end{aligned}$$

Since $Im(\alpha) > 0$, we have convergence and

$$\begin{aligned}
\hat{f}(u) &= 2\pi i \lim_{t\to\infty} \int_0^t e^{2\pi i(\alpha - u)x} dx \\
&= \frac{1}{u - \alpha}.
\end{aligned}$$

$\square$

The following lemma will also be a useful tool for evaluating the closed forms of the sums (1.2) and (1.3).

Lemma 2.5 *Let* ρ *and* β *be two complex numbers such that* $0 \leq \left|\frac{Im(\beta)}{2}\right| < Re(\rho)$*. Let* $h(x) = \frac{\pi}{\rho} e^{-2\pi\rho|x|} e^{-2\pi i \frac{\beta}{2} x}$*, and* $\gamma = \rho^2 + \frac{\beta^2}{4}$ *then,*

$$\hat{h}(u) = \frac{1}{u^2 + \beta u + \gamma}.$$

Proof: For given $h(x) = \frac{\pi}{\rho} e^{-2\pi\rho|x|} e^{-2\pi i \frac{\beta}{2} x}$, we have,

$$\begin{aligned}
\hat{h}(u) &= \int_{-\infty}^{\infty} \frac{\pi}{\rho} e^{-2\pi\rho|x|} e^{-2\pi i \frac{\beta}{2} x} e^{-2\pi i u x} dx \\
&= \lim_{t\to\infty} \left(\int_{-t}^{0} \frac{\pi}{\rho} e^{2\pi\rho x} e^{-2\pi i \frac{\beta}{2} x} e^{-2\pi i u x} dx \right) + \lim_{s\to\infty} \left(\int_{0}^{s} \frac{\pi}{\rho} e^{-2\pi\rho x} e^{-2\pi i \frac{\beta}{2} x} e^{-2\pi i u x} dx \right) \\
&= \frac{\pi}{\rho} \left(\lim_{t\to\infty} \int_{-t}^{0} e^{2\pi\left(\rho - i\left(u + \frac{\beta}{2}\right)\right)x} dx + \lim_{s\to\infty} \int_{0}^{t} e^{-2\pi\left(\rho + i\left(u + \frac{\beta}{2}\right)\right)x} dx \right).
\end{aligned}$$

Let $I_1 = \lim_{t\to\infty} \int_{-t}^{0} e^{2\pi\left(\rho - i\left(u + \frac{\beta}{2}\right)\right)x} dx$ and $I_2 = \lim_{s\to\infty} \int_{0}^{s} e^{-2\pi\left(\rho + i\left(u + \frac{\beta}{2}\right)\right)x} dx$. For I_1 to converge, we need $Re\left(2\pi\left(\rho - i\left(u + \frac{\beta}{2}\right)\right)\right) > 0$. Since u is real, we only need to verify $Re\left(\rho - i\frac{\beta}{2}\right) > 0$. We have

$$Re\left(Re(\rho) + iIm(\rho) - i\left(\frac{Re(\beta)}{2} + i\frac{Im(\beta)}{2} \right) \right) = Re(\rho) + \frac{Im(\beta)}{2},$$

which is clearly positive because $0 \leq \left|\frac{Im(\beta)}{2}\right| < Re(\rho)$.
For I_2, we need $Re\left(2\pi\left(\rho + i\left(u + \frac{\beta}{2}\right)\right)\right) > 0$ for convergence. We only need to check $Re\left(\rho + i\frac{\beta}{2}\right) > 0$. We have

$$Re\left(Re(\rho) + iIm(\rho) + i\left(\frac{Re(\beta)}{2} + i\frac{Im(\beta)}{2} \right) \right) = Re(\rho) - \frac{Im(\beta)}{2},$$

which is positive again by $0 \le \left|\frac{Im(\beta)}{2}\right| < Re(\rho)$.

Since I_1 and I_2 both converge, we have that

$$\begin{aligned}
\hat{h}(u) &= \frac{\pi}{\rho}\left(\lim_{t\to\infty}\int_{-t}^{0} e^{2\pi\left(\rho-i\left(u+\frac{\beta}{2}\right)\right)x}dx + \lim_{s\to\infty}\int_{0}^{s} e^{-2\pi\left(\rho+i\left(u+\frac{\beta}{2}\right)\right)x}dx\right)\\
&= \frac{\pi}{2\pi\rho}\left(\frac{1}{\left(\rho - i\left(u+\frac{\beta}{2}\right)\right)} + \frac{1}{\left(\rho + i\left(u+\frac{\beta}{2}\right)\right)}\right)\\
&= \frac{1}{2\rho}\left(\frac{2\rho}{\rho^2 + \left(u+\frac{\beta}{2}\right)^2}\right)\\
&= \frac{1}{\rho^2 + \left(u+\frac{\beta}{2}\right)^2}\\
&= \frac{1}{\rho^2 + u^2 + \beta u + \frac{\beta^2}{4}}\\
&= \frac{1}{u^2 + \beta u + \gamma},
\end{aligned}$$

where $\gamma = \rho^2 + \frac{\beta^2}{4}$. □

We now compute a function g whose Fourier transform is of the form $\frac{1}{u^3+Bu^2+Cu+D}$. This shall be helpful in computing the sum later.

Theorem 2.6 *Suppose $u^3 + Bu^2 + Cu + D$ with $B, C, D \in \mathbb{C}$ can be factored as*

$$u^3 + Bu^2 + Cu + D = (u-\alpha)(u^2+\beta u+\gamma) = (u-\alpha)(u-\mu)(u-\nu)$$

where $\alpha, \beta, \gamma, \rho, \mu, \nu \in \mathbb{C}$, α, μ and ν are distinct and $\gamma = \rho^2 + \frac{\beta^2}{4}, \mu = -\frac{\beta}{2} - i\rho, \nu = -\frac{\beta}{2} + i\rho$ such that

1. *$Im(\alpha) > 0$*

2. *$0 \le \left|\frac{Im(\beta)}{2}\right| < Re(\rho)$.*

Then for

$$g(x) = \begin{cases} \frac{\pi}{\rho}\left(\frac{e^{2\pi i\alpha x}}{\mu-\alpha} + \frac{e^{2\pi i\nu x}}{\nu-\alpha} - \frac{e^{2\pi i\alpha x}}{\nu-\alpha}\right), x > 0 \\ \frac{\pi}{\rho}\left(\frac{e^{-2\pi i\mu x}}{\mu-\alpha}\right), x \leq 0 \end{cases}$$

we have

$$\widehat{g}(u) = \frac{1}{u^3 + Bu^2 + Cu + D}.$$

Proof: Under the given assumptions, Lemma 2.4, implies that for $f(x) = 2\pi i e^{2\pi i\alpha x}H(x)$, $\hat{f}(u) = \frac{1}{u-\alpha}$ and Lemma 2.5 implies that for $h(x) = \frac{\pi}{\rho}e^{-2\pi\rho|x|}e^{-2\pi i\frac{\beta}{2}x}$, then,

$$\hat{h}(u) = \frac{1}{u^2 + \beta u + \gamma}.$$

Clearly, $\hat{g}(u) = \hat{f}(u)\hat{h}(u)$, so by Theorem 2.2, we only need to compute $g(x) = (f * h)(x)$. By Fourier convolution,

$$\begin{aligned} g(x) &= (f*h)(x) = \int_{-\infty}^{\infty} f(y)h(x-y)dy \\ &= \int_{-\infty}^{\infty} 2\pi i e^{2\pi i\alpha y}H(y)\frac{\pi}{\rho}e^{-2\pi\rho|x-y|}e^{-2\pi i\frac{\beta}{2}(x-y)}dy \\ &= \frac{2\pi^2 i}{\rho}\lim_{t\to\infty}\int_0^t e^{2\pi i\alpha y}e^{-2\pi\rho|x-y|}e^{-2\pi i\frac{\beta}{2}(x-y)}dy. \end{aligned}$$

We have two cases depending on x.

Case I: When $x > 0$ we have $g(x)$ equal to

$$\frac{2\pi^2 i}{\rho}\left(\int_0^x e^{2\pi i\alpha y}e^{-2\pi\rho(x-y)}e^{-2\pi i\frac{\beta}{2}(x-y)}dy + \lim_{t\to\infty}\int_x^t e^{2\pi i\alpha y}e^{-2\pi\rho(y-x)}e^{-2\pi i\frac{\beta}{2}(x-y)}dy\right)$$

$$= \frac{2\pi^2 i}{\rho}\left(\int_0^x e^{\left(2\pi\rho+2\pi i\frac{\beta}{2}+2\pi i\alpha\right)y}e^{\left(-2\pi\rho-2\pi i\frac{\beta}{2}\right)x}dy + \lim_{t\to\infty}\int_x^t e^{\left(-2\pi\rho+2\pi i\frac{\beta}{2}+2\pi i\alpha\right)y}e^{\left(2\pi\rho-2\pi i\frac{\beta}{2}\right)x}dy\right)$$

$$= \frac{2\pi^2 i}{\rho}\left(\int_0^x e^{\left(2\pi\rho+2\pi i\frac{\beta}{2}+2\pi i\alpha\right)y}e^{2\pi i\nu x}dy + \lim_{t\to\infty}\int_x^t e^{-\left(2\pi\rho-2\pi i\frac{\beta}{2}-2\pi i\alpha\right)y}e^{2\pi i\mu x}dy\right).$$

The above converges when $Re(2\pi\rho - 2\pi i\frac{\beta}{2} - 2\pi i\alpha) > 0$, or equivalently,

$$Re(Re(\rho) + iIm(\rho) - i\left(Re\left(\frac{\beta}{2}\right) + iIm\left(\frac{\beta}{2}\right)\right) - i(Re(\alpha) + iIm(\alpha)) > 0,$$

$$\Leftrightarrow Re(Re(\rho) + iIm(\rho) - iRe\left(\frac{\beta}{2}\right) + Im\left(\frac{\beta}{2}\right) - Re(\alpha)i + Im(\alpha)) > 0,$$

$$\Leftrightarrow Re(\rho) + Im\left(\frac{\beta}{2}\right) + Im(\alpha) > 0,$$

which is true. Therefore,

$$\begin{aligned}
g(x) &= \frac{2\pi^2 i}{\rho}\left(\int_0^x e^{\left(2\pi\rho+2\pi i\frac{\beta}{2}+2\pi i\alpha\right)y} e^{2\pi i\nu x}dy + \lim_{t\to\infty}\int_x^t e^{-\left(2\pi\rho-2\pi i\frac{\beta}{2}-2\pi i\alpha\right)y} e^{2\pi i\mu x}dy\right)\\
&= \frac{2\pi^2 i}{\rho}\left(\frac{e^{2\pi i\alpha x}}{2\pi\rho+2\pi i\frac{\beta}{2}+2\pi i\alpha} - \frac{e^{2\pi i\nu x}}{2\pi\rho+2\pi i\frac{\beta}{2}+2\pi i\alpha} + \frac{e^{2\pi i\alpha x}}{2\pi\rho-2\pi i\frac{\beta}{2}-2\pi i\alpha}\right)\\
&= \frac{\pi i}{\rho}\left(\frac{e^{2\pi i\alpha x}}{\rho+i\frac{\beta}{2}+i\alpha} - \frac{e^{2\pi i\nu x}}{\rho+i\frac{\beta}{2}+i\alpha} + \frac{e^{2\pi i\alpha x}}{\rho-i\frac{\beta}{2}-i\alpha}\right)\\
&= \frac{\pi}{\rho}\left(\frac{e^{2\pi i\alpha x}}{\mu-\alpha} + \frac{e^{2\pi i\nu x}}{\nu-\alpha} - \frac{e^{2\pi i\alpha x}}{\nu-\alpha}\right).
\end{aligned}$$

Case II: When $x \leq 0$,

$$\begin{aligned}
g(x) &= \frac{2\pi^2 i}{\rho}\lim_{t\to\infty}\int_0^t e^{2\pi i\alpha y} e^{-2\pi\rho(y-x)} e^{-2\pi i\frac{\beta}{2}(x-y)}dy\\
&= \frac{2\pi^2 i}{\rho}\lim_{t\to\infty}\int_0^t e^{\left(-2\pi\rho+2\pi i\frac{\beta}{2}+2\pi i\alpha\right)y} e^{\left(2\pi\rho-2\pi i\frac{\beta}{2}\right)x}dy\\
&= \frac{2\pi^2 i}{\rho}\lim_{t\to\infty}\int_0^t e^{-\left(2\pi\rho-2\pi i\frac{\beta}{2}-2\pi i\alpha\right)y} e^{2\pi i\mu x}dy.
\end{aligned}$$

The above converges when $Re\left(2\pi\rho - 2\pi i\frac{\beta}{2} - 2\pi i\alpha\right) > 0$, which holds by the similar argument as for the case $x > 0$. Therefore,

$$\begin{aligned}
g(x) &= \frac{2\pi^2 i}{\rho}\lim_{t\to\infty}\int_0^t e^{-\left(2\pi\rho-2\pi i\frac{\beta}{2}-2\pi i\alpha\right)y} e^{2\pi i\mu x}dy\\
&= \frac{2\pi^2 i}{\rho}\left(\frac{e^{2\pi i\mu x}}{2\pi\rho-2\pi i\frac{\beta}{2}-2\pi i\alpha}\right)\\
&= \frac{\pi}{\rho}\left(\frac{e^{2\pi i\mu x}}{\mu-\alpha}\right).
\end{aligned}$$

Thus,

$$g(x) = \begin{cases} \frac{\pi}{\rho}\left(\frac{e^{2\pi i\alpha x}}{\mu-\alpha} + \frac{e^{2\pi i\nu x}}{\nu-\alpha} - \frac{e^{2\pi i\alpha x}}{\nu-\alpha}\right), x > 0 \\ \frac{\pi}{\rho}\left(\frac{e^{2\pi i\mu x}}{\mu-\alpha}\right), x \leq 0. \end{cases}$$

□

Now, that we have Lemmas 2.4 and 2.5 and Theorem 2.6, we shall now use these along with Poisson summation to evaluate the sum $\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$.

Theorem 2.7 *Let $u^3 + Bu^2 + Cu + D$ be as in Theorem 2.6, and using the same notation as there along with $\alpha, \mu, \nu \notin \mathbb{Z}$, we have:*

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^3 + Bn^2 + Cn + D} = \frac{2\pi i}{(\nu-\mu)(\mu-\alpha)(\alpha-\nu)}\left(\frac{\nu-\mu}{e^{2\pi i\alpha}-1} + \frac{\alpha-\nu}{e^{2\pi i\mu}-1} + \frac{\mu-\alpha}{e^{2\pi i\nu}-1}\right).$$

Proof: We observe that $\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$ is exactly $\sum_{n\in\mathbb{Z}} \hat{g}(n)$ where $g(n)$ is defined in Theorem 2.6. By Poisson summation with $v = 0$, we know that $\sum_{n\in\mathbb{Z}} \hat{g}(n) = \sum_{n\in\mathbb{Z}} g(n)$. Therefore, from Theorem 2.6, we have

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^3 + Bn^2 + Cn + D} = \sum_{n\in\mathbb{Z}} g(n) = \sum_{n\geq 1} g(n) + \sum_{n\leq 0} g(n).$$

We compute the two terms in the sum separately. For the first term

$$\sum_{n\geq 1} g(n) = \sum_{n\geq 1}\left(\frac{\pi}{\rho}\left(\frac{e^{2\pi i\alpha n}}{\mu-\alpha} + \frac{e^{2\pi i\nu n}}{\nu-\alpha} - \frac{e^{2\pi i\alpha n}}{\nu-\alpha}\right)\right)$$
$$= \frac{\pi}{\rho}\left(\frac{1}{\mu-\alpha}\sum_{n\geq 1} e^{2\pi i\alpha n} + \frac{1}{\nu-\alpha}\sum_{n\geq 1} e^{2\pi i\nu n} - \frac{1}{\nu-\alpha}\sum_{n\geq 1} e^{2\pi i\alpha n}\right)$$
$$= \frac{\pi}{\rho}\left(\frac{1}{\mu-\alpha}\left(-1 + \sum_{n\geq 0} e^{2\pi i\alpha n}\right) + \frac{1}{\nu-\alpha}\left(-1 + \sum_{n\geq 0} e^{2\pi i\nu n}\right) - \frac{1}{\nu-\alpha}\left(-1 + \sum_{n\geq 0} e^{2\pi i\alpha n}\right)\right)$$
$$= \frac{\pi}{\rho}\left(\frac{1}{\mu-\alpha}\left(-1 + \sum_{n\geq 0} e^{2\pi i\alpha n}\right) + \frac{1}{\nu-\alpha}\left(-1 + \sum_{n\geq 0} e^{2\pi i\nu n} + 1 - \sum_{n\geq 0} e^{2\pi i\alpha n}\right)\right)$$
$$= \frac{\pi}{\rho}\left(\frac{1}{\mu-\alpha}\left(-1 + \sum_{n\geq 0} e^{2\pi i\alpha n}\right) + \frac{1}{\nu-\alpha}\left(\sum_{n\geq 0} e^{2\pi i\nu n} - \sum_{n\geq 0} e^{2\pi i\alpha n}\right)\right).$$

For these to converge, we need $Re(-2\pi i\alpha) > 0$ and $Re(-2\pi i\nu) > 0$. They are equivalent to

$Im(\alpha) > 0$ and $Re(iRe\left(\frac{\beta}{2}\right) - Im\left(\frac{\beta}{2}\right) + Re(\rho) + iIm(\rho)) = Re(\rho) - Im(\frac{\beta}{2}) > 0$, which are true. Therefore,

$$\sum_{n\geq 1} g(n) = \frac{\pi}{\rho}\left(\frac{1}{\mu-\alpha}\left(-1+\sum_{n\geq 0} e^{2\pi i\alpha n}\right) + \frac{1}{\nu-\alpha}\left(\sum_{n\geq 0} e^{2\pi i\nu n} - \sum_{n\geq 0} e^{2\pi i\alpha n}\right)\right)$$

and we evaluate each convergent geometric series to obtain,

$$\sum_{n\geq 1} g(n) = \frac{\pi}{\rho}\left(\frac{1}{\mu-\alpha}\left(-1+\frac{1}{1-e^{2\pi i\alpha}}\right) + \frac{1}{\nu-\alpha}\left(\frac{1}{1-e^{2\pi i\nu}} - \frac{1}{1-e^{2\pi i\alpha}}\right)\right).$$

For the second term in the sum,

$$\begin{aligned}\sum_{n\leq 0} g(n) &= \sum_{n\geq 0} g(-n) = \sum_{n\geq 0}\left(\frac{\pi}{\rho}\left(\frac{e^{2\pi i\mu(-n)}}{\mu-\alpha}\right)\right) \\ &= \frac{\pi}{\rho}\left(\frac{1}{\mu-\alpha}\left(\sum_{n\geq 0} e^{-2\pi i\mu n}\right)\right).\end{aligned}$$

For this to converge, we need $Re(2\pi i\mu) > 0$. This is equivalent to

$$Re(-iRe\left(\frac{\beta}{2}\right) + Im\left(\frac{\beta}{2}\right) + Re(\rho) + iIm(\rho)) = Re(\rho) + Im\left(\frac{\beta}{2}\right) > 0,$$

which is true. Therefore,

$$\sum_{n\leq 0} g(n) = \frac{\pi}{\rho}\left(\frac{1}{\mu-\alpha}\left(\sum_{n\geq 0} e^{-2\pi i\mu n}\right)\right)$$

and we evaluate the geometric series to obtain,

$$\begin{aligned}
\sum_{n\leq 0} g(n) &= \frac{\pi}{\rho}\left(\frac{1}{\mu-\alpha}\left(\frac{1}{1-e^{-2\pi i\mu}}\right)\right)\\
&= \frac{\pi}{\rho}\left(\frac{1}{\mu-\alpha}\left(\frac{e^{2\pi i\mu}}{e^{2\pi i\mu}-1}\right)\right)\\
&= \frac{\pi}{\rho}\left(\frac{1}{\mu-\alpha}\left(\frac{e^{2\pi i\mu}-1+1}{e^{2\pi i\mu}-1}\right)\right)\\
&= \frac{\pi}{\rho}\left(\frac{1}{\mu-\alpha}\left(1+\frac{1}{e^{2\pi i\mu}-1}\right)\right)\\
&= \frac{\pi}{\rho}\left(\frac{1}{\mu-\alpha}\left(1-\frac{1}{1-e^{2\pi i\mu}}\right)\right).
\end{aligned}$$

Thus, $\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$ is equal to

$$\frac{\pi}{\rho}\left(\frac{1}{\mu-\alpha}\left(-1+\frac{1}{1-e^{2\pi i\alpha}}\right)+\frac{1}{\nu-\alpha}\left(\frac{1}{1-e^{2\pi i\nu}}-\frac{1}{1-e^{2\pi i\alpha}}\right)\right)+\frac{\pi}{\rho}\left(\frac{1}{\mu-\alpha}\left(1-\frac{1}{1-e^{2\pi i\mu}}\right)\right)$$

which simplifies to

$$\frac{\pi}{\rho}\left(\frac{1}{\mu-\alpha}\left(\frac{1}{1-e^{2\pi i\alpha}}-\frac{1}{1-e^{2\pi i\mu}}\right)+\frac{1}{\nu-\alpha}\left(\frac{1}{1-e^{2\pi i\nu}}-\frac{1}{1-e^{2\pi i\alpha}}\right)\right).$$

We have that $\mu=-\frac{\beta}{2}-i\rho$ and $\nu=-\frac{\beta}{2}+i\rho$. This implies $\mu-\nu=-2i\rho$, and thus $\rho=\frac{\mu-\nu}{-2i}$; which implies that $\displaystyle\sum_{n\in\mathbb{Z}}\frac{1}{n^3+Bn^2+Cn+D}$ is equal to

$$\begin{aligned}
&\frac{-2\pi i}{(\mu-\nu)}\left(\frac{1}{\mu-\alpha}\left(\frac{1}{1-e^{2\pi i\alpha}}-\frac{1}{1-e^{2\pi i\mu}}\right)+\frac{1}{\nu-\alpha}\left(\frac{1}{1-e^{2\pi i\nu}}-\frac{1}{1-e^{2\pi i\alpha}}\right)\right)\\
=\ &\frac{-2\pi i}{(\mu-\nu)(\mu-\alpha)(\nu-\alpha)}\left(\frac{\nu-\alpha}{1-e^{2\pi i\alpha}}-\frac{\nu-\alpha}{1-e^{2\pi i\mu}}+\frac{\mu-\alpha}{1-e^{2\pi i\nu}}-\frac{\mu-\alpha}{1-e^{2\pi i\alpha}}\right)\\
=\ &\frac{-2\pi i}{(\mu-\nu)(\mu-\alpha)(\nu-\alpha)}\left(\frac{\nu-\mu}{1-e^{2\pi i\alpha}}-\frac{\nu-\alpha}{1-e^{2\pi i\mu}}+\frac{\mu-\alpha}{1-e^{2\pi i\nu}}\right)\\
=\ &\frac{2\pi i}{(\nu-\mu)(\mu-\alpha)(\alpha-\nu)}\left(\frac{\nu-\mu}{e^{2\pi i\alpha}-1}+\frac{\alpha-\nu}{e^{2\pi i\mu}-1}+\frac{\mu-\alpha}{e^{2\pi i\nu}-1}\right).
\end{aligned}$$

□

It should be noted that the roots of the cubic $u^3 + Bu^2 + Cu + D$ in 2.7, namely, α, μ, and ν cannot be integers or else, the sum (1.2) will have division by 0. We shall now attempt to compute the sum $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$.

2.3 Computing a closed form for $\displaystyle\sum_{n\in\mathbb{Z}} \frac{1}{(n^3 + Bn^2 + Cn + D)^k}$.

In this section, we investigate the sum $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$ and attempt to find its closed form using the results derived in the earlier sections. In computing the closed form for the sum, we will make use of discriminant of the cubic equation, so we begin the discussion by analyzing an important property of discriminants of cubic equations.

If α, μ and ν are roots of the cubic equation $ax^3 + bx^2 + cx + d = 0$, then its discriminant Δ is given by

$$\Delta = (a^2(\alpha - \mu)(\alpha - \nu)(\mu - \nu))^2.$$

Suppose $x^3 + Bx^2 + Cx + D$ can be factored as

$$x^3 + Bx^2 + Cx + D = (x - \alpha)(x^2 + \beta x + \gamma)$$

where $\alpha \in \mathbb{C}$ is a root. Then, writing $\gamma = \rho^2 + \frac{\beta^2}{4}$, we have the other two roots being

$$\mu = -\frac{\beta}{2} - i\rho,\ \nu = -\frac{\beta}{2} + i\rho.$$

Recalling from Cardano's formula for finding roots of a cubic equation, we have that $x^3 + Bx^2 + Cx + D = 0$ has roots

$$\begin{aligned} x_1 &= \sqrt[3]{M} + \sqrt[3]{N} - \frac{B}{3}, \\ x_2 &= \sqrt[3]{M}\omega + \sqrt[3]{N}\omega^2 - \frac{B}{3}, \\ x_3 &= \sqrt[3]{M}\omega^2 + \sqrt[3]{N}\omega - \frac{B}{3} \end{aligned}$$

where $M = \frac{-q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}, N = \frac{-q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}$, $p = \left(C - \frac{1}{3}B^2\right)$ and $q = \left(\frac{2}{27}B^3 - \frac{1}{3}BC + D\right)$.

Supposing $\alpha, \mu, \nu \notin \mathbb{Z}$ and are distinct, let us consider a case where $\alpha = x_3$. Then,

$$x_1 = i\left(\rho + i\frac{\beta}{2}\right) = \mu, \tag{2.1}$$

$$x_2 = -i\left(\rho - i\frac{\beta}{2}\right) = \nu. \tag{2.2}$$

Clearly, one can observe that x_1, x_2 and x_3 are functions of q. Also, from Cardano's formula for finding roots of the cubic, we have

$$q = \frac{2}{27}B^3 - \frac{1}{3}BC + D$$

which implies

$$D = q + \frac{BC}{3} - \frac{2B^3}{27}.$$

Considering a case where $x = n$ in the above discussion, the closed form in Theorem 2.7 has that $\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$ is equal to

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+q+\frac{BC}{3}-\frac{2B^3}{27})} \\
= \;& \frac{2\pi i}{(\nu-\mu)(\mu-\alpha)(\alpha-\nu)}\left(\frac{\nu-\mu}{e^{2\pi i\alpha}-1}+\frac{\alpha-\nu}{e^{2\pi i\mu}-1}+\frac{\mu-\alpha}{e^{2\pi i\nu}-1}\right) \\
= \;& \frac{2\pi i}{\sqrt{\Delta}}\left(\frac{\nu-\mu}{e^{2\pi i\alpha}-1}+\frac{\alpha-\nu}{e^{2\pi i\mu}-1}+\frac{\mu-\alpha}{e^{2\pi i\nu}-1}\right).
\end{aligned}$$

From (1.10), we have that $\Delta = -(27q^2+4p^3)$, therefore, the previous equation can be further modified as

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+q+\frac{BC}{3}-\frac{2B^3}{27})} \\
= \;& \frac{2\pi i}{\sqrt{-(27q^2+4p^3)}}\left(\frac{\nu-\mu}{e^{2\pi i\alpha}-1}+\frac{\alpha-\nu}{e^{2\pi i\mu}-1}+\frac{\mu-\alpha}{e^{2\pi i\nu}-1}\right) \\
= \;& \frac{2\pi}{\sqrt{(27q^2+4p^3)}}\left(\frac{\nu-\mu}{e^{2\pi i\alpha}-1}+\frac{\alpha-\nu}{e^{2\pi i\mu}-1}+\frac{\mu-\alpha}{e^{2\pi i\nu}-1}\right). \qquad (2.3)
\end{aligned}$$

Clearly, the closed form of the sum in (2.3) can be written as a composition of different functions of q and to obtain a closed form for $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$, we can differentiate (2.3) with respect to $q, k-1$ times. Equation (2.3) can be written as composition of functions of q as

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D} = \sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+q+\frac{BC}{3}-\frac{2B^3}{27})} =$$

$$2\pi[(f_1(f_2(f_3)))((f_6-f_5)(f_4(f_7))+(f_7-f_6)(f_4(f_5))+(f_5-f_7)(f_4(f_6)))] \qquad (2.4)$$

where $f_1(q) = \frac{1}{q}, \quad f_2(q) = \sqrt{q}, \quad f_3(q) = 27q^2+4p^3, \quad f_4(q) = \frac{1}{e^{2\pi iq}-1}$,
$f_5(q) = \mu = x_1, \quad f_6(q) = \nu = x_2, \quad f_7(q) = \alpha = x_3$.

To differentiate (2.3) with respect to q, we need to differentiate this composition of functions. For this, we will need to make use of some lemmas involving higher order chain rule and product rule. Stated below are some well known lemmas that shall prove to be helpful in differentiating the composition of functions of q in (2.4).

Lemma 2.8 (Faà di Bruno's Formula) *If F and G are functions with a sufficient number of derivatives, then the m^{th} derivative of $F(G)$ is given by*

$$(F(G))^{(m)} = \sum_{b_1,\ldots,b_m} \frac{m!F^{(b_1+\cdots+b_m)}(G)}{b_1!\cdots b_m!} \prod_{j=1}^{m} \left(\frac{G^{(j)}}{j!}\right)^{b_j}$$

where the sum is over all nonnegative integers $b_1, \ldots, b_m$ such that $b_1 + 2b_2 + \cdots + mb_m = m$.

Faá di Bruno's Formula is a higher order chain rule and a proof can be found in [8]. In the formula above, when $G^{(j)} = 0$ and $b_j = 0$, we interpret $\left(\frac{G^{(j)}}{j!}\right)^{b_j} = 1$. The following lemma is a higher order product rule which can also be found in [13].

Lemma 2.9 *If G and H are functions with a sufficient number of derivatives, then*

$$(GH)^{(m)} = \sum_{i=0}^{m} \binom{m}{i} G^{(m-i)} H^{(i)}.$$

Proof. We shall prove this lemma using principle of induction. We shall induct on m. Clearly, the lemma holds for $m = 0$. For $m > 0$,

$$\begin{aligned}
(GH)^{(m)} &= ((GH)')^{(m-1)} \\
&= (G'H)^{(m-1)} + (GH')^{(m-1)} \\
&= \sum_{i=0}^{m-1} \binom{m-1}{i} G^{(m-i)} H^{(i)} + \sum_{i=0}^{m-1} \binom{m-1}{i} G^{(m-1-i)} H^{(i+1)} \\
&= G^{(m)}H + \sum_{i=1}^{m-1} \binom{m-1}{i} G^{(m-i)} H^{(i)} + \sum_{i=0}^{m-1} \binom{m-1}{i} G^{(m-1-i)} H^{(i+1)}.
\end{aligned}$$

Replacing i by $i-1$ in the second sum, we get

$$(GH)^{(m)} = G^{(m)}H + \sum_{i=1}^{m-1} \binom{m-1}{i} G^{(m-i)}H^{(i)} + \sum_{i=1}^{m} \binom{m-1}{i-1} G^{(m-i)}H^{(i)}.$$

This last sum can be written as

$$G^{(m)}H + \sum_{i=1}^{m-1} \left(\binom{m-1}{i} + \binom{m-1}{i-1} \right) G^{(m-i)}H^{(i)} + GH^{(m)}.$$

We shall use Pascal's identity for binomial coefficients to simplify the previous equation. Pascal's Identity simply states that for any natural number n and any $0 \leq k \leq n$, we have

$$\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}.$$

Using Pascal's Identity, we find that

$$\begin{aligned} (GH)^{(m)} &= G^{(m)}H + \left(\sum_{i=1}^{m-1} \binom{m}{i} G^{(m-i)}H^{(i)} \right) + GH^{(m)} \\ &= \sum_{i=0}^{m} \binom{m}{i} G^{(m-i)}H^{(i)} \end{aligned}$$

□

We next examine derivatives of the various functions which comprise (2.4). The first two are well known.

Lemma 2.10 *For a function $F(x) = \frac{1}{x}$, we have*

$$F^{(m)}(x) = \frac{(-1)^m m!}{x^{m+1}}.$$

Proof. We shall prove this lemma using principle of induction. We shall use induction on m. Given $F(x) = \frac{1}{x}$, the result holds true for $m = 0$ clearly. For $m > 0$, we have

$$\begin{aligned}
F^{(m)}(x) &= (F^{(m-1)}(x))' \\
&= \left(\frac{(-1)^{m-1}(m-1)!}{x^{m-1+1}}\right)' \\
&= (-1)^{m-1}(m-1)!(x^{-m})' \\
&= (-1)^m m! x^{-m-1} \\
&= \frac{(-1)^m m!}{x^{m+1}}.
\end{aligned}$$

This completes the proof. □

Lemma 2.11 *Given a function $G(x) = \sqrt{x+a}$, we have for $m > 0$*

$$G^{(m)}(x) = \frac{(-1)^{m-1}(2m-2)!(x+a)^{-\frac{(2m-1)}{2}}}{2^{2m-1}(m-1)!}.$$

Proof: We shall prove this lemma using principle of induction. We shall use induction on m. Given $G(x) = \sqrt{x+a}$, the result holds true for $m = 1$ clearly. For $m > 2$, we have

$$\begin{aligned}
G^{(m)}(x) &= (G^{(m-1)}(x))' \\
&= \left(\frac{(-1)^{m-1-1}(2(m-1)-2)!(x+a)^{-\frac{(2(m-1)-1)}{2}}}{2^{2(m-1)-1}(m-1-1)!}\right)' \\
&= \frac{(-1)^{m-2}(2m-4)!}{2^{2m-3}(m-2)!}\left((x+a)^{-\frac{(2m-3)}{2}}\right)' \\
&= \frac{(-1)^{m-2}(2m-4)!}{2^{2m-3}(m-2)!}(-\frac{(2m-3)}{2})\left((x+a)^{-\frac{(2m-3)}{2}-1}\right) \\
&= \frac{(-1)^{m-1}(2m-3)!(x+a)^{-\frac{(2m-1)}{2}}}{2^{2m-2}(m-2)!} \\
&= \frac{(-1)^{m-1}(2m-2)!(x+a)^{-\frac{(2m-1)}{2}}}{2^{2m-1}(m-1)!}.
\end{aligned}$$

This completes the proof. □

The next lemma is followed from Lemma 3 in [13] and Theorem 3.1 in [24]. The lemma involves Stirling numbers of the second kind, denoted $S(n,k)$. These numbers are defined as the number of ways of partitioning n objects into k non-empty parts. The Stirling numbers of the second kind satisfy the recurrence relation

$$S(n+1,k) = kS(n,k) + S(n,k-1),$$

for any natural number n and $0 \leq k \leq n$. This recurrence relation can be proved using the principle of inclusion-exclusion.

Lemma 2.12 *For $m \geq 0$,*

$$\left(\frac{1}{e^{ax}-1}\right)^{(m)} = (-1)^m a^m \sum_{k=1}^{m+1} \frac{(k-1)!S(m+1,k)}{(e^{ax}-1)^k}$$

where $S(m+1,k) \in \mathbb{Z}$ is a Stirling number of the second kind.

Proof: We shall prove this result by principle of induction. We shall use induction on m. Clearly, the lemma holds true for $m = 0$ since $S(1,1) = 1$. Let us suppose that it is true for $m-1$. By induction we have

$$\left(\left(\frac{1}{e^{ax}-1}\right)^{(m-1)}\right)' = (-1)^{m-1}a^{m-1}\left(\sum_{k=1}^{m} \frac{(k-1)!S(m,k)}{(e^{ax}-1)^k}\right)'$$

which equals

$$(-1)^m a^m \sum_{k=1}^{m} k!S(m,k)\frac{e^{ax}}{(e^{ax}-1)^{k+1}} = (-1)^m a^m \sum_{k=1}^{m} k!S(m,k)\frac{e^{ax}-1+1}{(e^{ax}-1)^{k+1}}.$$

Writing $(e^{ax}-1+1)/(e^{ax}-1)^{k+1} = 1/(e^{ax}-1)^k + 1/(e^{ax}-1)^{k+1}$, our sum can be written as

$$(-1)^m a^m \left(\frac{1}{e^{ax}-1} + \sum_{k=2}^{m} \frac{(k-1)!(kS(m,k)+S(m,k-1))}{(e^{ax}-1)^k} + \frac{m!}{(e^{ax}-1)^{m+1}} \right)$$

which is same as

$$(-1)^m a^m \left(\frac{S(m+1,1)}{e^{ax}-1} + \sum_{k=2}^{m} \frac{(k-1)!(kS(m,k)+S(m,k-1))}{(e^{ax}-1)^k} + \frac{m!S(m+1,m+1)}{(e^{ax}-1)^{m+1}} \right)$$

because $S(m+1,1)=1$ and $S(m+1,m+1)=1$. Using recurrence relation for the Stirling numbers,$S(n+1,k)=kS(n,k)+S(n,k-1)$, we have the sum equal to

$$(-1)^m a^m \left(\frac{S(m+1,1)}{e^{ax}-1} + \sum_{k=2}^{m} \frac{(k-1)!S(m+1,k)}{(e^{ax}-1)^k} + \frac{m!S(m+1,m+1)}{(e^{ax}-1)^{m+1}} \right)$$

$$= (-1)^m a^m \sum_{k=1}^{m+1} \frac{(k-1)!S(m+1,k)}{(e^{ax}-1)^k}.$$

This completes the proof. □

The combinatorial significance of the Stirling numbers of the second kind in the formula given in Lemma 2.12 is not clear and could be taken up for research for future work.

From (2.4) we have that $\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$ is equal to

$$2\pi[(f_1(f_2(f_3)))((f_6-f_5)(f_4(f_7))+(f_7-f_6)(f_4(f_5))+(f_5-f_7)(f_4(f_6)))]$$

To obtain a closed form for $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$,we shall differentiate $\frac{1}{n^3+Bn^2+Cn+D}$ with respect to $q,(k-1)$ times. We shall do this step by step by differentiating smaller compositions of functions embedded within (2.4) because it is not feasible to differentiate nearly twelve compositions of seven functions altogether.

Step 1.1: Finding $\frac{1}{(n^3+Bn^2+Cn+D)^k}$

Differentiating (2.4) with respect to $q,\ k-1$ times, we get

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k} = \sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+q+\frac{BC}{3}-\frac{2B^3}{27})^k}$$

$$= 2\pi[(f_1(f_2(f_3)))((f_6-f_5)(f_4(f_7))+(f_7-f_6)(f_4(f_5))+(f_5-f_7)(f_4(f_6)))]^{(k-1)}$$

We let $f_8 = f_1(f_2(f_3))$ and $f_9 = ((f_6-f_5)(f_4(f_7))+(f_7-f_6)(f_4(f_5))+(f_5-f_7)(f_4(f_6))$. This implies

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k} = ((f_8)(f_9))^{(k-1)}$$

Using Lemma 2.9 we get

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k} = \sum_{i_1=0}^{k-1} \binom{k-1}{i_1} f_8^{(k-1-i_1)} f_9^{(i_1)}$$

Step 1.2: Finding ${f_8}^{(k-1-i_1)}$

$${f_8}^{(k-1-i_1)} = f_1(f_2(f_3))^{(k-1-i_1)}$$

We let $f_{10} = f_2(f_3)$ and using Lemma 2.8 we find that ${f_8}^{(k-1-i_1)}$ is equal to

$$f_1(f_2(f_3))^{(k-1-i_1)} = f_1(f_{10})^{(k-1-i_1)}$$

$$= \sum_{b_{1,1},b_{1,2}\ldots,b_{1,(k-1-i_1)}} \frac{(k-1-i_1)!\,f_1^{(b_{1,1}+b_{1,2}+\cdots+b_{1,(k-1-i_1)})}(f_{10})}{b_{1,1}!b_{1,2}!\cdots b_{1,(k-1-i_1)}!} \prod_{j_1=1}^{k-1-i_1} \left(\frac{f_{10}^{(j_1)}}{j_1!}\right)^{b_{1,j_1}}.$$

We let $R_1 = b_{1,1}+b_{1,2}+\cdots+b_{1,(k-1-i_1)}$ and use Lemma 2.10 to get that ${f_8}^{(k-1-i_1)}$ is equal to

$$\sum_{b_{1,1},b_{1,2}\ldots,b_{1,(k-1-i_1)}} \frac{(k-1-i_1)!f_1^{(R_1)}(f_{10})}{b_{1,1}!b_{1,2}!\cdots b_{1,(k-1-i_1)}!} \prod_{j_1=1}^{(k-1-i_1)} \left(\frac{f_{10}^{(j_1)}}{j_1!}\right)^{b_{1,j_1}}$$

$$= \sum_{b_{1,1},b_{1,2}\ldots,b_{1,(k-1-i_1)}} \frac{(k-1-i_1)!(-1)^{(R_1)}R_1!}{b_{1,1}!b_{1,2}!\cdots b_{1,(k-1-i_1)}!(f_{10})^{R_1+1}} \prod_{j_1=1}^{(k-1-i_1)} \left(\frac{f_{10}^{(j_1)}}{j_1!}\right)^{b_{1,j_1}}$$

To complete these calculations, we need to know higher derivatives of f_{10}, which are computed next.

Step 1.3: Finding $f_{10}^{(j_1)}$

$$f_{10}^{(j_1)} = (f_2(f_3))^{(j_1)}$$

Lemma 2.8 will be used once again here to find the higher derivative. Using the Lemma, we have that $f_{10}^{(j_1)}$ is equal to

$$\sum_{b_{2,1},b_{2,2}\ldots,b_{2,j_1}} \frac{(j_1)!f_2^{(b_{2,1}+b_{2,2}+\cdots+b_{2,j_1})}(f_3)}{b_{2,1}!b_{2,2}!\cdots b_{2,j_1}!} \prod_{j_2=1}^{j_1} \left(\frac{f_3^{(j_2)}}{j_2!}\right)^{b_{2,j_2}}$$

We let $R_2 = b_{2,1} + b_{2,2} + \cdots + b_{2,j_2}$ and using Lemma 2.11 with $a = 0$ we find that $f_{10}^{(j_1)}$

$$\sum_{b_{2,1},b_{2,2}\ldots,b_{2,j_1}} \frac{(j_1)!f_2^{(R_2)}(f_3)}{b_{2,1!}b_{2,2}!\cdots b_{2,j_1}!} \prod_{j_2=1}^{j_1} \left(\frac{f_3^{(j_2)}}{j_2!}\right)^{b_{2,j_2}}$$

$$= \sum_{b_{2,1},b_{2,2}\ldots,b_{2,j_1}} \frac{(j_1)!}{b_{2,1}!b_{2,2}!\cdots b_{2,j_1}!} \frac{(-1)^{R_2-1}(2R_2-2)!(f_3)^{-\frac{(2R_2-1)}{2}}}{2^{2R_2-1}(R_2-1)!} \prod_{j_2=1}^{j_1} \left(\frac{f_3^{(j_2)}}{j_2!}\right)^{b_{2,j_2}}$$

where $f_3(q) = 27q^2 + 4p^3$ and $f_3^{(j_2)} = \begin{cases} 54q, & j_2 = 1, \\ 54, & j_2 = 2, \\ 0, & j_2 > 2. \end{cases}$

Next we compute $f_9^{(i_1)}$.

Step 1.4: Finding $f_9^{(i_1)}$

$$\begin{aligned} f_9^{(i_1)} &= ((f_6 - f_5)(f_4(f_7)) + (f_7 - f_6)(f_4(f_5)) + (f_5 - f_7)(f_4(f_6))^{(i_1)} \\ &= ((f_6 - f_5)(f_4(f_7))^{(i_1)} + (f_7 - f_6)(f_4(f_5))^{(i_1)} + (f_5 - f_7)(f_4(f_6))^{(i_1)} \end{aligned}$$

Step 1.5: Finding $((f_6 - f_5)(f_4(f_7))^{(i_1)}$

This be calculated with the help of Lemma 2.9, which gives us

$$((f_6 - f_5)(f_4(f_7))^{(i_1)} = \sum_{i_2=0}^{i_1} \binom{i_2}{i_1} (f_6 - f_5)^{(i_2-i_1)} (f_4(f_7))^{(i_2)}$$

where $f_5 = x_1, f_6 = x_2$ and $(f_6 - f_5)^{(i_2-i_1)} = f_6^{(i_2-i_1)} - f_5^{(i_2-i_1)}$. Higher derivatives of f_5 and f_6 are computed later in this section and $(f_4(f_7))^{(i_2)}$ is computed next.

Step 1.6: Finding $(f_4(f_7))^{(i_2)}$

Using Lemma 2.8, we have

$$(f_4(f_7))^{(i_2)} = \sum_{b_{3,1}, b_{3,2} \dots, b_{3,i_2}} \frac{(i_2)! f_4^{(b_{3,1}+b_{3,2}+\cdots+b_{3,i_2})}(f_7)}{b_{3,1}! b_{3,2}! \cdots b_{3,i_2}!} \prod_{j_3=1}^{i_2} \left(\frac{f_7^{(j_3)}}{j_3!} \right)^{b_{3,j_3}}$$

Letting $R_3 = b_{3,1} + b_{3,2} + \cdots + b_{3,i_2}$ and use of Lemma 2.12 with $a = 2\pi i$ simplifies this equation to $(f_4(f_7))^{(i_2)}$ is equal to

$$\begin{aligned} &= \sum_{b_{3,1}, b_{3,2} \dots, b_{3,i_2}} \frac{(i_2)! f_4^{(R_3)}(f_7)}{b_{3,1}! b_{3,2}! \cdots b_{3,i_2}!} \prod_{j_3=1}^{i_2} \left(\frac{f_7^{(j_3)}}{j_3!} \right)^{b_{3,j_3}} \\ &= \sum_{b_{3,1}, b_{3,2}, \dots, b_{3,i_2}} \frac{(i_2)!(-1)^{R_3}(2\pi i)^{R_3}}{b_{3,1}! b_{3,2}! \cdots b_{3,i_2}!} \sum_{i_3=1}^{R_3+1} \frac{(i_3-1)! S(R_3+1, i_3)}{(e^{2\pi i f_7(q)} - 1)^{i_3}} \prod_{j_3=1}^{i_2} \left(\frac{f_7^{(j_3)}}{j_3!} \right)^{b_{3,j_3}} \end{aligned}$$

where $f_7(q) = x_3$ and its higher derivatives are computed towards later parts of this section.

Step 1.7: Finding $((f_7 - f_6)(f_4(f_5))^{(i_1)}$

This will be calculated with the help of Lemma 2.9, which gives us

$$((f_7 - f_6)(f_4(f_5))^{(i_1)} = \sum_{i_4=0}^{i_1} \binom{i_4}{i_1} (f_7 - f_6)^{(i_4 - i_1)} (f_4(f_5))^{(i_4)}$$

where $f_6 = x_2, f_7 = x_3$ and $(f_7 - f_6)^{(i_4 - i_1)} = f_7^{(i_4 - i_1)} - f_6^{(i_4 - i_1)}$. Higher derivatives of f_6 and f_7 are computed later in this section and $(f_4(f_5))^{(i_4)}$ is computed next.

Step 1.8: Finding $(f_4(f_5))^{(i_4)}$

Using Lemma 2.8, we have

$$(f_4(f_5))^{(i_4)} = \sum_{b_{4,1}, b_{4,2}, \ldots, b_{4,i_4}} \frac{(i_4)! f_4^{(b_{4,1} + b_{4,2} + \cdots + b_{4,i_4})}(f_5)}{b_{4,1}! b_{4,2}! \cdots b_{4,i_4}!} \prod_{j_4=1}^{i_4} \left(\frac{f_5^{(j_4)}}{j_4!} \right)^{b_{4,j_4}}$$

Letting $R_4 = b_{4,1} + b_{4,2} + \cdots + b_{4,i_4}$ and use of Lemma 2.12 with $a = 2\pi i$ simplifies this equation to $(f_4(f_5))^{(i_4)}$ is equal to

$$\begin{aligned}
&= \sum_{b_{4,1}, b_{4,2} \ldots, b_{4,i_4}} \frac{(i_4)! f_4^{(R_4)}(f_5)}{b_{4,1}! b_{4,2}! \cdots b_{4,i_4}!} \prod_{j_4=1}^{i_4} \left(\frac{f_5^{(j_4)}}{j_4!} \right)^{b_{4,j_4}} \\
&= \sum_{b_{4,1}, b_{4,2}, \ldots, b_{4,i_4}} \frac{(i_4)! (-1)^{R_4} (2\pi i)^{R_4}}{b_{4,1}! b_{4,2}! \cdots b_{4,i_4}!} \sum_{i_5=1}^{R_4+1} \frac{(i_5 - 1)! S(R_4 + 1, i_5)}{(e^{2\pi i f_5(q)} - 1)^{i_5}} \prod_{j_4=1}^{i_4} \left(\frac{f_5^{(j_4)}}{j_4!} \right)^{b_{4,j_4}}
\end{aligned}$$

where $f_5(q) = x_1$ and its higher derivatives are computed towards later parts of this section.

Step 1.9: Finding $((f_5 - f_7)(f_4(f_6))^{(i_1)}$

This be calculated with the help of Lemma 2.9, which gives us

$$((f_5 - f_7)(f_4(f_6))^{(i_1)} = \sum_{i_6=0}^{i_1} \binom{i_6}{i_1} (f_5 - f_7)^{(i_6 - i_1)} (f_4(f_6))^{(i_6)}$$

where $f_5 = x_1, f_7 = x_3$ and $(f_5 - f_7)^{(i_6-i_1)} = f_5^{(i_6-i_1)} - f_7^{(i_6-i_1)}$. Higher derivatives of f_7 and f_5 are computed towards the later parts of this section and $(f_4(f_6))^{(i_6)}$ is computed next.

Step 1.10: Finding $(f_4(f_6))^{(i_6)}$

Using Lemma 2.8, we have

$$(f_4(f_6))^{(i_6)} = \sum_{b_{5,1},b_{5,2}\dots,b_{5,i_6}} \frac{(i_6)!f_4^{(b_{5,1}+b_{5,2}+\cdots+b_{5,i_6})}(f_6)}{b_{5,1}!b_{5,2}!\cdots b_{5,i_6}!} \prod_{j_5=1}^{i_6} \left(\frac{f_6^{(j_5)}}{j_5!}\right)^{b_{5,j_5}}$$

Letting $R_5 = b_{5,1} + b_{5,2} + \cdots + b_{5,i_6}$ and use of Lemma 2.12 with $a = 2\pi i$ simplifies this equation to $(f_4(f_6))^{(i_6)}$ is equal to

$$\begin{aligned}
&= \sum_{b_{5,1},b_{5,2}\dots,b_{5,i_6}} \frac{(i_6)!f_4^{(R_5)}(f_6)}{b_{5,1}!b_{5,2}!\cdots b_{5,i_6}!} \prod_{j_5=1}^{i_6} \left(\frac{f_6^{(j_5)}}{j_5!}\right)^{b_{5,j_5}} \\
&= \sum_{b_{5,1},b_{5,2}\dots,b_{5,i_6}} \frac{(i_6)!(-1)^{R_5}(2\pi i)^{R_3}}{b_{5,1}!b_{5,2}!\cdots b_{5,i_6}!} \sum_{i_7=1}^{R_5+1} \frac{(i_7-1)!S(R_5+1,i_7)}{(e^{2\pi i f_6(q)}-1)^{i_7}} \prod_{j_5=1}^{i_6} \left(\frac{f_6^{(j_5)}}{j_5!}\right)^{b_{5,j_5}}
\end{aligned}$$

where $f_6(q) = x_2$ and its higher derivatives are computed next in this section.

The closed form for $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$ can be obtained by combining the calculations carried out in Steps 1.1 - 1.10. To get simplified closed form, we need to further calculate higher derivatives of f_5, f_6 and f_7 i.e. x_1, x_2 and x_3 respectively. Therefore, we shall now attempt to compute higher derivatives of x_1, x_2 and x_3. We have that

$$x_1 = \sqrt[3]{M} + \sqrt[3]{N} - \frac{B}{3}, \;\; x_2 = \sqrt[3]{M}\omega + \sqrt[3]{N}\omega^2 - \frac{B}{3}, \;\; x_3 = \sqrt[3]{M}\omega^2 + \sqrt[3]{N}\omega - \frac{B}{3}$$

where $M = \frac{-q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}, N = \frac{-q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}$, $p = \left(C - \frac{1}{3}B^2\right)$ and $q = \left(\frac{2}{27}B^3 - \frac{1}{3}BC + D\right)$. Observing x_1, x_2 and x_3 carefully, one can easily deduce that to compute their higher derivatives, it is enough to compute higher derivatives of any one of them and others can be calculated in similar fashion. Also, to compute their higher derivatives, it is enough to compute higher derivatives of $\sqrt[3]{M}$ and $\sqrt[3]{N}$. Therefore, we shall now attempt to compute higher derivatives of $\sqrt[3]{M}$ and $\sqrt[3]{N}$. Let us write

$$g_{\pm 1}(q) = \frac{-q}{2} \pm \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}$$

Then, $g_{+1} = M$ and $g_{-1} = N$. To compute n^{th} derivative of $\sqrt[3]{M}$ and $\sqrt[3]{N}$, it is enough to compute n^{th} derivative of $\sqrt[3]{g_{\pm 1}}$. We can write $\sqrt[3]{g_{\pm 1}}$ as composition of functions as follows:

$$\sqrt[3]{g_{\pm 1}} = g_5(g_{-2} \pm g_4(g_3(g_{+2})))$$

where $g_{\pm 2} = \pm\frac{q}{2}$, $g_3 = q^2$, $g_4 = \sqrt{q + \left(\frac{p}{3}\right)^3}$ and $g_5 = \sqrt[3]{q}$. Clearly, $g^{(1)}_{\pm 2}(q) = \pm\frac{1}{2}$ and $g^{(m)}_{\pm 1}(q) = 0$ for $m > 1$. Similarly, it is obvious that $g^{(1)}_3(q) = 2q, g^{(2)}_3(q) = 2$ and $g^{(m)}_3(q) = 0$ for $m > 2$. Higher derivatives of g_4 can be computed using Lemma 2.11 and those of g_5 are discussed below in the following well-known lemma.

Lemma 2.13 *Given a function $H(x) = \sqrt[3]{x}$, we have*

$$H^{(m)}(x) = \left(\prod_{k=1}^{m}(4 - 3k)\right)\frac{1}{3^m}x^{\frac{(1-3m)}{3}}.$$

Proof: We shall prove this by principle of induction. We shall use induction on m. For $H(x) = \sqrt[3]{x}$, the result holds true for $m = 0$ clearly since the empty product

$\prod_{k=1}^{0}$, is by convention, equal to 1. For $m > 0$, we have

$$\begin{aligned}
H^{(m)}(x) &= (H^{(m-1)}(x))' \\
&= \left(\prod_{k=1}^{m-1}(4-3k)\right)\frac{1}{3^{m-1}}\left(x^{\frac{(1-3(m-1))}{3}}\right)' \\
&= \left(\prod_{k=1}^{m-1}(4-3k)\right)\frac{1}{3^{m-1}}\frac{(1-3(m-1))}{3}\left(x^{\frac{(1-3(m-1))}{3}-1}\right) \\
&= \left(\prod_{k=1}^{m-1}(4-3k)\right)\frac{1}{3^{m-1}}\frac{(4-3m)}{3}\left(x^{\frac{(1-3(m-1))}{3}-1}\right) \\
&= \left(\prod_{k=1}^{m}(4-3k)\right)\frac{1}{3^{m}}x^{\frac{(1-3m)}{3}}.
\end{aligned}$$

This completes the proof. □

Now that we have the higher derivatives of $g_{\pm 2}, g_3, g_4$ and g_5, let us find the n^{th} derivative of $\sqrt[3]{g_{\pm 1}}$.

Step 2.1: Finding $\sqrt[3]{g_{\pm 1}}^{(n)}$.

We have that

$$\sqrt[3]{g_{\pm 1}} = g_5(g_{-2} \pm g_4(g_3(g_{+2})))$$

Thus,

$$\sqrt[3]{g_{\pm 1}}^{(n)} = (g_5(g_{-2} \pm g_4(g_3(g_{+2}))))^{(n)}$$

Letting $g_{\pm 6} = g_{-2} \pm g_4(g_3(g_{+2}))$ and using Faá Di Bruno's formula stated in Lemma 2.8, we find that

$$\sqrt[3]{g_{\pm 1}}^{(n)} = (g_5(g_{\pm 6}))^{(n)} = \sum_{b_{6,1},b_{6,2},\cdots b_{6,n}} \frac{n! g_5{}^{(b_{6,1}+b_{6,2}+\cdots+b_{6,n})}(g_{\pm 6})}{b_{6,1}! b_{6,2}! \cdots b_{6,n}!} \prod_{j_6=1}^{n}\left(\frac{g_{\pm 6}{}^{(j_6)}}{j_6!}\right)^{b_{6,j_6}}.$$

Letting $R_6 = b_{6,1} + b_{6,2} + \cdots + b_{6,n}$ and using Lemma 2.13 we get

$$\sqrt[3]{g_{\pm 1}}^{(n)} = \sum_{b_{6,1}, b_{6,2}, \cdots b_{6,n}} \frac{n!(g_{\pm 6}(q))^{\frac{(1-3R_6)}{3}}}{b_{6,1}!b_{6,2}! \cdots b_{6,n}!3^{R_6}} \prod_{k=1}^{R_6}(1-3(k-1)) \prod_{j_6=1}^{n} \left(\frac{{g_{\pm 6}}^{(j_6)}}{j_6!}\right)^{b_{6,j_6}}.$$

The higher derivatives of $g_{\pm 6}$ are computed next.

Step 2.2: Finding $(g_{\pm 6})^{(j_6)}$.

We have that

$$g_{\pm 6} = g_{-2} \pm g_4(g_3(g_{+2}))$$

Thus,

$$(g_{\pm 6})^{(j_6)} = (g_{-2} \pm g_4(g_3(g_{+2})))^{(j_6)}$$

$$(g_{\pm 6})^{(j_6)} = (g_{-2})^{(j_6)} \pm (g_4(g_3(g_{+2})))^{(j_6)}$$

$$(g_{\pm 6})^{(j_6)} = \begin{cases} -\frac{1}{2} \pm (g_4(g_3(g_{+2})))^{(j_6)}, \ j_6 = 1 \\ \pm (g_4(g_3(g_{+2})))^{(j_6)}, \ j_6 > 1. \end{cases}$$

Step 2.3: Finding $(g_4(g_3(g_{+2})))^{(j_6)}$.

Letting $g_7 = g_3(g_{+2})$ and using Faá Di Bruno's formula, we find that

$$(g_4(g_3(g_{+2})))^{(j_6)} = \sum_{b_{7,1}, b_{7,2}, \cdots b_{7,j_6}} \frac{(j_6)!{g_4}^{(b_{7,1}+b_{7,2}+\cdots+b_{7,j_6})}(g_7)}{b_{7,1}!b_{7,2}! \cdots b_{7,j_6}!} \prod_{j_7=1}^{j_6} \left(\frac{(g_7)^{(j_7)}}{j_7!}\right)^{b_{7,j_7}}.$$

Letting $R_7 = b_{7,1} + b_{7,2} + \cdots + b_{7,j_6}$ and using Lemma (2.11) with $a = \left(\frac{p}{3}\right)^3$ we get,
$(g_4(g_3(g_{+2})))^{(j_6)} =$

$$\sum_{b_{7,1}, b_{7,2}, \cdots b_{7,j_6}} \frac{(j_6)!(-1)^{R_7-1}(2R_7-2)!}{b_{7,1}!b_{7,2}! \cdots b_7, j_6!2^{2R_7-1}(R_7-1)!} \left(\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3\right)^{-\frac{(2R_7-1)}{2}} \prod_{j_7=1}^{j_6} \left(\frac{(g_7)^{(j_7)}}{j_7!}\right)^{b_{7,j_7}}.$$

Step 2.4: Finding $g_7^{(j_7)}$.

$$g_7 = g_3(g_{+2})$$

which implies

$$g_7^{(j_7)} = g_3(g_{+2})^{(j_7)}$$

Using Faá Di Bruno's formula, we find that

$$g_7^{(j_7)} = g_3(g_{+2})^{(j_7)} = \sum_{b_{8,1}, b_{8,2}, \cdots b_{8,j_7}} \frac{(j_7)! {g_3}^{(b_{8,1}+b_{8,2}+\cdots+b_{8,j_7})}(g_{+2})}{b_{8,1}! b_{8,2}! \cdots b_{8,j_7}!} \prod_{j_8=1}^{j_7} \left(\frac{(g_{+2})^{(j_8)}}{j_8!} \right)^{b_{8,j_8}}.$$

Letting $R_8 = b_{8,1} + b_{8,2} + \cdots + b_{8,j_7}$, we get

$$g_7^{(j_7)} = g_3(g_{+2})^{(j_7)} = \sum_{b_{8,1}, b_{8,2}, \cdots b_{8,j_7}} \frac{(j_7)! {g_3}^{(R_8)}(g_{+2})}{b_{8,1}! b_{8,2}! \cdots b_{8,j_7}!} \prod_{j_8=1}^{j_7} \left(\frac{(g_{+2})^{(j_8)}}{j_8!} \right)^{b_{8,j_8}},$$

where ${g_3}^{(R_8)} = \begin{cases} 2q, & R_8 = 1, \\ 2, & R_8 = 2, \\ 0, & R_8 > 2 \end{cases}$
and $(g_{+2})^{(j_8)} = \begin{cases} \frac{1}{2}, & j_8 = 1, \\ 0, & j_8 > 1. \end{cases}$

Clearly, it is not feasible to write higher derivatives of $\sqrt[3]{M}$ and $\sqrt[3]{N}$ explicitly here and hence the higher derivatives of x_1, x_2 and x_3. However, these can be computed with the help of computer by building a software or writing a program in which steps 2.1-2.4 can be coded as iterative steps for the calculations and the user can input different values of n. Once we have the n^{th} derivatives of x_1, x_2 and x_3, we can easily compute $(k-1)^{th}$ derivative of $\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$ similarly by writing steps 1.1 -1.10 into the program and inputting values for k. This shall give us the closed form for the sum $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$.

In the following section, we shall investigate a new approach to compute the closed forms for the sums (1.2) and (1.3) and attempt to compute the closed forms using the new method.

Chapter 3

Evaluating the sum using the cotangent function

In this chapter, we will derive many relations found in Chapter 2, using a different method. Namely, we will make use of the cotangent function and its relation to exponential functions to compute closed forms for the sums $\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$ and $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$.

As noted in [18], an important relation is derived from the Hadamard Product for $\sin(\pi z)$,

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

Taking the logarithmic we obtain,

$$\log \sin(\pi z) = \log \pi z + \sum_{n\geq 1} \log \left(1 - \frac{z^2}{n^2}\right).$$

Differentiating the previous equation with respect to z, we get,

$$\pi \cot(\pi z) = \frac{1}{z} + \sum_{n\geq 1} \frac{2z}{z^2 - n^2}.$$

Using partial fractions, we can write the last sum as

$$\begin{aligned}\pi \cot(\pi z) &= \frac{1}{z} + \sum_{n\geq 1}\left(\frac{1}{z-n} + \frac{1}{z+n}\right)\\ &= \sum_{n\in\mathbb{Z}} \frac{1}{n+z}.\end{aligned}$$

Also, it is well known that $\cos x = \frac{e^{ix}+e{-ix}}{2}$ and $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$. This implies

$$\cot x = i\left(\frac{e^{ix}+e{-ix}}{e^{ix}-e^{-ix}}\right) = i\left(\frac{e^{2ix}+1}{e^{2ix}-1}\right).$$

From the above discussion, we deduce that

$$\sum_{n\in\mathbb{Z}} \frac{1}{n+z} = \pi\cot(\pi z) = \pi i\left(\frac{e^{2\pi iz}+1}{e^{2\pi iz}-1}\right). \tag{3.1}$$

We shall use this formula to compute closed forms for the sums (1.2) and (1.3).

3.1 Computing a closed form for $\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$

If $\alpha, \mu, \nu \in \mathbb{C}$ are distinct non-integer roots of the cubic polynomial x^3+Bx^2+Cx+D, then we can write the sum $\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$ using partial fractions as

$$\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D} = \sum_{n\in\mathbb{Z}} \frac{c_1}{n-\alpha} + \sum_{n\in\mathbb{Z}} \frac{c_2}{n-\mu} + \sum_{n\in\mathbb{Z}} \frac{c_3}{n-\nu} \tag{3.2}$$

where $c_1, c_2, c_3 \in \mathbb{C}$ are non-zero numbers and can be computed using the following lemma that also appears in [18].

Lemma 3.1 *For two polynomials $A(x), B(x) \in \mathbb{C}[x]$ with deg$(A) <$ deg(B), $A(x) \neq 0$*

and $B(x)$ *having distinct roots* $\alpha_1, \alpha_2, \ldots \alpha_m$*, we have,*

$$\frac{A(x)}{B(x)} = \sum_{i=1}^{m} c_i \frac{1}{x - \alpha_i},$$

where $c_i = \frac{A(\alpha_i)}{B'(\alpha_i)}$.

Proof: We have by partial fractions,

$$\frac{A(x)}{B(x)} = \sum_{i=1}^{m} c_i \frac{1}{x - \alpha_i}.$$

Without loss of generality, we compute c_1 and note that the other $c_i, 2 \leq i \leq m$ follow the same pattern. Since α_1 is the root of $B(x)$, this implies $B(\alpha_1) = 0$, and thus, we can re-write the previous equation as

$$\frac{A(x)}{B(x) - B(\alpha_1)} = \sum_{i=1}^{m} c_i \frac{1}{x - \alpha_i}$$

Multiplying $(x - \alpha_1)$ on both sides of the equation, we get

$$\frac{A(x)}{B(x) - B(\alpha_1)}(x - \alpha_1) = \sum_{i=1}^{m} c_i \frac{1}{x - \alpha_i}(x - \alpha_1)$$

which is same as

$$A(x)\frac{x - \alpha_1}{B(x) - B(\alpha_1)} = c_1 + \sum_{i=2}^{m} c_i \frac{1}{x - \alpha_i}(x - \alpha_1).$$

Taking the limit as x approaches α_1, we get

$$\lim_{x \to \alpha_1} A(x)\frac{x - \alpha_1}{B(x) - B(\alpha_1)} = \lim_{x \to \alpha_1} \left(c_1 + \sum_{i=2}^{m} c_i \frac{1}{x - \alpha_i}(x - \alpha_1) \right)$$

which implies

$$\frac{A(\alpha_1)}{B'(\alpha_1)} = c_1.$$

The coefficients $c_2, c_3, \ldots, c_m$ can be computed similarly. $\square$

We shall use Lemma (3.1) for $A(x) = 1$ and $B(x) = x^3 + Bx^2 + Cx + D$ to compute coefficients c_1, c_2, c_3 in (3.2). Clearly, $A(x)$ is constant, so we only need to find $B'(x)$ for $x = \alpha, \mu, \nu$. We have,

$$B(x) = x^3 + Bx^2 + Cx + D = (x - \alpha)(x - \mu)(x - \nu)$$

and computing the derivatives gives

$$B'(x) = 3x^2 + 2Bx + C = (x - \mu)(x - \nu) + (x - \alpha)(x - \nu) + (x - \alpha)(x - \mu).$$

Thus,

$$\begin{aligned} B'(\alpha) &= 3\alpha^2 + 2B\alpha + C = (\alpha - \mu)(\alpha - \nu), \\ B'(\mu) &= 3\mu^2 + 2B\mu + C = (\mu - \alpha)(\mu - \nu), \\ B'(\nu) &= 3\nu^2 + 2B\nu + C = (\nu - \alpha)(\nu - \mu). \end{aligned}$$

Using Lemma 3.1, we have the partial fractions for $\frac{1}{x^3+Bx^2+Cx+D}$ equal to

$$\frac{1}{(\alpha-\mu)(\alpha-\nu)}\frac{1}{(x-\alpha)} + \frac{1}{(\mu-\alpha)(\mu-\nu)}\frac{1}{(x-\mu)} + \frac{1}{(\nu-\alpha)(\nu-\mu)}\frac{1}{(x-\nu)}$$

and so the sum $\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$ is equal to

$$\left[\frac{1}{(\alpha-\mu)(\alpha-\nu)}\sum_{n\in\mathbb{Z}}\frac{1}{(n-\alpha)}\right] + \left[\frac{1}{(\mu-\alpha)(\mu-\nu)}\sum_{n\in\mathbb{Z}}\frac{1}{(n-\mu)}\right] + \left[\frac{1}{(\nu-\alpha)(\nu-\mu)}\sum_{n\in\mathbb{Z}}\frac{1}{(n-\nu)}\right].$$

To compute this sum further, we shall now use Equation (3.1) to relate the sum to the cotangent function. We have that the sum $\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$ is equal to

$$\begin{aligned} & \frac{\pi\cot(-\pi\alpha)}{(\alpha-\mu)(\alpha-\nu)} + \frac{\pi\cot(-\pi\mu)}{(\mu-\alpha)(\mu-\nu)} + \frac{\pi\cot(-\pi\nu)}{(\nu-\alpha)(\nu-\mu)} \\ = & -\frac{\pi\cot(\pi\alpha)}{(\alpha-\mu)(\alpha-\nu)} - \frac{\pi\cot(\pi\mu)}{(\mu-\alpha)(\mu-\nu)} - \frac{\pi\cot(\pi\nu)}{(\nu-\alpha)(\nu-\mu)}. \end{aligned}$$

Replacing the cotangent function with its exponential equivalent, we obtain $\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$ is equal to

$$-\pi i\left[\frac{1}{(\alpha-\mu)(\alpha-\nu)}\left(\frac{e^{2\pi i\alpha}+1}{e^{2\pi i\alpha}-1}\right) + \frac{1}{(\mu-\alpha)(\mu-\nu)}\left(\frac{e^{2\pi i\mu}+1}{e^{2\pi i\mu}-1}\right) + \frac{1}{(\nu-\alpha)(\nu-\mu)}\left(\frac{e^{2\pi i\nu}+1}{e^{2\pi i\nu}-1}\right)\right].$$

We make some modifications to this sum to get a "better" closed form. The sum $\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$ can be written

$$\begin{aligned} & -\pi i\left[\frac{1}{(\alpha-\mu)(\alpha-\nu)}\left(\frac{e^{2\pi i\alpha}-1+1+1}{e^{2\pi i\alpha}-1}\right) + \frac{1}{(\mu-\alpha)(\mu-\nu)}\left(\frac{e^{2\pi i\mu}-1+1+1}{e^{2\pi i\mu}-1}\right)\right. \\ & \left. + \frac{1}{(\nu-\alpha)(\nu-\mu)}\left(\frac{e^{2\pi i\nu}-1+1+1}{e^{2\pi i\nu}-1}\right)\right] \\ = & -\pi i\left[\frac{1}{(\alpha-\mu)(\alpha-\nu)}\left(1+\frac{2}{e^{2\pi i\alpha}-1}\right) + \frac{1}{(\mu-\alpha)(\mu-\nu)}\left(1+\frac{2}{e^{2\pi i\mu}-1}\right)\right. \\ & \left. + \frac{1}{(\nu-\alpha)(\nu-\mu)}\left(1+\frac{2}{e^{2\pi i\nu}-1}\right)\right] \\ = & -\pi i\left[\frac{1}{(\alpha-\mu)(\alpha-\nu)} + \frac{1}{(\mu-\alpha)(\mu-\nu)} + \frac{1}{(\nu-\alpha)(\nu-\mu)}\right. \\ & + \frac{1}{(\alpha-\mu)(\alpha-\nu)}\left(\frac{2}{e^{2\pi i\alpha}-1}\right) + \frac{1}{(\mu-\alpha)(\mu-\nu)}\left(\frac{2}{e^{2\pi i\mu}-1}\right) \\ & \left. + \frac{1}{(\nu-\alpha)(\nu-\mu)}\left(\frac{2}{e^{2\pi i\nu}-1}\right)\right]. \end{aligned}$$

The first three terms combine to 0 and we obtain, $\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$ is equal to

$$\begin{aligned}
& -2\pi i \left[\frac{1}{(\alpha-\mu)(\alpha-\nu)}\left(\frac{1}{e^{2\pi i\alpha}-1}\right) + \frac{1}{(\mu-\alpha)(\mu-\nu)}\left(\frac{1}{e^{2\pi i\mu}-1}\right)\right.\\
& \left. + \frac{1}{(\nu-\alpha)(\nu-\mu)}\left(\frac{1}{e^{2\pi i\nu}-1}\right)\right]\\
= \; & -2\pi i \left[\frac{1}{(\alpha-\mu)(\nu-\alpha)}\left(\frac{1}{1-e^{2\pi i\alpha}}\right) + \frac{1}{(\alpha-\mu)(\mu-\nu)}\left(\frac{1}{1-e^{2\pi i\mu}}\right)\right.\\
& \left. + \frac{1}{(\nu-\alpha)(\mu-\nu)}\left(\frac{1}{1-e^{2\pi i\nu}}\right)\right]\\
= \; & \frac{-2\pi i}{(\alpha-\mu)(\mu-\nu)(\nu-\alpha)}\left(\frac{\mu-\nu}{1-e^{2\pi i\alpha}} + \frac{\nu-\alpha}{1-e^{2\pi i\mu}} + \frac{\alpha-\mu}{1-e^{2\pi i\nu}}\right)\\
= \; & \frac{2\pi i}{(\nu-\mu)(\mu-\alpha)(\alpha-\nu)}\left(\frac{\nu-\mu}{e^{2\pi i\alpha}-1} + \frac{\alpha-\nu}{e^{2\pi i\mu}-1} + \frac{\mu-\alpha}{e^{2\pi i\nu}-1}\right).
\end{aligned}$$

This gives us the closed form for the sum (1.2) which is exactly same as the one found using Fourier Analysis in Theorem 2.7 but with no convergence conditions. The results derived in this section are encapsulated in the following theorem.

Theorem 3.2 *If $\alpha, \mu, \nu \in \mathbb{C} \setminus \mathbb{Z}$ are the distinct roots of the cubic x^3+Bx^2+Cx+D, then, we have, $\displaystyle\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$ is equal to*

$$\frac{2\pi i}{(\nu-\mu)(\mu-\alpha)(\alpha-\nu)}\left(\frac{\nu-\mu}{e^{2\pi i\alpha}-1} + \frac{\alpha-\nu}{e^{2\pi i\mu}-1} + \frac{\mu-\alpha}{e^{2\pi i\nu}-1}\right).$$

Following the same approach of using the cotangent function to compute the closed form, we shall now attempt to find a closed form for the sum $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$ in the next section.

3.2 Computing a closed form for $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$.

Let $\alpha, \mu, \nu \in \mathbb{C}/\mathbb{Z}$ be the distinct roots of the cubic polynomial (x^3+Bx^2+Cx+D). If for the sake of simplicity of writing, we write $\alpha = \alpha_1, \quad \mu = \alpha_2, \quad \nu = \alpha_3$, then, we can write that the sum $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$ is equal to

$$\sum_{n\in\mathbb{Z}} \frac{1}{((n-\alpha_1)(n-\alpha_2)(n-\alpha_3))^k} = \sum_{n\in\mathbb{Z}} \frac{1}{(n-\alpha_1)^k(n-\alpha_2)^k(n-\alpha_3)^k}.$$

Drawing parallels with (3.2) in the previous section, we can split this sum using partial fractions and see that $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$ is equal to

$$\sum_{n\in\mathbb{Z}} \sum_{i=1}^{3} \sum_{j=1}^{k} \frac{c_{i,j}}{(n-\alpha_i)^j}$$

for constants $c_{i,j} \in \mathbb{C}$ where $1 \leq i \leq 3$ and $1 \leq j \leq k$. It is obvious that for higher values of k it is challenging to compute the $c_{i,j}$'s by hand as we do not have any elegant explicit formula to compute these; but it is not that these coefficients cannot be determined. One can compute these coefficients with the help of computer by writing a program that has computational steps coded into it as iterations as no formula is known at this time, in comparison to Lemma 3.1. Once we have these coefficients, it becomes possible to find the closed form that we are looking for. At this point, we have that $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$ is equal to

$$\sum_{i=1}^{3} \sum_{j=1}^{k} c_{i,j} \sum_{n\in\mathbb{Z}} \frac{1}{(n-\alpha_i)^j}. \tag{3.3}$$

To simplify this sum further, we need to determine what the sum $\sum_{n\in\mathbb{Z}} \frac{1}{(n-\alpha_i)^j}$ is equal to. This sum can be evaluated with the help of (3.1).

Lemma 3.3 *For $j \geq 1$, we have*

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n-z)^j} = \begin{cases} -\pi i \left(1 + \frac{2}{e^{2\pi i z}-1}\right), j = 1, \\ \frac{(-2\pi i)^j}{(j-1)!} \sum_{t=1}^{j} \frac{(t-1)! S(j,t)}{(e^{2\pi i z}-1)^t}, j \geq 2; \end{cases}$$

where $S(j,t) \in \mathbb{Z}$ is a Stirling number of the second kind.

Proof: Replacing z by $(-z)$ in (3.1), we get

$$\sum_{n\in\mathbb{Z}} \frac{1}{n-z} = -\pi \cot(\pi z) = -\pi i \left(\frac{e^{2\pi i z}+1}{e^{2\pi i z}-1}\right). \tag{3.4}$$

In this lemma, we have two cases depending on j.

Case I: When $j = 1$, we have from (3.4) that

$$\sum_{n\in\mathbb{Z}} \frac{1}{n-z} = -\pi \cot(\pi z) = -\pi i \left(\frac{e^{2\pi i z}+1}{e^{2\pi i z}-1}\right).$$

This implies

$$\begin{aligned} \sum_{n\in\mathbb{Z}} \frac{1}{n-z} &= -\pi i \left(\frac{e^{2\pi i z}-1+1+1}{e^{2\pi i z}-1}\right) \\ &= -\pi i \left(1 + \frac{2}{e^{2\pi i z}-1}\right) \\ &= \frac{-\pi i}{(j-1)!} \left(1 + \frac{2}{e^{2\pi i z}-1}\right)^{(j-1)}. \end{aligned}$$

Case II: When $j \geq 2$, by computing the $(j-1)^{st}$ derivative of (3.4), we get

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \frac{1}{(n-z)^j} &= \frac{(-\pi\cot(\pi z))^{(j-1)}}{(j-1)!} \\ &= \frac{-\pi i}{(j-1)!}\left(1+\frac{2}{e^{2\pi i z}-1}\right)^{(j-1)}.\end{aligned}$$

For $j \geq 2$, we have that $j-1 \geq 1$ and we see that

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n-z)^j} = \frac{-2\pi i}{(j-1)!}\left(\frac{1}{e^{2\pi i z}-1}\right)^{(j-1)}.$$

From Lemma 2.12 we have that for $m \geq 0$,

$$\left(\frac{1}{e^{ax}-1}\right)^{(m)} = (-1)^m a^m \sum_{t=1}^{m+1} \frac{(t-1)!S(m+1,t)}{(e^{ax}-1)^t}.$$

Using this lemma for $x = z, a = 2\pi i, m = j-1$, we find that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} \frac{1}{(n-z)^j} &= \frac{-2\pi i}{(j-1)!}(-1)^{j-1}(2\pi i)^{j-1} \sum_{t=1}^{j-1+1} \frac{(t-1)!S(j-1+1,t)}{(e^{2\pi i z}-1)^t} \\ &= \frac{(-2\pi i)^j}{(j-1)!} \sum_{t=1}^{j} \frac{(t-1)!S(j,t)}{(e^{2\pi i z}-1)^t}.\end{aligned}$$

This completes the proof. □

The closed form for the sum (1.3) is computed in the following theorem.

Theorem 3.4 *If $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C} \setminus \mathbb{Z}$ are distinct roots of the cubic $x^3 + Bx^2 + Cx + D$, then the sum $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$ is equal to*

$$\sum_{i=1}^{3} \left(-\pi i c_{i,1} + \sum_{j=1}^{k} c_{i,j} \frac{(-2\pi i)^j}{(j-1)!} \sum_{t=1}^{j} \frac{(t-1)!S(j,t)}{(e^{2\pi i \alpha_i}-1)^t}\right);$$

where $c_{i,j} \in \mathbb{C}, 1 \leq i \leq 3$ and $1 \leq j \leq k$.

Proof: From (3.3), we have that

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k} &= \sum_{i=1}^{3}\sum_{j=1}^{k} c_{i,j} \sum_{n\in\mathbb{Z}} \frac{1}{(n-\alpha_i)^j} \\
&= \sum_{i=1}^{3} c_{i,1} \sum_{n\in\mathbb{Z}} \frac{1}{(n-\alpha_i)} + \sum_{i=1}^{3}\sum_{j=2}^{k} c_{i,j} \sum_{n\in\mathbb{Z}} \frac{1}{(n-\alpha_i)^j}.
\end{aligned}$$

Using Lemma 3.3 for $z=\alpha_i$, we find that the sum $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$ is equal to

$$\begin{aligned}
&\sum_{i=1}^{3} c_{i,1}\left(-\pi i\left(1+\frac{2}{e^{2\pi i\alpha_i}-1}\right)\right) + \sum_{i=1}^{3}\sum_{j=2}^{k} c_{i,j}\frac{(-2\pi i)^j}{(j-1)!}\sum_{t=1}^{j}\frac{(t-1)!S(j,t)}{(e^{2\pi i\alpha_i}-1)^t} \\
= \ & -\pi i\sum_{i=1}^{3} c_{i,1} + \sum_{i=1}^{3}\sum_{j=1}^{k} c_{i,j}\frac{(-2\pi i)^j}{(j-1)!}\sum_{t=1}^{j}\frac{(t-1)!S(j,t)}{(e^{2\pi i\alpha_i}-1)^t} \\
= \ & \sum_{i=1}^{3}\left(-\pi i c_{i,1} + \sum_{j=1}^{k} c_{i,j}\frac{(-2\pi i)^j}{(j-1)!}\sum_{t=1}^{j}\frac{(t-1)!S(j,t)}{(e^{2\pi i\alpha_i}-1)^t}\right).
\end{aligned}$$

This completes the proof. □

Now that we have computed the closed forms for the sums (1.2) and (1.3) using Fourier analysis and by connecting the sums to the identities involving well-known trigonometric functions and combinatorial numbers, we shall compare the two approaches and results derived using each in the next chapter.

Chapter 4

Conclusions and future directions

This chapter reviews the goals of the research conducted. The results derived in the earlier chapters are summarized and compared. Also, a section of this chapter is allocated to discussion of further research.

4.1 Objectives of research

This thesis has been formulated for partial fulfillment of completion of Master of Science degree program in Mathematics at Wilfrid Laurier University. The thesis has provided me with an excellent opportunity to learn to apply the conceptual understanding of Mathematics to conduct and practice research for the abstract mathematical problems in detail. The results derived for writing of this thesis incorporated the theoretical knowledge gained through the graduate courses taken at Laurier. One such course was that of the reading course in Combinatorics (MA685) that I took with Dr. Chester Weatherby in Fall term (2015). Various concepts learnt through this course helped me build conceptual background that was required to yield results for the thesis.

This thesis is aimed to evaluate closed forms for the sums $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$ where $B, C, D \in \mathbb{C}$ and $k \in \mathbb{Z}$. Sums of this form are related to an important function in Mathematics, the Riemann zeta function, $\zeta(s)$. Though several properties of $\zeta(s)$ have been studied and analyzed, there are still fundamental postulations related to it, that remain unproved. So, it is important to explore mathematical problems involving or relating to $\zeta(s)$. The computation of the closed forms for the sums $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$ is a step forward of studying the Riemann zeta function.

4.2 Summary of thesis

The thesis begins with building mathematical background and motivation for the problem in question in Chapter 1. A section in this chapter analyzes the cubic polynomials of the form $ax^3 + bx^2 + cx + d$ and various methods of finding their roots. One of these methods that is used in calculations in later chapters is that of Cardano's method. Cardano's method is discussed in detail and it is learnt that using this method, the roots obtained for the cubic polynomial $ax^3 + bx^2 + cx + d$ are of the form

$$\begin{aligned} x_1 &= \sqrt[3]{M} + \sqrt[3]{N} - \frac{b}{3a}, \\ x_2 &= \sqrt[3]{M}\omega + \sqrt[3]{N}\omega^2 - \frac{b}{3a} \\ x_3 &= \sqrt[3]{M}\omega^2 + \sqrt[3]{N}\omega - \frac{b}{3a} \end{aligned}$$

where $M = \frac{-q}{2} + \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}, N = \frac{-q}{2} - \sqrt{\left(\frac{q}{2}\right)^2 + \left(\frac{p}{3}\right)^3}$, $p = \frac{c}{a} - \frac{1}{3}\left(\frac{b}{a}\right)^2$ and $q = \frac{2}{27}\left(\frac{b}{a}\right)^3 - \frac{1}{3}\frac{bc}{a^2} + \frac{d}{a}$.

In chapter 2, we review fundamental concepts of Fourier analysis that are used to compute the closed forms for the sums $\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$ and $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$. The closed form computed for the sum $\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$ is given by the Theorem 2.7, which is re-written below.

Theorem 4.1 *Suppose $\alpha, \mu, \nu \in \mathbb{C} \setminus \mathbb{Z}$ are the distinct roots of the cubic $u^3 + Bu^2 +$*

$Cu + D$ where $B, C, D \in \mathbb{C}$ and the cubic can be factored as,

$$u^3 + Bu^2 + Cu + D = (u - \alpha)(u^2 + \beta u + \gamma) = (u - \alpha)(u - \mu)(u - \nu)$$

where $\alpha, \beta, \gamma, \rho, \mu, \nu \in \mathbb{C}$ and $\gamma = \rho^2 + \frac{\beta^2}{4}, \mu = -\frac{\beta}{2} - i\rho, \nu = -\frac{\beta}{2} + i\rho$ such that

1. *$Im(\alpha) > 0$*
2. *$0 \leq \left|\frac{Im(\beta)}{2}\right| < Re(\rho)$.*

Then,

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n^3 + Bn^2 + Cn + D)} = \frac{2\pi i}{(\nu - \mu)(\mu - \alpha)(\alpha - \nu)} \left(\frac{\nu - \mu}{e^{2\pi i \alpha} - 1} + \frac{\alpha - \nu}{e^{2\pi i \mu} - 1} + \frac{\mu - \alpha}{e^{2\pi i \nu} - 1}\right).$$

For the sum $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$, the previous sum was written as a composition of functions of q as follows

$$\sum_{n\in\mathbb{Z}} \frac{1}{(n^3 + Bn^2 + Cn + D)} = \sum_{n\in\mathbb{Z}} \frac{1}{(n^3 + Bn^2 + Cn + q + \frac{BC}{3} - \frac{2B^3}{27})} =$$

$$2\pi[(f_1(f_2(f_3)))((f_6 - f_5)(f_4(f_7)) + (f_7 - f_6)(f_4(f_5)) + (f_5 - f_7)(f_4(f_6)))]$$

where $f_1(q) = \frac{1}{q}$, $f_2(q) = \sqrt{q}$, $f_3(q) = 27q^2 + 4p^3$, $f_4(q) = \frac{1}{e^{2\pi i q} - 1}$, $f_5(q) = \mu = x_1$, $f_6(q) = \nu = x_2$, $f_7(q) = \alpha = x_3$. It was then differentiated with respect to q, $k - 1$ times using concepts of Faá Di Bruno's formula, product rule for higher order derivatives and various lemmas involving generalised derivatives for parts that make the whole. Since differentiating the sum meant differentiating the composition of seven different functions of q, and applying multiple applications of Faá Di Bruno's formula, product rule for higher order derivatives, etc., it was not feasible to write an explicit closed form for the sum (1.3) by doing computations by hand. The differentiation was carried out step by step followed by differentiation of the roots α, μ and ν, which too involved decomposing the roots into functions of q and differentiating these in further steps. The closed form in this case using Fourier methods, can only be obtained if the information exhibited in the individual steps is programmed and different values of k can be input into the computer to compute the closed forms for the sum (1.3) for different values of k.

In Chapter 3, we evaluated the closed forms for the sums (1.2) and (1.3), but by using a different approach, that involved connecting the sums to the cotangent function and its exponential equivalents. The closed forms computed using this method are outlined in Theorems 3.2 and 3.4, that are mentioned below.

Theorem 4.2 (Theorem 3.2) *If $\alpha, \mu, \nu \in \mathbb{C} \setminus \mathbb{Z}$ are the distinct roots of the cubic $x^3 + Bx^2 + Cx + D$, then, we have, $\displaystyle\sum_{n\in\mathbb{Z}} \frac{1}{n^3 + Bn^2 + Cn + D}$ is equal to*

$$\frac{2\pi i}{(\nu-\mu)(\mu-\alpha)(\alpha-\nu)}\left(\frac{\nu-\mu}{e^{2\pi i\alpha}-1}+\frac{\alpha-\nu}{e^{2\pi i\mu}-1}+\frac{\mu-\alpha}{e^{2\pi i\nu}-1}\right).$$

Theorem 4.3 (Theorem 3.4) *If $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C} \setminus \mathbb{Z}$ are distinct roots of the cubic $x^3 + Bx^2 + Cx + D$, then, the sum $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$ is equal to*

$$\sum_{i=1}^{3}\left(-\pi i c_{i,1} + \sum_{j=1}^{k} c_{i,j}\frac{(-2\pi i)^j}{(j-1)!}\sum_{t=1}^{j}\frac{(t-1)!S(j,t)}{(e^{2\pi i\alpha_i}-1)^t}\right);$$

where $c_{i,j} \in \mathbb{C}, 1 \leq i \leq 3$ and $1 \leq j \leq k$.

The closed forms for the sums (1.2) and (1.3) computed using the two different approaches, namely, by using tools and techniques of Fourier Analysis and by connecting the sums to cotangent function, will be compared in the next section.

4.3 Comparison of the results derived

The closed forms obtained for the sum $\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$ using approach of Fourier analysis in Theorem 2.7 was the same as the one computed by connecting the sum to the cotangent function and its exponential equivalents in Theorem 3.2; the only difference being that the latter did not have any conditions associated with it, while the former did have certain conditions on the roots of the cubic. So, on comparing the two methods, it will not be wrong to say that using the cotangent method to compute the closed form for the sum $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$ is the "better" method for computing the closed form in the case of $k = 1$.

For $k > 1$, it was found that by using Fourier analysis it was not feasible to write the closed form for the sum explicitly by making the computations by hand for the purpose of this thesis, but nevertheless it is not impossible to calculate the closed form. One can easily compute the closed form for the sum (1.3) by building a software with the computations programmed into it and inputting different values of k. On the other hand, in the closed form evaluated using the cotangent function method in Theorem 3.4, the coefficients $c_{i,j}$'$s \in \mathbb{C}, 1 \leq i \leq 3$ and $1 \leq j \leq k$ were unknown for higher values of k, as no elegant formula to compute these $c_{i,j}$'s for higher values of k is known to the writer at this time. If the coefficients $c_{i,j}$'s were known, we would get a really "nice" closed form for the sum (1.3). Therefore, as far as the methods of computing the closed form for the sum (1.3) are concerned, it is the approach of using cotangent function to compute the closed form that is a "better" approach. Nonetheless, the approach of using the tools and techniques of Fourier analysis is also useful. It helps us to get insight into another way of computing the sum in closed and explicit form in terms of roots of the cubic in question.

4.4 Future work

The research conducted for the formulation of this thesis can be advanced in number of ways. One, as was mentioned in the previous section, the closed forms obtained for the sum $\sum_{n\in\mathbb{Z}} \frac{1}{n^3+Bn^2+Cn+D}$ using approach of Fourier analysis in Theorem 2.7 had certain conditions associated with the roots of the cubic, namely

1. $Im(\alpha) > 0$

2. $0 \leq \left|\frac{Im(\beta)}{2}\right| < Re(\rho)$.

It could be attempted to evaluate the closed form for the sum (1.2) using the approach of Fourier analysis without imposing any conditions on the roots of the cubic polynomial.

Second, the closed form evaluated using the cotangent function method in Theorem 3.4 gave us a "nice" closed form but the coefficients $c_{i,j}$'$s, 1 \leq i \leq k, 1 \leq j \leq 3$ could not be computed for higher values of k. Future work could be directed toward seeking a formula for computing the coefficients $c_{i,j}$'s for higher values of k.

Third, the closed forms were computed for the sums $\sum_{n\in\mathbb{Z}} \frac{1}{(n^3+Bn^2+Cn+D)^k}$ where the cubic $x^3 + Bx^2 + Cx + D$ had complex but non-integral and distinct roots. It would be interesting to explore the sums related to cubic polynomials with integral and repeated roots.

Fourth, Lemma 2.12 involved Stirling numbers of the second kind. The future work for this thesis could be directed towards researching for the combinatorial significance of these numbers.

Fifth, based on the approaches followed to compute the sums (1.2) and (1.3), this thesis research could also be extended by attempting to compute closed forms for the sums $\sum_{n\in\mathbb{Z}} \frac{1}{n^4+Bn^3+Cn^2+Dn+E}$ or $\sum_{n\in\mathbb{Z}} \frac{1}{P(n)}$ where $B, C, D, E \in \mathbb{Z}$ and $P(n)$ is a polynomial of degree $m \in \mathbb{Z}$ where $m > 3$.

The future research described above would enable us to explore the Reimann zeta function, $\zeta(s)$, further.

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