Portfolio Selection in Gaussian and Non-Gaussian Worlds

Jing Wang
Wilfrid Laurier University

Portfolio Selection in Gaussian and Non-Gaussian Worlds

Jing Wang

Dalian University of Technology, M.Sc. 2003

THESIS

Submitted to the Department of Mathematics

in partial fulfilment of the requirements for

Master of Science in Mathematics

Wilfrid Laurier University

2007

Your file Votre référence
ISBN: 978-0-494-32285-7
Our file Notre référence
ISBN: 978-0-494-32285-7

Abstract

In the case of minimizing risk with a given level of expected return, we discuss the portfolio selection problem when the asset returns are characterized by a Gaussian distribution and heavy tailed distribution.

More specifically, under the Gaussian assumption, we give the explicit solutions to the problems of minimizing risk variance, CaR and EaR respectively. When a compound Poisson process is assumed, we derive explicit solutions to the variance, CaR and EaR. Furthermore, we give the explicit solution for the CaR when a Lévy distribution is considered.

For the more realistic process–normal inverse process, we are able to obtain the analytical solution for the EaR with the help of the explicit form of its probability density function.

Moreover, we give numerical results using Monte Carlo simulation for each risk measure discussed above by assuming that the stock returns follow Gaussian and Compound Poisson models, respectively. Finally, we give a comparison of the risk curves between these two processes and characterize the sensitivity of the risk curves for various values of the model parameters.

Acknowledgement

During the past year there were so many people who helped and supported me with my courses and thesis.

Foremost, I would like to thank my supervisors Prof. George Lai and Prof. Joe Campolieti for providing me with the opportunity to complete my Master's thesis at Wilfrid Laurier University. They have been always available to advise me and give me guidance and support. I am very grateful to Prof. Lai for his extreme patience. He is a very nice professor and has always been supportive and caring for the students. I am deeply grateful to Prof. Campolieti for his excellent job in the course "Financial Math". I had no any background before taking this course, while with his effort, I not only learned stochastic process, but also got the basis and principles of financial math. In addition, he checked this thesis so carefully from spelling to the ideas when he was really busy. I am so appreciated.

I especially want to thank Prof. Kathie Cameron and Prof. Yuming Chen on this spot. I gained much more than the pure knowledge from their courses "Optimization" and "Orthogonal System" respectively. Their attitudes and efforts to the job and the students deserve our respect. I am so lucky and so happy to know them. Meeting them was a turning point of my whole life. They might don't know how important what they helped me is for me, but I know.

I would like to thank Prof. Roderick Melnik, Prof. Ross Cressmam and Prof. Zilin Wang. By taking their courses, I extended and enhanced my knowledge of Modeling, Real analysis and Statistics. They are very knowledgeable and very responsible for our students. Without their efforts to the teaching, I won't be able to gain so much from the courses.

I want to thank Chair Syd Bulman-Fleming and Graduate officer Anthony Bonato who give me and other graduate students the most assistance for our living and studying at Laurier. I also want to give special thank to Ms. June Aleong and Ms.

Tao Gong. They have been always patient to my questions and requirements. I got lots of help from them, particularly when I was doing teaching assistant. They did a great job.

The last but not the least, I would like to thank all the classmates and friends at Laurier. I had a lot of fun to be with them and enjoyed the just past year very much. Thank them to share their ideas and happiness with me.

Contents

List of Figures

Chapter 1

Introduction

One of the basic questions in mathematical finance is how to choose an optimal investment strategy in a securities market. More precisely, an investor endowed with a given initial capital $W(0) = w$ has to decide how many shares of which asset he/she should hold at what time instant to maximize his/her utility of consumption during the time interval $[0, T]$ and total wealth at the time horizon T. The main object of portfolio selection theory is to derive solution methods for the portfolio problem.

In principle, these methods can be applied to general decision problems under uncertainty, but the models we consider here are typical of financial markets, such as infinitely divisible assets, trading possibilities in continuous time and so on.

1.1 Development of models for the underlying uncertainty

Prior to H. Markowitz's work [42] [36], investors focused on assessing the risks and rewards of individual securities in constructing their portfolios. Standard investment advice was to identify those securities that offers the best opportunities for gain with the least risk and then construct a portfolio from these. Following this advice, an investor might conclude that railroad stocks all offered good risk-reward characteristics and compile a portfolio entirely from these. While,in 1952, Markowitz [41] proposed that investors focus on selecting *portfolios* based on their overall risk-reward characteristics instead of merely compiling portfolios from securities that each individually have attractive risk-reward characteristics. He proposed the mean-variance model for the investment problem, and hence started the modern portfolio theory. In 1959, J. Tobin [67] expanded on Markowitz's work by adding a risk-free asset to the analysis. This made it possible to leverage portfolios on the efficient frontier.

The classical mean-variance model is easy to implement and demands no special knowledge beyond very basic stochastic models, therefore it still has great importance in real-life applications and is widely applied in the risk management departments of banks. However, it is only suited for one-period decision problems. Samulson [58] and Smith [64] formulated and solved a multi-period generalization, corresponding to lifetime planning of consumption and investment decisions, see also [26] and [50]. But their models are still confined in the discrete-time case. To overcome the limitations and problems raised by modelling the portfolio problem in a discrete time setting, a continuous time approach for modelling the stock prices and the actions of the investors is called for.

The work of Robert Merton [45] can be regarded as the real starting point of continuous- time portfolio theory, see also [49]. By applying standard methods and results from stochastic control theory to the portfolio problem he was able to obtain explicit solutions for some special examples. In his model, the standard Brownian process was assumed for the return distribution.

As pointed out in [47], the continuous trading solution will be a valid asymptotic approximation to the discrete trading solution provided that the dynamics have continuous sample paths. However, the Black-Scholes solution is not valid, when the stock price dynamics cannot be represented by a stochastic process with a continuous sample path. In 1976, Merton [48] proposed a Brownian motion compound Poisson process which suggested that the dynamics of the stock price motion should be a natural prototype process for the continuous component, i.e. a Gaussian process, plus a jump component, i.e. a Poisson driven process. The latter process allows for a positive probability of a stock price change of extraordinary magnitude, no matter how small the time interval between successive observations.

Although the Gaussian distribution is almost a "universal law" for random phenomena encountered in nature, it does not describe the behavior of asset returns in a very realistic way. One reason for this is that the distribution of real data is often leptokurtic, i.e. it exhibits smaller values than a normal law and has often heavy tails, in other words its kurtosis is higher than the kurtosis of the normal distribution. Even some of the jump-diffusion models cannot represent this heavy tails phenomenon very well.

For this reason, Fama [21], Mandelbrot and Samuelson [40] proposed models for stock prices ranging from log-normally distributed ones to models having stable Pareto and other α stable distributions. Two important properties of α stable distributions are stability or invariance under addition and the fact that these distributions are the only possible limiting distributions for sums of i.i.d. random variables. They have concluded, from economic and statistical points of view, that the distribution of price changes following a stable distribution with characteristic exponent $\alpha < 2$ seemed to fit the data better than the Gaussian distribution. However, in practice, its "infinite" variance makes the α stable process showing extremely erratic behavior even for very large samples. One may feel that it is nonsense to talk about infinite variances when dealing with real world variables. In particular, when it comes down to the classical mean-variance model for the portfolio selection problem, it seems useless. Actually α stable distributions with infinite variances do not provide the only alternative to the Gaussian distribution.

Eberlein and Keller [18] showed, for instance, the fit of the generalized hyperbolic distribution to financial data in a very convincing way. Generalized hyperbolic distribution, as contributed by Barndorff(1977) [6], can be fitted to the empirical distributions with high accuracy.

In the 1990s, the normal inverse Gaussian process was applied considerably in finance; see, for example, [8] [9], and [56]. It has remarkable been shown that the NIG law has much potential for capturing key styhlsed features of observational series from finance and it can be used as a building element for the construction of analytically and statistically tractable stochastic processes. In this thesis we consider the application of such models to portfolio optimization.

1.2 Development of risk measures

The role of risk in decision making under uncertainty has been investigated extensively in the field of finance [30]. Markowitz in [42] and [44] proposed variance as a risk measure and a mean-variance model for portfolio selection based on minimizing variance subject to a given level of mean return, see also [62] and [63]. But it is well known that the usual Markowitz-type efficient set analysis is highly sensitive to the estimates of the variances used [10] [11] [59]. Thus, if it is difficult to develop good estimates of variances because of erratic sampling behavior induced by long-tailed distributions of returns, one may feel forced to use an alternative measure of dispersion in portfolio analysis [66].

Another deficiency is the well-known fact that the variance as a risk measure is increasing with the time horizon. This is in contrast to the common wisdom of asset managers that in the long run stock investment leads to an almost sure gain over risk-less bond investment and hence the larger the planning horizon, the greater should be the investment in risky stocks. For this reason, we concentrate on the risk measures based on quantile and expected shortfall. In recent years, Value-at-risk (VaR) has been accepted as the benchmark risk measure, see [16]. VaR is a low quantile of the profit-loss-distribution of a portfolio, see [31]. Based on VaR, Emmer et al. [19] introduced the definition of CaR. In essence both measures describe the loss that is likely to be exceeded by a specified probability.

Sentana [61] analyzes the mean-variance frontier under a VaR constraint assuming elliptically symmetric distributions. Alexander and Baptista [3] assume either a normal distribution or a t-distribution when comparing VaR and standard deviation in the context of mean-VaR analysis. Yiu [70] analyzed the impact of a VaR constraint and Emmer [19] considered some other quantile constraints on asset allocation CaR while using numerical techniques. Kaplanski [32] developed an analytical tool for calculating the VaR of a portfolio composed of generally distributed assets. This analytical VaR can then be used to construct optimal portfolios in which the target function and constraints are expressed in terms of VaR.

However, either VaR or CaR fails to satisfy the coherence of a risk measure as pointed out by Artzen et al. [4]. They adopted an axiomatic approach to characterize economically coherent risk measures and suggested tail conditional expectation (TCE) as an alternative to VaR. The notion of coherent risk measure has been extended to convex risk measure [23], and has been generalized to more complex spaces of risk, which allows to take into consideration financial positions with different types of cash streams structures. For instance, Cheridito et al. [14] extended the definition and the representation of coherent and convex risk measures to the space of $c\grave{a}dl\grave{a}g$ processes. Jaschke [29] considered coherent risk measures on abstract spaces of risk including deterministic, stochastic, singely or multi-periodic cash stream structures. Based on these works, Imen [27] extend the tail conditional expectation to multi-asset portfolios, that is, the vector valued TCE. Based on TCE, also called tailed VaR, Li et al. [39] introduced a new risk measure–Earning-at-Risk which measures the difference between the expected future return and the tail conditional expectation. It naturally preserves the property of coherence, since it has a TCE basis.

1.3 Contribution of the thesis

Within the Gaussian process for the underlying uncertainty of the risky asset, Emmer [19] considered the optimization problems of maximizing the expected return with constrained variance and CaR. However, this is only one side of the coin. In fact, the dual problem, that is, minimizing the risk with the expected return constrained is also meaningful and important in practice, see [42] and [17]. Therefore, we consider the analytical optimal solutions to the dual problem for different risks.

By changing the risk measure to be EaR, Li et al. [38, 37] considered the minimizing EaR problem and showed the advantage of this measure by comparing the results generated for the EaR problem with those generated by the CaR problem. To make the discussion complete, we consider the problem of maximizing the return. It turns out that the optimal solution is consistent with the optimal solution to the problem of maximizing the return problem in [19].

In the Non-Gaussian case, Emmer [19] generated the analytical solutions for the problem of maximizing the return with constrained variance where the compound Poisson process, Normal Inverse Gaussian (NIG) process and variance gamma process were assumed, respectively. We extend these results to the minimization problem. Moreover we also consider a particular α-stable process where $\alpha = 1/2$.

For the NIG process, Emmer [19] derived an approximation formula for the risk measure CaR by using the approximation results in [5]. Besides extending their work to the EaR measure, we also give the analytical formula of EaR by using the probability density function for the NIG process.

Furthermore, besides the analytical work, we provide simulation results for the Gaussian and Compound Poisson processes and compare them. From the numerical results, we characterize the regularity of the change of the risk curve tendency with respect to changes in the model parameters. Also, we compare the curve tendencies with respect to the two processes.

Chapter 2

Portfolio Selection and Lévy process

2.1 Background of modern portfolio theory

The basic investment-choice problem for an individual is to determine the optimal allocation of his or her wealth among the available investment opportunities. The solution to the general problem of choosing the best investment mix is called portfolio selection theory. It is a mature field which grew out of the Markowitz's mean variance theory [42]. This theory relies on the numerical representation of the preference relation investors have for assets with random outcomes. When the mean variance theory was developed, variance was widely used as the measure of risk, since it allows a very detailed theoretical analysis of the properties of optimal portfolios(such as the efficient frontier) and the use of the quadratic optimization methods. The objective of the portfolio selection theory consists in the selection of an optimal allocation of an investor's wealth in different investment alternatives such that the investor obtains the best possible outcome at the end of the investment period.

The basic problem can be stated as follows: consider a continuous trading horizon $[0, T]$. Let w be the initial amount of wealth available to an investor across n

random assets and one risk-free asset, such as a bond. Assume that each one of the assets has an initial price $p_i(0)$, and a final price $p_i(T)$. The price processes $p_i(t)$ are non-negative random variables whose values become known to the investors at period T. Then we can define the return of the asset as

$$r_i(T) := \frac{p_i(T) - p_i(0)}{p_i(0)} = \frac{p_i(T)}{p_i(0)} - 1$$

Denote

$$R_i(T) := \frac{p_i(T)}{p_i(0)} = 1 + r_i(T), i = 0, \dots, n$$

, since the randomness only depends on $R_i(T)$, we may use $R_i(T)$ as the returns of the assets. Then the means, variances, and covariances are given by

$$E(R_i(T)) = \mu_i, i = 0, \dots, n$$

$$var(R_i(T)) = \sigma_i{}^2$$

$$Cov(R_i(T), R_j(T)) = \sigma_{ij}, i, j = 0, \dots, n$$

We assume that each asset is divisible, i.e. we can hold $\psi_i \in \mathbb{R}$ shares of asset. A negative position, i.e. $\psi_i < 0$ for some asset corresponds to a short selling.

Definition 2.1. An investor with initial wealth $w > 0$ is assumed to hold $\psi_i \geq 0$ shares of asset i, $i = 0, \dots, n$, with

$$\Sigma_{i=0}^{n} \psi_i \cdot p_i(0) = w.$$

Then the portfolio vector $\pi = (\pi_0, \dots, \pi_n)$ is defined as

$$\pi_i := \frac{\psi_i \cdot p_i(0)}{w}, i = 0, \dots, n$$

and $R := \Sigma_{i=0}^{n}\pi_i \cdot R_i(T)$ is called the corresponding portfolio return.

Remark 2.1. The components of the portfolio vector represent the fraction of the total wealth which are invested in the corresponding assets. In particular, we have $\Sigma_{i=0}^{n}\pi_i = \frac{\Sigma_{i=0}^{n}\psi_i \cdot p_i(0)}{w} = 1$.

Remark 2.2. If W_T denotes the final wealth corresponding to an initial wealth of w and a portfolio vector π, and

$$W_T = \Sigma_{i=0}^{n}\psi_i \cdot p_i(T)$$

then we have

$$R = \Sigma_{i=0}^{n}\pi_i \cdot R_i(T) = \Sigma_{i=0}^{n}\frac{\psi_i \cdot p_i(0)}{w} \cdot \frac{p_i(T)}{p_i(0)} = \frac{W_T}{w}$$

This justifies the definition of the portfolio return.

Remark 2.3. The mean and variance of the portfolio return are given by

$$E(R) = \Sigma_{i=0}^{n}\pi_i \cdot \mu_i = \pi \cdot \mu,$$

and

$$Var(R) = \Sigma_{i=0}^{n}\Sigma_{j=0}^{n}\pi_i \cdot \sigma_{ij} \cdot \pi_j = \pi'\sigma\pi$$

where μ is the mean vector and σ is the nonnegative semi-definite variance-covariance matrix.

When choosing a portfolio, an investor has the aim of obtaining a return as large as possible. If the mean of the portfolio return is the only criterion, then this will lead to investing the entire wealth into the asset with the highest mean return. However, this could be a very risky asset and thus, the return can have big fluctuations. To accommodate this fact, we introduce the idea of minimizing such a risk as a second criterion. As a measure of this risk, the portfolio variance was introduced by Markowitz [42]. His basic idea was to look for a balance between the wealth deviation risk and the return.

2.2 Formulation of mean variance model

In formulating our the mathematical model for the portfolio selection theory, we need to make some assumptions.

Assumption 1. Frictionless Markets.

There are no transaction costs or taxes, and all securities are perfectly divisible.

Assumption 2. Price Taker.

The investor believes that his actions cannot affect the probability distribution of returns on the available securities. Hence, if π_i is the fraction of the investor's initial wealth W_0 allocated to security i, then $\pi_0, \ldots, \pi_n$ uniquely determines the probability distribution of his terminal wealth. A risk-less security is defined to be a security or feasible portfolio of securities whose return per dollar over the period is known with certainty, i.e. regardless of outcome.

Assumption 3. No-Arbitrage Opportunities.

All risk-less securities must have the same return per dollar. This common return will be denoted by $R = 1 + r$.

Assumption 4. No-Institutional Restrictions.

Short-sales of all securities, with full use of proceeds, are allowed without restriction. If there exists a risk-less security, then the borrowing rate equals the lending rate.

With these assumption, the first idea is to require an upper bound for the portfolio return and then choose from the corresponding set the portfolio vector with minimal variance, i.e. we choose π as.

$$\begin{aligned} &\min_{\pi \in \mathbb{R}^{n+1}} var(R) \quad subject \quad to \\ &E(R) \geq C_1 \end{aligned} \tag{2.1}$$

In words: under all possible portfolios $\pi \in \mathbb{R}^{n+1}$, consider only those which satisfy the constraints, in particular those which yield at least an expected return of C_1. Then, among those portfolios determine the one with the smallest return variance.

The second idea is to set up an upper bound for the portfolio variance and then determine the portfolio vector with highest possible mean return from the remaining set. This corresponds to fining π as

$$\begin{aligned} &\max_{\pi \in \mathbb{R}^{n+1}} E(R) \quad subject \quad to \\ &var(R) \leq C_2 \end{aligned} \tag{2.2}$$

In words: under all possible portfolios $\pi \in \mathbb{R}^{n+1}$, consider only those which satisfy the constraints (which form the "feasible region"). In particular, portfolios those which have a variance that is below C_1. Then, among those portfolios determine the one with the largest expected return.

Since $R = \frac{W(T)}{w}$, where w is constant and π_0 corresponds to the fraction of the risk-less bond, with the assumption that no short selling is allowed (that is $\pi \geq \underline{0}$), we can equivalently consider

$$\begin{aligned} &\min_{\pi \in \mathbb{R}^{n}} var(W^{\pi}(T)) \quad subject \quad to \\ &E(W^{\pi}(T)) \geq C_1 \end{aligned} \tag{2.3}$$

Here, the components of $\pi = (\pi_1, \ldots, \pi_n)$ correspond to the fractions of the respective risky assets. This is slightly different from the π in (2.1) which is the vector including the fraction of the bond. C_1 here is the predetermined minimum level of including the fraction of the bond. C_1 here is the predetermined minimum level of the expected terminal wealth $E(W^{\pi}(T))$.

2.3 Other types of expected utility functions

The above $W^{\pi}(T)$ can be viewed as a kind of utility function. A utility function is a twice-differentiable function of wealth $U(W)$ defined for $W > 0$ which has the properties of $non - satiation$, i.e. positive first derivative, and risk aversion, i.e. non-negative second derivative [12]. The non-satiation property states that utility increases with wealth, that is, more wealth is preferred to less wealth. The risk aversion property states that the utility function is concave down, in other words, the marginal utility of wealth decreases as wealth increases. Utility functions provide a way to measure investor's preferences for wealth and the amount of risk they are willing to undertake in the hope of attaining greater wealth.

There are many kinds of utility functions [51] [22], for example: an investor might consider the square root utility function given by $U(W) = \sqrt{W}$, or he might consider the natural logarithm function as a utility function defined as $U(W) = \log(W)$. But there is no absolute right answer to such questions like: which utility function is the best one, or which is typical of most investors. Different investors have different patterns of risk aversion as functions of wealth.

Throughout this thesis, we are going to keep using $U(W) = W$ as the utility which is a very simple degenerate case. All the results in this thesis can be easily extended to a more general utility function $U(W)$.

2.4 Definition and construction of a Lévy process

Processes with independent and stationary increments are named Lévy process after the French mathematician Paul Lévy(1886-1971), who made the connection with infinitely divisible laws, characterized their distributions(Lévy-Khintchine formula) and described their structure (Lévy-Itô decomposition).

Lévy processes play a fundamental role in many fields of science, such as Physics, Engineering and Actuarial science. In particular, they have become more and more popular in Mathematical Finance recently because they can sometimes describe the observed reality of financial markets in a more accurate way than models based on Brownian motion.

In the "real" world, we observe that asset price processes have jumps and risk-managers have to take them into account. Moreover, the empirical distribution of asset returns exhibits fat tails and skewness behavior that deviates from normality. Hence, models that accurately fit return distributions are essential for the estimation of profit and loss distributions. In the "risk-neutral" world, we observe that implied volatilities are constant neither across strike nor across maturities as stipulated by the Black- Scholes model. Therefore, traders need models that can capture the behavior of the implied volatility smiles more accurately, in order to handle the risk of trades. Lévy processes provide us with an appropriate framework to adequately describe all these observations, both in the "real" and in the "risk-neutral" world.

After knowing the importance of the Lévy process with the application in finance, it is natural to ask what is a Lévy process and how to get it. To answer this question, we need to give a definition of a random walk.

Definition 2.2. Let $Y_i, i \geq 1$, be i.i.d. random variables, then $X_n = \Sigma_{i=1}^n Y_i, n \in \mathbb{N}$, is called a random walk.

By the definition, we can see that random walks have stationary and independent increments. Here stationarity means all the increments have identical distribution. Stochastic processes are collections of random variables $\{X_t, t \geq 0\}$. For us, all $\{X_t, t \geq 0\}$ take values in a common state space, which we will choose specifically as $\mathbb{R}$(or $[0, \infty)$, or $\mathbb{R}^d$ for some $d \geq 2$). We can think of X_t as the position of the stock price at time t, changing as t varies. It is natural to suppose that the price moves continuously in the sense that $t \mapsto X_t$ is continuous almost surely, but we know that sometimes the stock price has jumps for some $t > 0$, and hence we need to turn to another continuity definition to describe the process in this case, that is, the right continuity.

Definition 2.3. Suppose the limits of

$$\Delta X_t = X_{t+} - X_{t-} = \lim_{\varepsilon \to 0+} (X_{t+\varepsilon} - X_{t-\varepsilon})$$

exist for all $t \geq 0$ and that in fact $X_{t+} = X_t$, i.e. that $t \mapsto X_t$ is right continuous with left limits X_{t-} for all$t \geq 0$ almost surely, then the path $t \mapsto X_t$ is called a random right-continuous function.

A Lévy process is a right continuous process with stationary and independent increments. More precisely,

Definition 2.4. A real-valued stochastic process $X = (X_t)_{t \geq 0}$ is called a Lévy process if

(a) for all $n \geq 1$ and $0 \leq t_0 < t_1 < \ldots < t_n$, the random variables $X_{t_0}, X_{t_1} - X_{t_0}, \ldots X_{t_n} - X_{t_{n-1}}$ are independent (i.e. independent increments),

(b) $X_{t+s} - X_t$ has the same distribution as X_s for all $s, t \geq 0$ (i.e. stationary increments),

(c) the paths $t \mapsto X_t$ are right-continuous with left limits almost surely.

One way to obtain the Lévy process is by the central limit theorem (CLT). Since CLT is well known, we don't put the theorem itself here.

Theorem 2.1. (Khintchine) Let $Y_i^{(n)}$, $i = 1, \ldots, n$ be i.i.d. with distribution changing with $n \geq 1$, and such that

$$Y_i^{(n)} \to 0, \quad in \quad probability, \quad as \quad n \to \infty$$

if

$$S_n^{(n)} \to X \quad in \quad distribution$$

where X has the infinitely divisible distribution.

Definition 2.5. A one-dimensional distribution ν is called an infinitely divisible distribution if for every $n = 1, 2, \ldots$ we can find a distribution ν_N such that

$$\nu = {\nu_N}^n$$

Remark 2.4. Suppose $X_n, n = 1, 2, \ldots$ and X are real valued random variables. We say that $X_n \to X$ as $n \to \infty$ in probability, if for any $\epsilon > 0$,

$$P(|X_n - X| > \epsilon) \to 0, \quad as \quad n \to \infty$$

Remark 2.5. Suppose that $X_n, n = 1, 2, \ldots$ and X are real valued random variables with distribution functions $F_n, n = 1, 2, \ldots$ and F, respectively. We say that X_n converges to X in distribution, if for all x,

$$F_n(x) \to F(x), \quad as \quad n \to \infty$$

Theorem 2.2. In the above theorem of Khintchine, $i \geq 1$, $n \geq 1$,

$$X_t^{(n)} = S_{nt}^{(n)} \to X_t \quad in \quad distribution, \quad as \quad n \to \infty$$

Furthermore, $X^{(n)} \to X$ where X is a Lévy process.

Actually, Lévy process is not only attained as limits, by restricting the increment distributions to infinitely divisible distributions, we can construct Lévy process with given infinitely divisible increment distribution.

We restrict the increment distributions to infinitely divisible because it can be parameterized nicely, see the following theorem:

Theorem 2.3. (Lévy-Khintchine) A random variable X is infinitely divisible if and only if its characteristic function $E(e^{iuX})$ is of the form

$$\exp\{i\alpha u - \frac{1}{2}\beta^2 u^2 + \int_{\mathbb{R}} (e^{iux} - 1 - iux1_{\{|x|\leq 1\}})\nu(dx)\}$$

where $\alpha \in \mathbb{R}$, $\beta^2 \geq 0$ and ν is a measure on $\mathbb{R}$ such that $\int_{\mathbb{R}}(1 \wedge x^2)\nu(ds) < \infty$. Here (α, β^2, ν) is called the three components of Lévy process.

Next, we give an very important decomposition theorem:

Theorem 2.4. (Lévy-Itô) Let X_t be a Lévy process, whose distribution is parameterized by (α, β^2, ν), then X_t is decomposed as

$$X_t = \alpha t + \beta B_t + J_t + M_t,$$

where B_t is a Brownian motion, and ΔX_t follows a Poisson process with intensity measure ν, $J_t = \Sigma_{s\leq t}\Delta X_s 1_{\{|\Delta X_s|>1\}}$, and M_t is a martingale (within the given probability measure) with jumps $\Delta M_t = \Delta X_t 1_{\{|\Delta X_s|\leq 1\}}$.

2.5 Several important examples of Lévy processes

2.5.1 Compound poisson process

A compound Poisson process with intensity λ and jump size distribution f is a continuous time stochastic process $\{X_t : t \geq 0\}$, given by

$$X_t = \Sigma_{i=1}^{N(t)} Y_i$$

where jumps sizes Y_i are i.i.d. with distribution f and $N(t)$ is a Poisson process with intensity λ, independent from $(Y_i)_{i\geq 1}$. A Poisson process itself can be seen as a compound Poisson process on $\mathbb{R}$ such that $Y_i \equiv 1$. This explains the origin of term "compound Poisson" in the definition.

Let $R(n) = \Sigma_{i=0}^{n} Y_i$, $n \geq 0$ be a random walk with step size distribution f. The compound Poisson process X_t can be obtained by changing the time of R with an independent Poisson process N_t. X_t thus describes the position of a random walk after a random number of time steps, given by N_t.

Using conditional expectation, the expected value of a compound Poisson process can be calculated as:

$$E(X_t) = E(E(X_t|N(t))) = E(N(t)E(Y)) = \lambda t E(Y),$$

and

$$\begin{aligned} var(X_t) &= E(var(X_t|N(t))) + var(E(X_t|N(t))) \\ &= var(Y)E(N(t)) + E(Y)^2 var(N(t)) \\ &= \lambda t E(Y^2) \end{aligned}$$

More generally, denote the characteristic function of f by $\hat{f}$, then

$$\begin{aligned} E(e^{iuX_t}) &= E(E(e^{iuX_t}|N(t))) \\ &= E((\hat{f})^{N(t)}) \\ &= e^{t\lambda(\hat{f}-1)} \\ &= \exp(t\lambda \textstyle\int_{\mathbb{R}} (e^{iux} - 1) f(dx)) \end{aligned}$$

2.5.2 Hyperbolic processes

In response to remarkable regularities discovered by geomorphologists in the 1940s, Barndorff-Nielsen [7] introduced the hyperbolic law for modeling the grain size distribution of diamonds from a large mining area in South West Africa. Almost twenty years later the hyperbolic law was found to provide a very good model for the distributions of daily stock returns from a number of leading German enterprisers, giving way to its use in stock price modeling and market risk measurement. The name of the distribution is derived from the fact that its log-density forms a hyperbola. Recall that the log-density of the normal distribution is a parabola. Hence the hyperbolic distribution provides the possibility of modeling heavier tails.

The hyperbolic distribution is defined as a normal variance-mean mixture where the mixing distribution is the generalized inverse Gaussian(GIG) law with parameter $\lambda = 1$, i.e. it is conditionally Gaussian. More precisely, a random variable Z has the hyperbolic distribution if:

$$(Z|Y) \sim N(\mu + \beta Y, Y)$$

where Y is a generalized inverse Gaussian random variable and $N(m, s^2)$ denotes the Gaussian normal distribution with mean m and variance s^2. The above relation implies that a hyperbolic random variable $Z \sim H(\psi, \beta, \chi, \mu)$ can be represented in the form:

$$Z \sim \mu + \beta Y + \sqrt{Y} N(0, 1) \tag{2.4}$$

with the characteristic function:

$$\phi_Z(u) = e^{iu\mu} \int_0^\infty e^{i\beta zu - zu^2/2} dF_Y(z) \tag{2.5}$$

Here $F_Y(z)$ denotes the distribution function of a generalized inverse Gaussian random variable Y with $\lambda = 1$. Hence the hyperbolic probability density function (pdf) is given by:

$$f_H(x) = \frac{\sqrt{\psi/\chi}}{2\sqrt{\psi + \beta^2} K_1(\sqrt{\psi\chi})} e^{-\sqrt{(\psi+\beta^2)(\chi+(x-\mu)^2)}\beta(x-\mu)},$$

where K_1 is the modified Bessel function of the second kind of order 1.

Let $\delta = \sqrt{\chi}$ and $\alpha = \sqrt{\psi + \beta^2}$, then the PDF of the hyperbolic $H(\alpha, \beta, \delta, \mu)$ law can be written as:

$$f_H(x) = \frac{\sqrt{\alpha^2 - \beta^2}}{2\alpha\delta K_1(\delta\sqrt{\alpha^2 - \beta^2})} e^{-\alpha\sqrt{\delta^2+(x-\mu)^2}+\beta(x-\mu)} \tag{2.6}$$

where $\delta > 0$ is the scale parameter, $\mu \in \mathbb{R}$ is the location parameter and $0 \leq |\beta| < \alpha$. The latter two parameters α and β determine the shape, with α being responsible for the steepness and β for the skewness. $\beta = 0$ gives a symmetric density.

2.5.3 Normal inverse Gaussian process

The hyperbolic law is a member of a more general class of generalized hyperbolic distributions, which also includes the normal-inverse Gaussian (NIG) and variance-gamma distributions as special cases. The generalized hyperbolic law can be represented as a normal variance-mean mixture where the mixing distribution is the generalized inverse Gaussian (GIG) law with any $\lambda \in \mathbb{R}$. The normal-inverse Gaussian distributions were introduced by Barndorff-Nielsen (1995) as a subclass of the generalized hyperbolic laws obtained for $\lambda = -1/2$. The density of the normal-inverse Gaussian distribution is given by:

$$f_{NIG}(x) = \frac{\alpha\delta}{\pi} e^{\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)} \frac{K_1(\alpha\sqrt{\delta^2+(x-\mu)^2})}{\sqrt{\delta^2+(x-\mu)^2}} \tag{2.7}$$

At the expense of four parameters, the NIG distribution is able to model symmetric and asymmetric distributions with possibly long tails in both directions. Its tail behavior is often classified as "semi-heavy", i.e. the tails are lighter than those of a non-Gaussian stable law, but much heavier than Gaussian. Obviously, the NIG distribution may not be adequate to deal with cases of extremely heavy tails such as those of Pareto or non-Gaussian stable laws. However, empirical experience suggests an excellent fit of the NIG law to financial data [69]. Moreover, the class of normal inverse Gaussian distributions possesses an appealing feature that the class of hyperbolic laws does not have. Namely, it is closed under convolution, i.e. a sum of two independent NIG random variables is also a NIG.

In recent years, the hyperbolic distribution has formed the basis for models of financial data, (see [8] and [18]). However, as already indicated, the hyperbolic distribution lacks the property of being closed under convolution in contrast to the NIG distribution. Also, the cumulants are not easily calculated like they are for the NIG distribution, see above. All in all, the NIG distribution seems to have much nicer probabilistic properties than the hyperbolic distribution.

2.5.4 α-stable process

As already mentioned above, it has been long known that financial asset returns are not normally distributed. Rather, the empirical observations exhibit excess kurtosis. This heavy-tailed or leptokurtic character of the distribution of price changes has been repeatedly observed in various financial and commodity markets.

In response to the empirical evidence Mandelbrot[40] and Fama[21] proposed the stable distribution as an alternative model to the Gaussian law. There are at least two good reasons for modeling financial variables using stable distributions. Firstly, they are supported by the Generalized Central Limit Theorem, which states that stable laws are the only possible limit distributions for properly normalized and centered sums of independent, identically distributed random variables. Secondly, stable distributions are leptokurtotic. Since they can accommodate the fat tails and asymmetry, they fit empirical distributions much better.

Stable laws-also called α-stable. They were introduced by Paul Lévy during his investigations of the behavior of sums of independent random variables in the early 20th century. The α-stable distribution requires four parameters for complete description: $\alpha \in (0,2], \beta \in [-1,1], \sigma > 0$ and $\mu \in \mathbb{R}$. The tail exponent α determines the rate at which the tails of the distribution taper off. When $\alpha = 2$, it is the Gaussian distribution. When $\alpha < 2$, the variance is infinite and the tails are asymptotically equivalent to a Pareto law, i.e. they exhibit a power-law behavior. More precisely, using a central limit theorem type argument it can be shown that

$$\begin{aligned} \lim_{x\to\infty} x^{\alpha} P(X > x) &= C_{\alpha}(1+\beta)\sigma^{\alpha} \\ \lim_{x\to\infty} x^{\alpha} P(X < -x) &= C_{\alpha}(1+\beta)\sigma^{\alpha} \end{aligned}$$

where C_{α} is a function of α only. When $\alpha > 1$, the mean of the distribution exists and is equal to μ. When the skewness parameter β is positive, the distribution is skewed to the right, i.e. the right tail is thicker. The last two parameters σ and μ are the usual scale and location parameters.

From a practitioner's point of view the crucial drawback of the stable distribution is that, with the exception of three special cases $\alpha = 0.5, 1, 2$, its probability density function and cumulative distribution function do not have closed form expressions. Hence, the α-stable distribution can be most conveniently described by its characteristic function $\phi(t)$. The most popular parameterization of the characteristic function of $S_{\alpha}(\sigma, \beta, \mu)$ is given by Samorodmitsky and Taqqu [57], and Weron [68]:

$$\log \phi(t) = \begin{cases} -\sigma^{\alpha}|t|^{\alpha}\{1 - i\beta sign(t)\tanh(\pi\alpha/2)\} + i\mu t &, \quad \alpha \neq 1, \\ -\sigma|t|\{1 + i\beta sign(t)\frac{2}{\pi}\log|t|\} + i\mu t &, \quad \alpha = 1. \end{cases}$$

If we consider a Brownian motion with drift $\hat{B}_t = B_t + \gamma t$ then this property is only verified up to a translation:

$$\forall a > 0, (\frac{\hat{B}_{at}}{\sqrt{a}})_{t\geq 0} =^{d} (B_t + \sqrt{a}\gamma t)_{t\geq 0}$$

A natural question is whether there exist other real-valued Lévy Processes that share this self-similarity property: a Lévy Process X_t is said to be self-similar if

$$\forall a > 0, , \exists b(a) > 0 : (\frac{X_{at}}{b(a)})_{t \geq 0} =^d (X_t)_{t \geq 0}$$

Since the characteristic function of X_t has the form $\Phi_{X_t}(z) = \exp(-t\psi(z))$, this property is equivalent to the following property of the characteristic function:

$$\forall a > 0, , \exists b(a) > 0 : \Phi_{X_t}(z)^a = \Phi_{X_t}(zb(a))$$

The distributions that verify this property are called strictly stable distributions. More formally, these are defined as follows,

Definition 2.6. A random variable $X \in \mathbb{R}$ is said to have a stable distribution if for every $a > 0$ there exist $b(a) > 0$ and $c(a) \in \mathbb{R}$ such that

$$\Phi_X(z)^a = \Phi_X(zb(a))e^{ic(a)z}.$$

It is said to have a strictly stable distribution if $c(a) = 1$

$$\Phi_X(z)^a = \Phi_X(zb(a)).$$

The name stable comes from the following stability under addition property: if X has stable distribution and $X^{(1)}, \ldots, X^{(n)}$ are independent copies of X then there exist a positive number c_n and a vector d such that

$$X^{(1)} + \cdots + X^{(n)} =^d c_n X + d$$

This property is clearly verified if the distribution of X is that of a self-similar Lévy

Process at a given time t.

It can be shown that for every stable distribution there exists a constant $\alpha \in (0, 2]$ such that $b(a) = a^{1/\alpha}$. This constant is called the index of stability and stable distributions with index α are also referred to as α-stable distributions. The only 2-stable distributions are Gaussian.

A self-similar Lévy Process therefore has strictly stable distribution at all times. For this reason, such processes are also called strictly stable Lévy Processes. A strictly α-stable Lévy Process satisfies:

$$\forall a > 0, (\frac{X_{at}}{a^{1/\alpha}}) =^d (X_t)_{t\geq 0},$$

i.e. Brownian motion is a 2-stable Lévy Process.

More generally, an α-stable Lévy Process satisfies this relation up to a translation:

$$\forall a > 0, \exists c \in \mathbb{R}^d : (X_{at})_{t\geq 0} =^d (a^{1/\alpha} X_t + ct)_{t\geq 0}$$

A stable Lévy Process defines a family of stable distributions and the converse is also true: every stable distribution is infinitely divisible and can be seen as the distribution at a given time of a stable Lévy Process.

Remark 2.6. A distribution on $\mathbb{R}^d$ is a α-stable with $0 < \alpha < 2$ if and only if if is infinitely divisible with characteristic triplet $(0, \nu, \gamma)$ and there exists a finite measure λ on S, a unit sphere of $\mathbb{R}^d$, such that

$$\nu(B) = \int_S \int_0^\infty 1_B(r\xi) \frac{dr}{r^{1+\alpha}} \lambda(d\xi) \tag{2.8}$$

Remark 2.7. For real-valued stable variables and Lévy Processes ($d = 1$) the above representation can be made explicit: if X is a real-valued α-stable variable with $0 < \alpha < 2$ then its Lévy measure is of the form

$$\nu(x) = \frac{A}{x^{1+\alpha}} 1_{x>0} + \frac{B}{|x|^{1+\alpha}} 1_{x<0}$$

for some positive constants A and B.

Chapter 3

Market Risk Assessment

Market Risk is the risk that a position will not be as profitable as an investor expected because of fluctuations in market prices or rates(e.g. equity prices, interest rates, currency rates or commodity prices). Market Risk can be defined as the uncertainty of the future market values of the portfolio's profits and losses resulting from adverse market movements of the market-risk factors.

Mathematically, for a fixed probability space $(\Omega, \mathcal{F}, \mathbb{P})$, given some convex cone $\mathcal{M} \subseteq (\Omega, \mathcal{F}, \mathbb{P})$ of random variables, a measure of risk with domain $\mathcal{M}$ is a mapping $\rho : \mathcal{M} \rightarrow \mathbb{R}$. Recall that $\mathcal{M}$ is a convex cone if $X_1 \in \mathcal{M}$ and $X_2 \in \mathcal{M}$ implies that $X_1 + X_2 \in \mathcal{M}$ and $\lambda X_1 \in \mathcal{M}$ for every $\lambda > 0$. Economically, a risk measure should capture the preferences of the decision maker.

Many possible definitions of risk have been proposed in the literature, because different investors adopt different investment strategies in seeking to realize their investment objectives. In some sense, risk itself is a subjective concept and this is probably the main characteristic of risk. Thus, even if we can identify some desirable features of an investment risk measure, probably no unique risk measure exists that can be used to solve every investor's problem. Loosely speaking, it is hard to discriminate between a "good" and a "bad" risk measure. Most of portfolio theory had based the concept of risk in strong connection with the investor's preferences and their utility function.

3.1 The original risk measure–Variance

An important risk measure that is widely used in financial economics is the standard deviation, also sometimes means variance risk measure. It was firstly considered by Markowitz [42] in his Modern Portfolio Theory and it was also a basis of the Capital Asset Pricing Model. One deficiency of the standard deviation measure for applications to Gaussian distribution is that it treats the negative and the positive deviations from the mean in the same way.

Even for the application to heavy-tailed distribution, it also shows some drawbacks. The heavy-tailed distributions are widely applied in insurance and are gradually penetrating in finance as well. However, for such distributions, as we will see in the following chapters, the standard deviation may not even exist. In the case that it does not exist, the risk measure provides little relevant probabilistic information. All these reasons motivate the research into other risk measures.

3.2 The most classical risk measure–VaR

A highly comprehensive market risk measure is the so-called Value-at-Risk(VaR). VaR describes the loss that can occur over a given period, at a given confidence level, due to exposure to market risk. Formally, it is defined as:

Definition 3.1. Given some confidence level $\alpha \in (0,1)$ and a risk time horizon $T > 0$, $VaR(\alpha, T)$ is the smallest real value such that the probability that the portfolio changes W exceeds that value is not larger than $1-\alpha$,

$$VaR_\alpha = \inf\{w \in \mathbb{R}, P(W > w) \leq 1 - \alpha\}. \tag{3.1}$$

The wide usage of the VaR stems from the fact that it is an easily interpretable summary measure of risk and also has an appealing rationale, as it allows its users to focus attention on "normal market conditions" in their routine operations. Note that VaR in the definition corresponds to a negative dollar value when a loss is observed at level $1 - \alpha$.

By noting

$$VaR_\alpha = \inf\{w \in \mathbb{R}, 1 - F_W(w) \leq 1 - \alpha\} = \inf\{w \in \mathbb{R}, F_W(w) \geq \alpha\}$$

we observe that the definition of VaR coincides with the definition of an α-quartile of the distribution of W in terms of a generalized inverse of the distribution function F_W.

If W denotes the aggregate claims of an insurance portfolio over a given reference period, P denotes the aggregate premium for this portfolio, and $Q_\alpha(W)$ denotes the α quantile of the wealth W, then $Q_\alpha(W) - P$ is the smallest "additional capital " required such that the insurer becomes technically insolvent, i.e. $W > Q_\alpha(W)$, with a probability of at most $1 - \alpha$. Note that here α is the confidence degree, e.g. 0.95. In addition, many books talk about the simulation of VaR, one may refer to [2] and [65].

3.3 A better risk measure–Tail VaR

A single quartile risk measure of a predetermined level α does not give any information about the thickness of the upper tail of the distribution function from $VaR_\alpha(W)$ on. A regulator for instance is not only concerned with the frequency of default, but also about the severity of default. Also shareholders and management should be concerned with the question "how bad is bad?" when they want to evaluate the risks at hand in a consistent way. Therefore, one often uses another risk measure which is called the Tail Value-at-Risk(TVaR) at level α.

Definition 3.2. Given a small number $\alpha \in (0,1)$, TVaR is the arithmetic average of the quantiles of W, from α on, i.e.

$$TVaR_{\alpha}(W) = \frac{1}{1-\alpha}\int_{\alpha}^{1} VaR_{\hat{\alpha}}(W)d\hat{\alpha}, \alpha \in (0,1). \tag{3.2}$$

Note that the $TVaR_{\alpha}$ is always larger than the corresponding quantile. From the above formula it follows immediately that the TVaR is a non-decreasing function of α.

Let W again denote the aggregate claims of an insurance portfolio over a given reference period and P the provision for this portfolio. Setting the capital equal to $TVaR_{\alpha}(W)-P$,we could define "bad times" as those where W takes a value in the interval $[Q_{\alpha}(W), TVaR_{\alpha}(W)]$. Hence, "bad times" are those where the aggregate claims exceed the threshold $Q_{\alpha}(W)$, but not using up all available capital.

3.4 An α-quantile based risk measure-CaR

Based on the definition of VaR, Emmer et al[19] considered capital at risk (CaR) as the risk measure. CaR is defined via the value at risk, that is, a low quantile $(1-\alpha)$of profit-loss distribution of a portfolio. The CaR of a portfolio is then commonly defined as the difference between the mean of the profit-loss distribution and the VaR. We think of the CaR as the capital reserve in equity. One of the advantage of this risk measure is that it allows one to derive explicit closed form solutions for the portfolio problem in either Gaussian world or some subclass of a Levy world[20].

Definition 3.3. (Capital at Risk) Let W_0 be the initial capital and T a given time horizon, let Z_{α} be the α-quantile of the standard normal distribution. For some portfolio $\pi \in \mathbb{R}^d$ and the corresponding terminal wealth W_T, the α-quantile of W_T

is given by

$$\rho(W,\pi,T) = W_0 \exp((\pi\prime(b - r\cdot 1) + r - \|\pi\prime\sigma\|^2)T + z_\alpha\|\pi\prime\sigma\|\sqrt{T})$$

that is, $\rho(W,\pi,T) = \inf\{z \in \mathbb{R} : P(W_T \leq z) \geq \alpha\}$, then define

$$\begin{aligned} CaR(W,\pi,T) &= W_0 \exp(rT) - \rho(W,\pi,T) \\ &= W_0 \exp(rT)(1 - \exp((\pi\prime(b - r\cdot 1) \\ &\quad + r - \|\pi\prime\sigma\|^2)T + z_\alpha\|\pi\prime\sigma\|\sqrt{T})), \end{aligned} \tag{3.3}$$

the capital at risk of the portfolio π(with initial capital W and time horizon T).

Remark 3.8. Here α is different from the one in the definition of VaR which denotes the confidence level. From the definition of CaR and the following, we assume α is a small number which actually denotes one minus confidence level. Just to ease the notation, we still use α, and thus $z_\alpha < 0$.

Remark 3.9. The definition of CaR limits the possibility of excess losses over the risk-less investment. One typically wants to have a positive CaR as the upper bound for the "likely losses".

3.5 A Tail-VaR based risk measure–EaR

However CaR also shows several disadvantages, since it is based on VaR. As stated in [4], VaR, and hence CaR doesn't behave nicely with respect to addition of risks, even independent ones, creating severe aggregation problems; the use of VaR doesn't encourage and, indeed, sometimes prohibits diversification, because VaR does not take into account the economic consequences of the events the probabilities of which it controls. As a replacement of VaR, Artzner introduced Tail Conditional Expectation(TCE), also called TVaR (as defined previously). For convenience, we use an alternate definition of TVaR presented in the following.

Definition 3.4.(Tail-VaR)

$$TVaR_\alpha(W(T)) = E(W(T)|W(T) \leq VaR_\alpha) \tag{3.4}$$

Intuitively, tail-VaR is a more appealing risk measure than the α-quantile. It gives the average severity of loss given that the loss exceeds its α-quantile. α-quantile, on the other hand, only provides a probabilistic statement for which the loss exceeds the α-quantile. Based on the properties of TVaR, Li et.al [38, 37] considered another risk measure: Earning at Risk(EaR).

Definition 3.5. The Earnings-at-Risk of a constant re-balanced portfolio investment strategy π relative to a risk measure TCE_α is defined as the difference between the mean terminal wealth and its associated risk measure:

$$EaR_\alpha := E(W(T)) - TVaR_\alpha(W(T)) \tag{3.5}$$

Remark 3.10. Note that there is an important distinction between the EaR proposed above and the CaR considered by Emmer et al.[19]. CaR is defined as the difference between the terminal wealth of the pure bond (risk-less) investment strategy and the risk measure VaR. CaR measures risk relative to a pure bond investment strategy while EaR measures risk relative to mean terminal wealth. The mean terminal wealth depends explicitly on the adopted investment strategy π while the pure bond strategy is independent of π. EaR therefore provides a trade-off between investing in the portfolio with position π and its expected shortfall as a result of adopting such an investment strategy. When formulated as an optimization problem, both the mean return and its risk measure are considered jointly. Hence it is a more relevant measure over CaR which only provides a trade-off between the risk-free investment and its associated risk measure.

Chapter 4

Portfolio Selection in Gaussian World

Consider a standard Black-Scholes type market consisting of one risk-less bond and several risky stocks. Their respective prices $(P_0(t))_{t\geq 0}$ and $(P_i(t))_{t\geq 0}$, $i = 1, \ldots, n$ evolve according to the SDE

$$dP_0(t) = P_0(t)rdt, P_0(0) = 1,$$

$$dP_i(t) = P_i(t)(b_i dt + \sum_{j=1}^{n} \sigma_{ij} dB_j(t)),\ P_i(0) = p_i,$$

where $i = 1, \ldots, n$ Here $B(t) = (B_1(t), \ldots, B_n(t))'$ is a standard n-dimensional Brownian motion, $r \in \mathbb{R}$ is the risk-less interest rate, $b = (b_1, \ldots, b_n)'$ the vector of stock-appreciation rates or the percentage of return, and $\sigma = (\sigma_{ij})_{1\leq i,j\leq n}$ is a volatility matrix.

Let $\pi(t) = (\pi_1(t), \ldots, \pi_n(t))' \in \mathbb{R}^n$ be an admissible portfolio process, in other words, $\pi_i(t)$ is the fraction of the wealth $W(t)$ that is invested in stock i. Denoting by $(W(t))_{t\geq 0}$ the wealth process, it follows the dynamic

$$dW(t) = W(t)((1 - \pi(t)'1)r + \pi(t)'b)dt + \pi(t)'\sigma dB(t), W(0) = w \qquad (4.1)$$

where $w \in \mathbb{R}$ denotes the initial capital of the investor and $1 = (1, \ldots, 1)'$ denotes the vector(of appropriate dimension) with unit components. The fraction of the investment in the bond is $\pi_0(t) = 1 - \pi(t)'1$.

Throughout the thesis, we restrict ourselves to the constantly-rebalanced portfolio (CRP)$\pi(t) = \pi = (\pi_1, \ldots, \pi_n)$ for all $t \in [0, T]$. By this assumption, we can obtain the explicit results in the Gaussian case. Moreover, the economic interpretation of the mathematical results is comparably easy. Here, it is important to point out that following a constant portfolio process does not mean that there is no trading. Since the stock prices evolve randomly, one has to trade at every time instant to keep the fractions of wealth invested in the different securities constant. Thus, following a constant portfolio process still means one must follow a dynamic trading strategy.

By solving SDE 4.1, we get the following explicit formulas for the wealth process for all $t \in [0, T]$, (see [35]for the derivation).

$$W(t) = w \exp((\pi'(b - r1) + r - \|\pi'\sigma\|^2/2)t + \pi'\sigma B(t)), \quad (4.2)$$

$$E(W(t)) = w \exp((\pi'(b - r1) + r)t), \quad (4.3)$$

$$var(W(t)) = w^2 \exp(2(\pi'(b - r1) + r)t)(\exp(\|\pi'\sigma\|^2 t) - 1). \quad (4.4)$$

4.1 Variance as the risk measure

Now consider the optimization problem:

$$\begin{aligned} &\min_{\pi \in \mathbb{R}^n} Var(W^\pi(T)) \quad subject \quad to \\ &E(W^\pi(T)) \geq C \end{aligned} \quad (4.5)$$

Since the pure bond policy yields a deterministic terminal wealth of $W(0)\exp(rT)$, it is natural to assume that $C \geq W(0)\exp(rT)$. With the assumption $b \neq r1$, we have the following result:

Theorem 4.5. The unique optimal policy of problem (4.5) is

$$\pi^* = \frac{(\sigma\sigma')^{-1}(b - r1)}{||\sigma^{-1}(b - r1)||^2}\hat{C} \tag{4.6}$$

where the $\hat{C} = \frac{1}{T}\ln(C/W(0)) - r$. The corresponding expected terminal wealth is $E(W^{\pi^*}(T)) = C$ and variance $var(W^{\pi^*}(T)) = C^2(\exp(\frac{\hat{C}^2 T}{||\sigma^{-1}(b-r1)||^2}) - 1)$

Proof. By the constraints $E(W^{\pi}(T)) \geq C$ and equation (4.3), we have the inequality $\pi'(b - r1) \geq \frac{1}{T}\ln(C/W(0)) - r$. To simplify the notation, we let $\hat{C} \equiv \frac{1}{T}\ln(C/W(0)) - r$. Then we have

$$\pi'(b - r1) \geq \hat{C}$$

Now consider the objective function. Since $W(0)\exp((\pi'(b - r1) + r)T) \geq C$, $W(0)^2\exp(2(\pi'(b - r1) + r)T)(\exp(||\pi'\sigma||^2 T) - 1) \geq C^2(\exp(||\pi'\sigma||^2 T) - 1)$. It is easy to see that, as $||\pi'\sigma|| \to \infty$, the right hand side of the above inequality goes to infinity, and thus $var(W(T)) \to \infty$. Therefore, minimizing the variance is equivalent to minimizing $||\pi'\sigma||$. Now consider the following optimization problem:

$$\begin{aligned} &\min_{\pi} ||\pi'\sigma||^2 \quad subject \;\; to \\ &\pi'(b - r1) \geq \hat{C} \end{aligned}$$

Define $f(\pi) = \pi'\sigma\sigma'\pi$ and $g(\pi) = \pi'(b - r1) - \hat{C}$. Then both $f(\pi)$ and $g(\pi)$ are differentiable with respect to π. Define $h(\pi) = f(\pi) + \lambda g(\pi)$,using the Lagrangian method, we get

$$\begin{cases} \frac{1}{2}\sigma\sigma'\pi + \lambda(b - r1) = 0 \\ \pi'(b - r1) - \hat{C} = 0 \end{cases}$$

By solving this system of equation, we get the unique optimal solution to problem

(4.1)

$$\pi^* = \frac{(\sigma\sigma')^{-1}(b - r1)}{[\sigma^{-1}(b - r1)]'[\sigma^{-1}(b - r1)]}\hat{C}$$

By the definition of norm and inner product, we can simplify the above solution as

$$\pi^* = \frac{(\sigma\sigma')^{-1}(b - r1)}{||\sigma^{-1}(b - r1)||^2}\hat{C} \tag{4.7}$$

The rest of the results are easy to verify.

4.2 CaR as the risk measure

Now let's consider the optimization problem

$$\min_\pi CaR(W^\pi(T)) \quad subject \quad to$$
$$E(W^\pi(T)) \geq C$$

for the same predetermined minimum attainable expected terminal wealth C. In Emmer et al.[19], they considered the opposite case, that is, maximizing the expectation of the terminal wealth with constrained risk. In order to make the comparison of the three risk measures easy to see, we choose to discuss the case of minimizing the risk given that expectation is greater than a lower bound. Instead of giving a theorem, we are going to obtain the unique solution by the following analysis.

Let $f(\pi) = (\pi\prime(b - r \cdot 1) - ||\pi\prime\sigma||^2/2)T + z_\alpha||\pi\prime\sigma||\sqrt{T}), \pi \in \mathbb{R}^n$, where z_α is the α-quantile of the standard normal distribution.

Remark 4.1. Note that here $z_\alpha < 0$, since we assume that $\alpha < 1/2$, e.g. 0.05. Thus, in the following we will use $-|z_\alpha|$ in place of z_α

Remark 4.2. Let $\varepsilon = ||\pi'\sigma||$, then $f(\pi) \to -\infty$ as $\varepsilon \to \infty$.

Remark 4.3. One obvious fact is that $\sup_\pi CaR(W(T)) = W(0)\exp(rT)$; in other words, the use of extremely risky strategies(in the sense of high norm $||\pi'\sigma||$)

can lead to a loss that is close to the total capital.

Remark 4.4. For any fixed positive ε, with the assumption that $b \neq r1$, it is easy to see that minimizing the function

$$\begin{aligned} f(\pi) &= (\pi'(b - r \cdot 1) - ||\pi'\sigma||^2/2)T + z_\alpha ||\pi'\sigma||\sqrt{T}) \\ &= z_\alpha \varepsilon \sqrt{T} - \varepsilon^2 T/2 + \pi'(b - r \cdot 1)T \end{aligned} \tag{4.8}$$

over the boundary of an ellipsoid $||\pi'\sigma|| = \varepsilon$ is equivalent to the problem

$$\begin{aligned} &\min_\pi \pi'(b - r \cdot 1) \quad subject \quad to \\ &||\pi'\sigma|| = \varepsilon \end{aligned} \tag{4.9}$$

Again by Lagrange's method, we get the explicit solution

$$\pi_\varepsilon^* = \varepsilon \frac{(\sigma\sigma')^{-1}(b - r1)}{||\sigma^{-1}(b - r1)||} \tag{4.10}$$

Denoting $\theta \equiv ||\sigma^{-1}(b - r1)||$, from equation (4.10), we get

$$\pi_\varepsilon^* \theta = (\sigma\sigma')^{-1}(b - r1)\varepsilon$$

and multiplying both sides by $\sigma\sigma'$, we have

$$(\sigma\sigma')\pi_\varepsilon^* \theta = (b - r1)\varepsilon$$

Then, multiplying both sides by $\pi_\varepsilon^{*\prime}$, we have

$$\pi_\varepsilon^{*\prime}\sigma\sigma'\theta = \pi_\varepsilon^{*\prime}(b - r1)\varepsilon$$

Furthermore we get

$$\pi_\varepsilon^{*\prime}(b - r1) = \varepsilon\theta \tag{4.11}$$

Substituting (4.11) back into (4.8), we get immediately that

$$\begin{aligned} f(\pi_\varepsilon^*) &= -\varepsilon^2 T/2 + \varepsilon\sqrt{T}(\theta\sqrt{T} - |z_\alpha|) \\ &= -\tfrac{1}{2}[\varepsilon\sqrt{T} - (\theta\sqrt{T} - |z_\alpha|)]^2 + \tfrac{1}{2}(\theta\sqrt{T} - |z_\alpha|)^2 \end{aligned} \tag{4.12}$$

Now, let's look at the above as a function of ε, that is,

$$g(\varepsilon) = -\frac{1}{2}[\varepsilon\sqrt{T} - (\theta\sqrt{T} - |z_\alpha|)]^2 + \frac{1}{2}(\theta\sqrt{T} - |z_\alpha|)^2$$

On one hand, if $\theta\sqrt{T} - |z_\alpha| < 0$, in the domain $[\theta\sqrt{T} - |z_\alpha|, +\infty]$, $g(\varepsilon)$ is decreasing. Since $\varepsilon \geq 0$, it obtains its maximum only when $\varepsilon = 0$. Therefore $f(\pi_\varepsilon^*) = g(\varepsilon) = 0$. In this case, $\pi_\varepsilon^* = 0$, and thus corresponds to a pure bond strategy.

On the other hand,if $\theta\sqrt{T} - |z_\alpha| \geq 0$, we need to consider the constraints of problem 4.2. By simplification, the constraints are equivalently changed to be $\varepsilon\sqrt{T} \geq \frac{\hat{C}\sqrt{T}}{\theta}$. Under this condition, we have two cases to consider.

First of all, if $\frac{\hat{C}\sqrt{T}}{\theta} \leq \theta\sqrt{T} - |z_\alpha|$, then by the properties of the quadratic equation, we get the maximum of $g(\varepsilon^*) = \frac{1}{2}(\theta\sqrt{T} - |z_\alpha|)^2$, where $\varepsilon^* = \theta - \frac{|z_\alpha|}{\sqrt{T}}$. And thus the minimal CaR is

$$CaR(W^{\pi^*}(T)) = W(0)\exp(rT)(1 - \exp(\frac{1}{2}(\theta\sqrt{T} - |z_\alpha|)^2)) \tag{4.13}$$

and

$$\pi^* = (\theta - \frac{|z_\alpha|}{\sqrt{T}})\frac{(\sigma\sigma')^{-1}(b - r1)}{\|\sigma^{-1}(b - r1)\|} \tag{4.14}$$

Secondly, if $\frac{\hat{C}\sqrt{T}}{\theta} > \theta\sqrt{T} - |z_\alpha|$, $g(\varepsilon)$ obtains its maximum when $\varepsilon^* = \frac{\hat{C}}{\theta}$. In this case

$$\pi^* = (\frac{\hat{C}}{\theta})\frac{(\sigma\sigma')^{-1}(b - r1)}{\|\sigma^{-1}(b - r1)\|} \tag{4.15}$$

and

$$CaR(W^{\pi^*}(T)) = W(0)\exp(rT)(1-\exp(-\frac{1}{2}\frac{\hat{C}^2T}{\theta}+\frac{\hat{C}T}{\theta}(\theta\sqrt{T}-|z_\alpha|))) \quad (4.16)$$

Remark 4.6. So far, we don't consider the case when $b = r1$. Actually, this case happens in a risk-neutral world. Under this condition,

$$f(\pi) = g(\varepsilon) = -\frac{1}{2}(\varepsilon\sqrt{T}+|z_\alpha|)^2 + |z_\alpha|^2/2$$

And so $g(\varepsilon)$ obtains its maximum only when $\varepsilon^* = 0$, that is, $\pi^* = 0$, and thus

$$CaR(W^{\pi^*}(T)) = 0$$

This result states that in a risk-neutral market the CaR of every strategy containing stock investment is bigger than the CaR of pure bond strategy.

4.3 EaR as the risk measure

Before getting into the discussion of the explicit solution to the mean-EaR optimization problem, let's give some detailed derivation of some results in [38, 37] which we did not explain clearly.

By equation4.2, given an α-level quantile $\rho_0(\pi)$, we have the following inequality

$$W(0)\exp((\pi'(b-r1)+r-||\pi'\sigma||^2/2)T+\pi'\sigma B(T)) \leq \rho_0(\pi)$$

Denoting $a = \pi'(b-r1)+r-||\pi'\sigma||^2/2$, by simple manipulation of the inequality, we have

$$z \equiv \frac{\pi'\sigma B(T)}{||\pi'\sigma||\sqrt{T}} \leq \frac{\ln\frac{\rho_0}{W(0)} - aT}{||\pi'\sigma||\sqrt{T}} \equiv z_\alpha$$

Since $B(T)$ is n-dimensional standard Brownian Motion, $z \sim N(0,1)$, and thus z_α is the α quantile of the standard normal distribution. Let Υ and $p(z)$be the distribution function and density function of z, respectively. Then:

$$\rho_0 = W(0)\exp(aT + z_\alpha \|\pi'\sigma\|\sqrt{T})$$

By the definition of TVaR in (3.2), we have

$$\begin{aligned} \rho &\equiv E(W^\pi(T)|W^\pi(T) \le \rho_0) \\ &= E(W^\pi(T)|z \le z_\alpha) \\ &= \tfrac{1}{\alpha}\int_{-\infty}^{z_\alpha} W^\pi(T)p(z)dz \\ &= W(0)\exp((\pi'(b - r1) + r)T)\tfrac{\Phi(z_\alpha - \|\pi'\sigma\|\sqrt{T})}{\alpha} \end{aligned}$$

By the definition of EaR, we have

$$\begin{aligned} EaR(W^\pi(T)) &:= E(W^\pi(T)) - \rho \\ &= W(0)\exp((\pi'(b - r1) + r)T)(1 - \tfrac{\Phi(z_\alpha - \|\pi'\sigma\|\sqrt{T})}{\alpha}) \end{aligned} \tag{4.17}$$

Next let's consider the following optimization problem:

$$\min_{\pi\in\mathbb{R}^n} EaR(W^\pi(T)) \quad subject \ \ to \quad E(W^\pi(T)) \ge C \tag{4.18}$$

Firstly, we give the extreme case of EaR:

Proposition 1.

1. $\min_{\pi\in\mathbb{R}^n} var(W(T)) = 0$ if and only if $\pi = 0$, where $0 = (0, \ldots, 0)_n$.

2. $\sup_{\pi\in\mathbb{R}^n} var(W(T)) = \begin{cases} W_0\exp(rT) & b = r1 \\ +\infty & otherwise \end{cases}$

For more details of the proof, please see [38, 37]. Here, we just present the main results. Firstly, give an important function defined as

$$f(\pi) = (\pi'(b - r1) + r)T + \ln(1 - \frac{\Phi(z_\alpha - \|\pi'\sigma\|\sqrt{T})}{\alpha}) \tag{4.19}$$

Then EaR can be expressed as $EaR(\pi) = W(0)\exp(f(\pi))$, if $\|\pi'\sigma\| > 0$. Furthermore, minimizing EaR is equivalent to minimizing $f(\pi)$, since EaR is an increasing function of $f(\pi)$, with the constraint $\|\pi'\sigma\| = \varepsilon$, for fixed ε.

With the above proposition, Li [38] give their main result as following.

Theorem 4.6. Assume that $b \neq r1$, then the unique optimal policy of problem 4.18 is

$$\pi^* = \varepsilon^* \frac{(\sigma\sigma')^{-1}(b - r1)}{\|\sigma^{-1}(b - r1)\|} \tag{4.20}$$

where

$$\varepsilon^* = \frac{\ln(C/W(0)) - rT}{\|\sigma^{-1}(b - r1)\|T} \tag{4.21}$$

The corresponding expected terminal wealth is $E(W^{\pi^*}(T)) = C$ and Earnings at Risk is

$$Ear(\pi^*) = C(1 - \frac{\Phi(z_\alpha - \varepsilon^*\sqrt{T})}{\alpha})$$

As a comparison, we give the explicit solution to the dual problem by giving the following theorem and proof.

Theorem 4.7. For the following optimization problem:

$$\max_{\pi \in \mathbb{R}^n} E(W^\pi(T)) \quad subject \ \ to \quad EaR(W^\pi(T)) \leq C \tag{4.22}$$

If $b \neq r1$, then the unique optimal policy of the problem is given by

$$\pi^* = \varepsilon^* \frac{(\sigma\sigma')^{-1}(b - r1)}{\|\sigma^{-1}(b - r1)\|} \tag{4.23}$$

where ε^* is the unique solution of the equation

$$\|\sigma^{-1}(b-r1)\|\varepsilon - \left(\frac{1}{T}\ln\frac{C/w}{1-\frac{\Phi(z_\alpha-\varepsilon\sqrt{T})}{\alpha}} - r\right) = 0 \tag{4.24}$$

The corresponding maximal expected terminal wealth under the variance constraint equals

$$E(W^{\pi^*}) = w\exp((\varepsilon^*\|\sigma^{-1}(b-r1)\| + r)T)$$

Proof. Let $\|\pi'\sigma\| = \varepsilon$, then the constraint

$$\begin{array}{lcll} EaR(W^\pi(T)) & \leq & C & \Leftrightarrow \\ w\exp((\pi'(b-r1)+r)T)(1-\frac{\Phi(z_\alpha-\varepsilon\sqrt{T})}{\alpha}) & \leq & C & \Leftrightarrow \\ (\pi'(b-r1)+r)T+\ln(1-\frac{\Phi(z_\alpha-\varepsilon\sqrt{T})}{\alpha}) & \leq & \ln(\frac{C}{w}) & \Leftrightarrow \\ (\pi'(b-r1)+r)T & \leq & \ln(\frac{C}{w})-\ln(1-\frac{\Phi(z_\alpha-\varepsilon\sqrt{T})}{\alpha}) & \Leftrightarrow \\ \pi'(b-r1) & \leq & \frac{1}{T}\ln\frac{\frac{C}{w}}{1-\frac{\Phi(z_\alpha-\varepsilon\sqrt{T})}{\alpha}} - r & \end{array} \tag{4.25}$$

To simplify the notation, let $\hat{C} := \frac{1}{T}\ln\frac{\frac{C}{w}}{1-\frac{\Phi(z_\alpha-\varepsilon\sqrt{T})}{\alpha}} - r$, then it is easy to verify that $\hat{C}$ is a decreasing function of ε and as $\varepsilon \to 0$, $\hat{C} \to +\infty$; as $\varepsilon \to \infty$, $\hat{C} \to \frac{1}{T}\ln\frac{C}{w} - r$. Note that $\hat{C} \geq 0$ for any ε, since we assume that $C \leq w\exp(rT)$ in the previous sections.

Now let's consider the objective function $\max_{\pi\in\mathbb{R}^n} E(W^\pi(T))$. It is easy to see that $w\exp((\pi'(b-r1)+r)T)$ is an increasing function of π. With the assumption of no short selling, we have the equivalent objective function $\max_{\pi\in\mathbb{R}^n}\pi'(b-r1)$ and further

$$\max_{\pi\in\mathbb{R}^n}\pi'(b-r1)(b-r1)'\pi$$

Let $\theta = ||\sigma^{-1}(b - r1)||$, then

$$\begin{aligned} \pi'(b-r1)(b-r1)'\pi &= \pi'\sigma\sigma^{-1}(b-r1)(b-r1)'\sigma'^{-1}\sigma'\pi \\ &= \pi'\sigma\theta^2\sigma'\pi \\ &= \sigma^2\varepsilon^2 \end{aligned} \tag{4.26}$$

Thus $\pi'(b - r1)(b - r1)'\pi$ is an increasing function of ε. Since $||\pi'\sigma|| = \varepsilon$ is an increasing function of ε and by the inequality 4.25, $\sigma^2\varepsilon^2 \leq \hat{C}^2$, we have that $\sigma^2\varepsilon^2$ gets its maximum with $\hat{C}^2$ at the intersection point, that is, when ε^* is the unique solution of

$$\sigma^2\varepsilon^2 = \hat{C}^2. \tag{4.27}$$

On the other hand

$$\begin{aligned} \pi'(b-r1) &= \theta\varepsilon \\ \pi'\sigma\sigma^{-1}(b-r1) &= \theta\varepsilon \\ \pi'\sigma\sigma^{-1}(b-r1)\sigma^{-1}(b-r1)' &= \theta\varepsilon\sigma^{-1}(b-r1)' \\ \pi'\sigma\theta^2 &= \theta\varepsilon\sigma^{-1}(b-r1)' \\ \pi &= \tfrac{\varepsilon\sigma\sigma'^{-1}(b-r1)}{\theta} \end{aligned} \tag{4.28}$$

Therefore for the unique ε^* which satisfies equation

$$||\sigma^{-1}(b - r1)||\varepsilon - \left(\frac{1}{T}\ln\frac{C/w}{1 - \frac{\Phi(z_\alpha - \varepsilon\sqrt{T})}{\alpha}} - r\right) = 0$$

the optimal solution to the problem 4.22 is

$$\pi^* = \varepsilon^* \frac{(\sigma\sigma')^{-1}(b - r1)}{||\sigma^{-1}(b - r1)||}$$

and the maximum value of the objective function

$$E(W^{\pi^*}) = w\exp((\varepsilon^*\|\sigma^{-1}(b - r1)\| + r)T)$$

is easy to verify.

4.4 Summary

Remark 4.7. Comparing the Variance and CaR we can see that the existence of at least one stock with a mean rate of return different from the risk-less rate implies the existence of a stock and bond strategy with a negative CaR as soon as the time horizon T is large. Thus, even if the CaR were the only criterion by which to judge an investment strategy the pure bond investment would not be optimal if the time horizon were far away. On the one hand this fact is in line with empirical results on stock and bond markets. On the other hand this shows a remarkable difference between the behavior of the CaR and the variance at risk measures. Independent of the time horizon and the market coefficients, pure bond investment would always be optimal with respect to the variance of the corresponding wealth process.

Remark 4.8. One consequence of the analysis of the three risk measures is that for a given minimum level C of the expected terminal wealth $E(W^{\pi}(T))$, the optimal investment strategies for both the mean-EaR and the mean-variance problems are equivalent. In fact, it can also be shown that similar optimal π^* can also be obtained if we had considered the risk measure CaR as in the mean-CaR optimization problem. This implies all these risk measures yield similar optimal investment strategies as long as the preselected level C is identical.

Remark 4.9. We also can see a linkage between the EaR and the Variance of the terminal wealth. For instance, by fixing the level of EaR, we can get the highest attainable expected return and hence the optimal portfolio π^*. This, in turn, allows us to determine the corresponding minimum variance of terminal wealth using the result of theorem 1.

Chapter 5

Portfolio Selection in Non-Gaussian World

5.1 Analytical solutions to the optimization problem

In this section, we will use the notation in Emmer et al.[20] to make out context consistent. Consider a standard Black-Scholes type market consisting of a risk-less bond and several risky stocks, which follow an exponential Lévy process. The stock prices $P_i(t), i = 1, \ldots, n$ are given by the SDE

$$\begin{aligned} dP_i(t) &= P_i(t-)(b_i dt + d\hat{L}_i(t)) \\ &= P_i(t-)((b_i + \tfrac{1}{2}\Sigma_{j=1}^{n}(\sigma_{ij}\beta_{ij})^2)dt + \Sigma_{j=1}^{n}\sigma_{ij}dL_j(t) \\ &\quad + \exp(\Sigma_{j=1}^{n}\sigma_{ij}\Delta L_j(t)) - 1 - \Sigma_{j=1}^{n}\sigma_{ij}\Delta L_j(t)) \end{aligned} \tag{5.1}$$

The quantity $r \in \mathbb{R}$ is the risk-less interest rate and $\sigma = (\sigma_{ij})_{1 \le i,j \le N}$ is an invertible matrix, $b \in \mathbb{R}^n$ can be chosen such that each stock has the desired appreciation rate. Assume $L_i, i = 1, \cdots, n$ are independent. $L = (L_1, \ldots, L_n)$ has characteristic triplet $(\alpha, \beta'\beta, \nu)$, where $\alpha \in \mathbb{R}^n$, β is an arbitrary n-dimensional diagonal matrix. Since the components of βW are independent Wiener processes with possibly different variances, we allow for different scaling factors for the Wiener process and the non-Gaussian components. $\hat{L}_i$ is such that $\varepsilon(\hat{L}_i) = \exp(\Sigma_{j=1}^n \sigma_{ij} L_j(t))$, where ε denotes the **Stochastic Exponential** of a process. From the representation we see that jumps of $\hat{L}_i$ occur at the same time as those of $(\sigma L)_i = \Sigma_{j=1}^n \sigma_{ij} L_j(t)$, but a jump of size $\Sigma_{j=1}^n \sigma_{ij} \Delta L_j(t)$ is replaced by one of size $\exp(\Sigma_{j=1}^n \sigma_{ij} \Delta L_j(t)) - 1$ leading to the term $\exp(\Sigma_{j=1}^n \sigma_{ij} \Delta L_j(t)) - 1 - \Sigma_{j=1}^n \sigma_{ij} \Delta L_j(t)$, whereas the Brownian component remains the same.

In order to simplify the following discussion, we put the definition of the stochastic exponential and its relation with the ordinary exponential here [15].

Definition 5.1. Let $(\hat{L}_t)$ be a Lévy process with triplet (α, β^2, ν), then there exists a unique cadlag process $(Z_t)_{t \geq 0}$ such that

$$dZ_t = Z_t d\hat{L}_t$$

Z is called the **stochastic exponential** of $\hat{L}$ and is denoted by $Z = \varepsilon(\hat{L})$.

Following the above assumption, there exists another Lévy process $(L_t)_{t \geq 0}$, such that $Z_t = e^{L_t}$ where

$$L_t = \ln(Z_t) = \hat{L}_t - \frac{\beta^2 t}{2} + \Sigma_{0 \leq s \leq t} \{\ln(1 + \hat{L}_S) - \Delta \hat{L}_S\}$$

Let $\pi(t) = (\pi_1(t), \ldots, \pi_n(t)) \in \mathbb{R}^n$ be an admissible portfolio process, i.e. $\pi(t)$ is the fraction of the wealth $W^\pi(t)$, which is invested in asset i. The fraction of the investment in the bond is $\pi_0(t) = 1 - \pi(t)'1$, where $1 = (1, \ldots, 1)'$ denotes the vector having unit components. As stated in [20], we restrict ourselves to the constant portfolios; i.e. $\pi(t) = \pi$, for some fixed time horizon T. In order to avoid negative wealth we require that $\pi \in [0, 1]^n$, hence short selling is not allowed in this model. Denote the wealth process by $(W^\pi(t))_{t\geq 0}$, it follows the dynamic

$$dW^\pi(t) = W^\pi(t-)(((1-\pi'1)r + \pi' b)dt + \pi' d\hat{L}(t)), t > 0, W^\pi(0) = w$$

where $w \in \mathbb{R}$ denotes the initial capital of the investor.

From the result in [25], we know that if L is a real-valued Lévy process with characteristic triplet (α, β^2, ν), then also $\hat{L}$ defined by $\varepsilon(\hat{L}) = e^L$ is a Lévy process with characteristic triplet $(\hat{\alpha}, \hat{\beta^2}, \hat{\nu})$ given by

$$\hat{\alpha} - \alpha = \frac{1}{2}\beta^2 + \int ((e^x - 1)1_{\{|e^x-1|<1\}} - x1_{\{|x|<1\}})\nu(dx)$$

$$\hat{\beta^2} = \beta^2$$

$$\hat{\nu}(\Lambda) = \nu(\{x \in \mathbb{R} | e^x - 1 \in \Lambda\}) \text{ for any Borel set } \Lambda \subset \mathbb{R}.$$

Furthermore, Sato[60] gives the characteristic triplet $(\alpha_\pi, \beta_\pi^2, \nu_\pi)$ of an n-dimensional Lévy process, which also presents the relation between the characteristic triplets of an n-dimensional Lévy process with L and its linear transformation $\pi' L$ by the following formula:

$$\alpha_\pi = \pi'\alpha + \int \pi' x(1_{\{|\pi' x<1|\}} - 1_{\{|x|<1\}})\nu(dx), \tag{5.2}$$

$$\beta_\pi^2 = |\pi'\beta|^2, \tag{5.3}$$

$$\nu_\pi(\Lambda) = \nu(\{x \in \mathbb{R}^n | \pi' x \in \Lambda\}) \forall \Lambda \subset \mathbb{R}. \tag{5.4}$$

With the above results, we can derive the above solution to SDE by using Itô's formula [20].

$$\begin{aligned} W^\pi(t) &= w \exp(t(r + \pi'(b - r1)))\varepsilon(\pi'\hat{L}(t)) \\ &= w \exp(\alpha_W t + \pi'\sigma\beta W(t))\tilde{W}^\pi(t) \end{aligned} \tag{5.5}$$

where $\alpha_W = r + \pi'(b + [\sigma\beta]^2/2 - r1 + \sigma\alpha) - |\pi'\sigma\beta|^2/2 + \int_{\mathbb{R}^n}(\ell(x)1_{\{|\ell(x)|\le 1\}} - \pi'\sigma 1_{\{|x|\le 1\}})\nu(dx)$, and $\ell(x) := \ln(1 + \pi'(e^{\sigma x}) - 1)$.

$$\begin{aligned}\ln \tilde{W}^\pi(t) &= \int_0^t \int_{\mathbb{R}^n} \ell(x) 1_{\{|\ell(x)|>1\}} M_L(ds, dx) \\ &\quad + \int_0^t \int_{\mathbb{R}^n} \ell(x) 1_{\{|\ell(x)|\le 1\}} (M_L(ds, dx) - ds\nu_L(dx))\end{aligned}$$

The wealth process is again an exponential Lévy process. α_W together with $\beta_W^2 = |\pi'\sigma\beta|^2$ and $\nu_W(\Lambda)$ is the characteristic triplet of the Lévy process $\ln(W^\pi/w)$. Assume the k-th moment of the wealth process exists, then

$$E(W^\pi(t)^k) = w^k \exp((k\alpha_W + K^2\beta_W^2/2)t) E((\tilde{W}^\pi(t))^k)$$

and

$$E((\tilde{W}^\pi(t))^k) = \exp(\tilde{\mu_k} t)$$

where

$$\tilde{\mu_k} = \int_{\mathbb{R}^n} ((1 + \pi'(e^{\sigma x} - 1))^k - 1 - k\ell(x) 1_{\{|\ell(x)|\le 1\}})\nu(dx)$$

and ν is the Lévy measure of L. Now, we would like to find the mean and variance of the wealth process.

Firstly assume that a Lévy random variable $L(1)$ has moment generating function $\hat{f}(s) = E(\exp(sL(1)))$ such that $\hat{f}(e_i'\sigma) < \infty$ for $i = 1, \ldots, n$, where e_i is the i-th dimensional unit vector. Then it is easy to calculate

$$\begin{aligned}\hat{f}(e_i'\sigma) &= E(\exp(e_i'\sigma L(1))) \\ &= \exp((\sigma\alpha + (\sigma\beta)^2/2 + \int_{\mathbb{R}^n}(e^{\sigma x} - 1 - \sigma x 1_{\{|x|<1\}})\nu(dx))_i)\end{aligned}$$

Let

$$\ln \hat{f}(\sigma) := (\ln \hat{f}(e_1'\sigma), \ldots, \ln \hat{f}(e_n'\sigma))'$$

then

$$\begin{aligned} E(W^{\pi}(t)) &= w\exp(t(r+\pi'(b-r1+\ln\hat{f}(\sigma)))) \\ &= w\exp(t(r+\pi'(b-r1+(\sigma\beta)^2/2+ \\ &\qquad \sigma\alpha+\textstyle\int_{\mathbb{R}^n}(e^{\sigma x}-1-\sigma x 1_{\{|x|<1\}})\nu(dx))))). \end{aligned} \tag{5.6}$$

Let $A_{ij} := \hat{f}((e_i+e_j)'\sigma) - \hat{f}(e_i'\sigma) - \hat{f}(e_j'\sigma)$, and $A = (A_{ij}) 1 \le i, j \le n$, then

$$\pi' A \pi = |\pi'\sigma\beta|^2 + \int_{\mathbb{R}^n} (\pi'(e^{\sigma x}-1))^2 \nu(dx)$$

With this, we can calculate the variance

$$\begin{aligned} var(W^{\pi}(t)) &= w^2\exp(2t(r+\pi'(b-r1+(\sigma\beta)^2/2+ \\ &\quad \textstyle\int_{\mathbb{R}^n}(e^{\sigma x}-1-\sigma x 1_{\{|x|<1\}})\nu(dx)))) \cdot \\ &\quad \exp(t(|\pi'\sigma\beta|^2+\textstyle\int_{\mathbb{R}^n}(\pi'(e^{\sigma x}-1))^2\nu(dx))-1) \\ &= w^2\exp(2t(r+\pi'(b-r1+\ln\hat{f}(\sigma))))(\exp(t\pi' A\pi)-1) \end{aligned} \tag{5.7}$$

For more details, see [20].

Generally, if the moment generating function of a stochastic process exist, and hence the mean and variance exist, then we can easily calculate the optimal solution based on the above discuss. Sometimes, not all the stochastic processes have defined first or second moments, therefore we might not be able to get the explicit solution to the optimization problem 2.3. Next, we are going to consider the cases when the first and second moments of the process are well defined.

5.1.1 variance as the risk measure

In this section, we consider the following optimization problem using the variance as the risk measure

$$\min_{\pi\in[0,1]^n} var(W^{\pi}(T)) \quad subject \quad to \quad E(W^{\pi}(T)) \ge C$$

Assume matrix A is symmetric and positive definite. Denote $s := b - r1 + \ln \hat{f}(\sigma)$, and $\varepsilon^2 := \pi' A \pi$, where ε is a positive real number. Then

$$var(W^{\pi}(T)) = w^2 \exp(2T(r + \pi' s))(\exp(T\varepsilon^2) - 1)$$

It is easy to see that

$$\text{as } \varepsilon \to 0,\, g(\varepsilon) = var(W^{\pi}(T)) \to 0$$

and

$$\text{as } \varepsilon \to \infty,\, g(\varepsilon) = var(W^{\pi}(T)) \to \infty.$$

Noting that $g(\varepsilon)$is a strictly increasing function in ε. Therefore, for fixed ε, minimizing $var(W^{\pi}(T))$ is equivalent to minimizing ε. In other words, minimizing $\pi' A \pi$ whose value is ε^2. By the constraint $E(W^{\pi}(T)) \geq C$, we have $\pi' s \geq \frac{1}{T} \ln \frac{C}{w} - r$. By Lagrange's method, we get

$$\pi^* = \frac{\varepsilon^{*2} A^{-1} s}{\frac{1}{T} \ln \frac{C}{w} - r}$$

where

$$\varepsilon^{*2} = \frac{(\frac{1}{T} \ln \frac{C}{w} - r)^2}{s' A^{-1} s}.$$

Furthermore,

$$var(W^{\pi^*}(T)) = C^2 (\exp(\frac{T(\frac{1}{T} \ln \frac{C}{w} - r)^2}{s' A^{-1} s}) - 1).$$

Remark 5.1. Note that the minimal variance value only depends on the stocks via the value $s' A^{-1} s$. There is no explicit dependence on the number of stocks. Therefore, there is no difference between investment in our multi-stock market and a market consisting of the bond and just one stock with appropriate market coefficients b and σ. In the following we take $n = 1$, and without loss of generality, we choose $\sigma = 1$.

Now let's consider the Lévy process $(L(t))_{t \geq 0}$ ([20]) taken as the sum of the drift part $(\gamma t)_{t\geq 0}$ and the **Compound Poisson process** $(\bar{L}(t))_{t\geq 0} := \Sigma_{i=1}^{N(t)} Y_i$, as defined in section (5.2.1).

If the moment generating function $\hat{g}(n) := E(e^{nY}) < \infty$, then

$$\hat{f}(n) = \exp(\lambda(\hat{g}(n) - 1))$$

where $\lambda > 0$ is the Poisson intensity. It is easy to calculate

$$s'A^{-1}s = \frac{[b - r + \lambda(\hat{g}(1) - 1)]^2}{\lambda(\hat{g}(2) - 2\hat{g}(1) + 1)}$$

where $A = \lambda(\hat{g}(2) - 2\hat{g}(1) + 1)$, and $s = b - r + \lambda(\hat{g}(1) - 1)$. Since the drift part of $L(t)$ equals the sum of γ and $\lambda(\hat{g}(1) - 1)$, we may choose γ such that $\gamma + \lambda(\hat{g}(1) - 1) = 0$. In this case we get the same expectation as in the Black-Scholes model; that is, when the stochastic process of the stock price is assumed to be Gaussian. With this particular chosen γ, we calculate the expectation and variance as follows:

$$E(W^{\pi}(T)) = w\exp((\pi'(b - r1) + r)T) = w\exp(T(\pi b + (1 - \pi)r))$$

$$var(W^{\pi}(T)) = w^2\exp(2T(\pi b + (1 - \pi)r))(\exp(\pi^2 T(\lambda(\hat{g}(2) - 2\hat{g}(1) + 1))) - 1)$$

and the minimal risk is given by

$$var(W^{\pi^*}(T)) = C^2(\exp(\frac{T(\frac{1}{T}\ln\frac{C}{w} - r)^2\lambda(\hat{g}(2) - 2\hat{g}(1) + 1)}{[b - r + \lambda(\hat{g}(1) - 1)]^2}) - 1)$$

Remark 5.2. Note that although we made an assumption that only one bond and one stock is involved, this optimal solution can be viewed as the solution to the general multi-stock case.

Remark 5.3. We will see more clearly in a later discussion that variance as the risk measure does not ask explicit probability density function. While, CaR and EaR as the risk measure must require explicit pdf which sometimes is hard to be satisfied.

5.1.2 CaR as the risk measure

The explicit form of the Lévy measure shows that if $\alpha > 1$, the α-stable distributions on $\mathbb{R}$ never admit a second moment, and they only admit a fist moment. Therefore, to get the first and second moments of an α-stable Lévy process, we shall limit our attention on $\alpha \leq 1$. When $\alpha = 1$, $S_1(\sigma, 0, \mu)$ gives us the Cauchy distribution

$$f_C(x) = \frac{\sigma}{\pi((x-\mu)^2 + \sigma^2)}$$

But, the Cauchy distribution doesn't belong to exponential families under investigation. So let's consider another case, that is, $\alpha = 1/2$. This gives us the Lévy distribution $S_{1/2}(\sigma, 1, \mu)$ with density:

$$f_L(x) \equiv (\frac{\sigma}{2\pi})^{1/2} \frac{1}{(x-\mu)^{3/2}} \exp\{-\frac{\sigma}{2(x-\mu)}\} 1_{x>\mu} \tag{5.8}$$

Now calculate the moment generating function

$$\begin{aligned} \hat{f}(s) &= \int_{\mu}^{\infty} e^{sx} f_L(x) dx \\ &= \int_{\mu}^{\infty} e^{sx} (\tfrac{\sigma}{2\pi})^{1/2} \tfrac{1}{(x-\mu)^{3/2}} \exp\{-\tfrac{\sigma}{2(x-\mu)}\} 1_{x>\mu} dx \end{aligned} \tag{5.9}$$

However, we find that the first moment and second moment are not defined, therefore we can't consider the portfolio selection problem with variance as the risk measure. Since the density of the Lévy distribution has explicit form, it is natural to consider the portfolio selection problem with CaR as the risk measure. Obviously, this will be an unconstrained optimization problem, since the expectation does not exist. One can use a special optimization algorithm to solve it as, for example, Newton's method.

Let $\phi(X_\alpha) := \int_\mu^{X_\alpha} f_L(x)dx$ be the cumulative distribution function of the Lévy distribution. Then its α-quantile is $X_\alpha = \phi^{-1}(\alpha)$. It is easy to calculate the α-quantile of the wealth process $W^\pi(T)$, which is given by the formula

$$VaR_\alpha(W^\pi(T)) = W_\alpha^\pi = w\exp(T(r + \pi'(b - r1)) + \phi^{-1}(\alpha))$$

Then by the definition of CaR, we have

$$\begin{aligned} CaR_\alpha(W^\pi(T)) &= w\exp(rT) - VaR_\alpha(W^\pi(T)) \\ &= w\exp(rT)(1 - \exp(\pi'(b - r1)T + \phi^{-1}(\alpha))) \end{aligned}$$

However, we are not able to extend this result to the risk measure EaR, since the first moment does not exist. From this example we can see that, one of the advantages of CaR over the variance and the EaR as the risk measure is that it has a wider application. EaR asks that the random process at least has a first moment, while variance has a stronger requirement that both first and second moments must exist.

5.1.3 EaR as the risk measure

The advantage of the application of the NIG process in finance has been stated clearly in section (5.2.3). Since it has explicit pdf and the first moment is well defined, we are able to use it to calculate EaR, and hence solve the constrained optimization problem 4.18.

In order to clarify our discussion, we are going to give a more detailed explanation of the pdf of the NIG process. First of all, we give the standard definition of the Gamma function, see [18], [56] and [60]:

Definition 5.2. For complex number z with positive real part, the **Gamma**

function is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt$$

Using the definition of Gamma function, we have:

Definition 5.3. The **Modified Bessel function of the first kind** is given by

$$I_v(x) = (\frac{x}{2})^v \Sigma_{k=0}^\infty \frac{(\frac{x}{2})^{2k}}{k!\Gamma(k+1+v)}$$

Definition 5.4. The **Modified Bessel function of the second kind of order 1** is defined as

$$K_1(x) = \frac{\pi}{2}(\frac{I_{-1}(x) - I_1(x)}{\sin(\pi)})$$

Finally, the pdf of the NIG random variable in (2.7) is:

$$f_{NIG}(x; \alpha, \beta, \delta, \mu) = \frac{\alpha\delta}{\pi} e^{\delta\sqrt{\alpha^2-\beta^2}+\beta(x-\mu)} \frac{K_1(\alpha\sqrt{\delta^2 + (x-\mu)^2})}{\sqrt{\delta^2 + (x-\mu)^2}}$$

Based on the pdf of NIG process, we define the cumulative distribution function (cdf) as $F_{NIG}(x_{\hat{\alpha}}, \beta) = \int_{-\infty}^{x_\alpha} f_{NIG}(x)dx$. It is easy to see that the $\hat{\alpha}$-quantile of NIG is given by $F_{NIG}^{-1}(\hat{\alpha}, \beta)$. Here we sacrifice the notion of the α a little bit. And note that we use another parameter β in the definition of $F_{NIG}(x_{\hat{\alpha}}, \beta)$, since it is a family of distributions with variable β. By the results generated by (5.5), (5.6) and (5.7), we have:

$$W^\pi(t) = w\exp(t(r + \pi'(b - r1)))\varepsilon(\pi'\hat{L}(t)) \quad (5.10)$$

$$E(W^\pi(t)) = w\exp(t(r + \pi'(b - r1 + \ln\hat{f}(\sigma)))) \quad (5.11)$$

Since we have assumed that only one stock is involved as the risky asset and the corresponding $\sigma = 1$ (see Remark(5.5) for the details), $\varepsilon(\pi'\hat{L}(t)) = \exp(\pi L_{NIG})$

and $\hat{f}(\sigma) = E(e^{L_{NIG}})$.

Firstly, let's calculate $E(e^L) \equiv E(e^{L_{NIG}})$.

$$\begin{aligned}
\hat{f}(1) = E(e^L) &= \int_{-\infty}^{\infty} e^l f_{NIG}(l) dl \\
&= \int_{-\infty}^{\infty} \frac{\alpha\delta}{\pi} e^l e^{\delta\sqrt{\alpha^2-\beta^2}+\beta(l-\mu)} \frac{K_1(\alpha\sqrt{\delta^2+(l-\mu)^2})}{\sqrt{\delta^2+(l-\mu)^2}} dl \\
&= e^{\mu} \int_{-\infty}^{\infty} \frac{\alpha\delta}{\pi} e^{\delta\sqrt{\alpha^2-\beta^2}+(\beta+1)(l-\mu)} \frac{K_1(\alpha\sqrt{\delta^2+(l-\mu)^2})}{\sqrt{\delta^2+(l-\mu)^2}} dl \\
&= e^{\mu+\delta(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+1)^2})} \int_{-\infty}^{\infty} f_{NIG}(l : \alpha, \beta+1, \delta, \mu) dl \\
&= e^{\mu+\delta(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+1)^2})}
\end{aligned}$$

Therefore

$$\ln \hat{f}(1) = \mu + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta+1)^2})$$

Furthermore, we have

$$W^{\pi}(T) = w \exp(T(r + \pi(b - r)))e^{\pi L} \tag{5.12}$$

$$E(W^{\pi}(T)) = w \exp(T(r + \pi(b - r) + \mu + \delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta+1)^2}))) \tag{5.13}$$

Now let's calculate the $\hat{\alpha}$-quantile, $W_{\hat{\alpha}}$, of the wealth process $W^{\pi}(T)$.

$$\begin{aligned}
\hat{\alpha} &= P(W^{\pi}(T) \leq W_{\hat{\alpha}}) \\
&= P(w \exp(T(r + \pi(b - r)))e^{\pi L} \leq W_{\hat{\alpha}}) \\
&= P(\pi L \leq \ln \tfrac{W_{\hat{\alpha}}}{w} - T(r + \pi(b - r))) \\
&= P(L \leq \tfrac{1}{\pi}(\ln W_{\hat{\alpha}}/w - T(r + \pi(b - r))))
\end{aligned} \tag{5.14}$$

By the definition of the cdf of NIG, we have $\frac{1}{\pi}(\ln W_{\hat{\alpha}}/w - T(r + \pi(b - r))) = F_{NIG}^{-1}(\hat{\alpha},)$, and hence $W_{\hat{\alpha}} = w \exp(T(r + \pi(b - r)) + \pi F_{NIG}^{-1}(\hat{\alpha}))$.

With this derivation, let's calculate the conditional expectation, that is, the

$TVaR_{\hat{\alpha}}$. To make the notation easy, denote $a := w\exp(T(r+\pi(b-r)))$:

$$\begin{aligned}
E(W^\pi(T)|W^\pi(T) \leq W_{\hat{\alpha}}) &= E(W^\pi(T)|L \leq l_{\hat{\alpha}}) \\
&= \tfrac{1}{\hat{\alpha}} \int_{-\infty}^{l_{\hat{\alpha}}} W^\pi(T) f_{NIG}(l) dl \\
&= \tfrac{a}{\hat{\alpha}} \int_{-\infty}^{l_{\hat{\alpha}}} \tfrac{\alpha\delta}{\pi} e^{\pi l} e^{\delta\sqrt{\alpha^2-\beta^2}+\beta(l-\mu)} \tfrac{K_1(\alpha\sqrt{\delta^2+(l-\mu)^2})}{\sqrt{\delta^2+(l-\mu)^2}} dl \\
&= \tfrac{a}{\hat{\alpha}} e^{\pi\mu+\delta(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+\pi)^2})} \\
&\quad \int_{-\infty}^{l_{\hat{\alpha}}} f_{NIG}(l;\alpha,\beta+\pi,\delta,\mu) dl \\
&= \tfrac{a}{\hat{\alpha}} e^{\pi\mu+\delta(\sqrt{\alpha^2-\beta^2}-\sqrt{\alpha^2-(\beta+\pi)^2})} F_{NIG}(l_{\hat{\alpha}}, \beta+\pi)
\end{aligned} \tag{5.15}$$

Then, with the equations 5.13 and 5.15, we can obtain the formula for EaR by its definition 3.5.

$$EaR_{\hat{\alpha}} = a e^{\delta\sqrt{\alpha^2-\beta^2}} \left(e^{\mu-\delta\sqrt{\alpha^2-(\beta+1)^2}} - \frac{F_{NIG}(l_{\hat{\alpha}}, \beta+\pi)}{\hat{\alpha}} e^{\pi\mu-\delta\sqrt{\alpha^2-(\beta+\pi)^2}}\right) \tag{5.16}$$

After getting the explicit risk formula, we substitute it into the optimization problem (4.18) and by the previous method or other optimization methods, we get the optimal portfolio. Since solving this optimization problem is out of our topic of this thesis, we simply stop here.

5.2 Semi-analytical solutions to the optimization problem

The calculation of the CaR and EaR involves the α quantile of the exponential of the NIG process, which is quite a complicated object. To calculate its distribution explicitly is certainly not possible. One can possibly approximate its density using the inverse fast fourier transform method. However, Asmussen [5] proposed an alternative approximation method based on a weak limit theorem. In this section, we are going to approximate some Lévy processes by some well defined process, for example, Gaussian, which makes the calculation of the α quantile much easier. Here, we call the solution semi-analytical , since the calculation is based on the approximated process.

Let $X = \{X(t) : t \geq 0\}$ be a Lévy process with characteristic function of the form

$$E(e^{iuX(t)}) = \exp\{t(i\alpha u - \beta^2 u^2/2) + \int_{-\infty}^{\infty} (e^{iux} - 1 - iux\mathbb{1}_{|x|\leq 1})\nu(dx)\} \quad (5.17)$$

where $\alpha \in \mathbb{R}$, $\beta^2 \geq 0$ and ν is a Lévy measure. With standard terminology, we refer to the case $\int_{|x|\leq 1} |x|\nu(dx) < \infty$ as the finite variation case, and $\int_{|x|\leq 1} |x|\nu(dx) = \infty$ as the compensated case. The term corresponding to $x\mathbb{1}(|x| \leq 1)$ in (5.17) represents a centering which is necessary for convergence in the compensated case and may be deleted in the bounded variation case (say X is a subordinator). $X(t)$ is the independent sum of a drift term αt, a Brownian component $\beta W(t)$, and a compensated pure jump part with Lévy measure ν, having the interpretation that a jump of size x occurs at rate $\nu(dx)$ (See [60]).

For simulation, the generation of the Brownian part is a standard topic. For the jump part, the most straightforward case is the compound Poisson case $||\nu|| < \infty$ where jumps have distribution $\nu/||\nu||$ and can be simulated at the epochs of a Poisson process with rate $||\nu||$. When $||\nu|| = \infty$, one could attempt to generate a discrete skeleton. This is straightforward if the marginal distribution of $X(t)$ is easily simulated for any t as for the stable case, the Gamma case, or the inverse Gaussian case. However, most often marginal distributions are not easily simulated and in practice, one often as an approximation simulates a Lévy process obtained by neglecting jumps with absolute size smaller than ϵ. In the finite variation case, this may be implemented by simply removing such jumps, leading to

$$X_0^{\epsilon}(t) := X(t) - \Sigma_{s\leq t}\Delta X(s)\mathbb{1}_{(|\Delta(X(s))|<\epsilon)} \quad (5.18)$$

In the compensated case, the idea leads instead to

$$X_1^\epsilon(t) := \mu_\epsilon t + bW(t) + N^\epsilon(t) \tag{5.19}$$

where

$$\mu_\epsilon := a - \int_{\epsilon \le |x| \le 1} x\nu(dx)$$

and

$$N^\epsilon(t) := \Sigma_{s \le t} \Delta X(s) 1(|\Delta(X(s))| \ge \epsilon)$$

is a compound Poisson process with jump measure $\nu_{|x| \ge \epsilon}$ and independent of the standard Brownian motion W_t. In the finite variation case, 5.19 means that X_1^ϵ is obtained from X by replacing small jumps by their expected value rather than just only removing them. That is ,

$$X_1^\epsilon(t) := X_0^\epsilon(t) + E(X(t) - X_0^\epsilon(t))$$

A further improvement is to incorporate also the contribution from the variation of small jumps, which leads to

$$X_2^\epsilon(t) := \mu_\epsilon t + (b^2 + \sigma^2(\epsilon))^{1/2} W(t) + N^\epsilon(t) \tag{5.20}$$

where

$$\sigma^2(\epsilon) := \int_{|x| < \epsilon} x^2 \nu(dx)$$

Notice a Brownian term appears in $X_2^\epsilon(t)$ even when the original process X does not have one. This implicitly assumes that the error

$$X_\epsilon(t) := X(t) - X_1^\epsilon(t)$$

is approximately normal, as has been suggested on intuitive grounds in some particular cases [56].

$X_\epsilon(t)$ is a Lévy process with characteristic function

$$E(e^{iuX_\epsilon(t)}) = \exp(t\int_{|x|<\epsilon}(e^{iux}-1-iux)\nu(dx))$$

Consequently, $E(X_\epsilon) = 0$ and $var(X_\epsilon(1)) = \sigma^2(\epsilon)$. $\sigma(\epsilon)^{-1}X_\epsilon$ is convergent in distribution to a standard Brownian motion W_t equipped with the uniform metric. For the specific definition of **convergence in distribution**, denoted by $\xrightarrow{d}$, please see Remark(5.2.). One of the properties of this convergence is that *for every function* $f : D[0,1] \longrightarrow \mathbb{R}$*, that is continuous with respect to the uniform metric, bounded, and measurable with respect to the projection* σ*-field, one has* $E(f(\sigma(\epsilon)^{-1}X_\epsilon)) \xrightarrow{d} E(f(w))$ *in distribution, as* $\epsilon \longrightarrow \infty$.

Theorem 5.1. $\sigma(\epsilon)^{-1}X_\epsilon \xrightarrow{d} W$ as $\epsilon \longrightarrow 0$ if and only if for each $\kappa > 0$

$$\sigma(\kappa\sigma(\epsilon)\wedge\epsilon) \sim \sigma(\epsilon)$$

as $\epsilon \longrightarrow 0$

The next theorem gives a more intuitive condition for the validity of the normal approximation.

Theorem 5.2. $\sigma(\kappa\sigma(\epsilon)\wedge\epsilon) \sim \sigma(\epsilon)$ is implied by

$$\lim_{\epsilon\to 0}\frac{\epsilon}{\sigma(\epsilon)} = 0 \tag{5.21}$$

This means that the normal approximation holds when the dispersion of the small jump part of a Lévy process converges slower to zero than the level of truncation. Equivalently, the range $\frac{\epsilon}{\sigma(\epsilon)}$ of jumps of $\sigma(\epsilon)^{-1}X_\epsilon$ approaches 0.

Furthermore, to find the approximation for $\ln\varepsilon(\pi\hat{L})$, we need the main results in [20], which are stated below without proofs.

Theorem 5.3. LetZ^ϵ, $\epsilon > 0$, be real valued Lévy processes without Gaussian component and $Y^\epsilon = \ln \varepsilon(Z^\epsilon)$. be the logarithmic stochastic exponentials. Let $g : \mathbb{R} \longrightarrow \mathbb{R}^+$ with $g(\epsilon) \longrightarrow 0$ as $\epsilon \longrightarrow 0$. Let V be a Lévy process, then for $t \geq 0$, as $\epsilon \longrightarrow 0$, the following are equivalent

$$\frac{Z^\epsilon(t)}{g(\epsilon)} \xrightarrow{d} V(t)$$

$$\frac{Y^\epsilon(t)}{g(\epsilon)} \xrightarrow{d} V(t)$$

Theorem 5.4. Let L be a Lévy process and L_ϵ as given in 5.19. Let $\hat{L}$ be such that $\varepsilon \hat{L} = e^L$ and let L_ϵ and $\hat{L}_\epsilon$ be the same relationship as L and $\hat{L}$. Then

$$\sigma(\epsilon)^{-1}(L(t) - L_\epsilon(t)) \xrightarrow{d} V(t)$$

is equivalent to

$$(\pi\sigma(\epsilon))^{-1}(\ln \varepsilon(\pi\hat{L}(t)) - \ln \varepsilon(\pi\hat{L}_\epsilon(t))) \xrightarrow{d} V(t)$$

Theorem 5.5. With the same notation, we have the following result

$$\begin{aligned} \ln \varepsilon(\pi\hat{L}) &\approx \ln \varepsilon(\pi\hat{L}_\epsilon + \pi\sigma(\epsilon)V(t) \\ &= \gamma_\pi^\epsilon t + \pi\beta W(t) + M_\pi^\epsilon(t) + \pi\sigma(\epsilon)V(t) \end{aligned}$$

where

$$\gamma_\pi^\epsilon = \pi(\mu(\epsilon) + 1/2\beta^2(1 - \pi))$$

and

$$M_\pi^\epsilon(t) = \Sigma_{s\leq t} \ln(1 + \pi(e^{\Delta L(s) 1_{|\Delta L(s)|>\epsilon}} - 1))$$

if $V(t)$ is a Brownian motion, then

$$\ln \varepsilon(\pi \hat{L}) \approx \gamma_\pi^\epsilon t + \pi(\beta^2 + \sigma^2(\epsilon))^{1/2} W(t) + M_\pi^\epsilon(t) \tag{5.22}$$

Since throughout this thesis we are mainly interested in the Compound Poisson process, α-stable process and Normal inverse Gaussian process, we just present the analysis results of [5] related to these processes.

Remark 5.1 For compound Poisson process, the condition 5.21 doesn't hold, since $\nu(\mathbb{R}) < \infty$, $\sigma(\epsilon) = o(\epsilon)$

Remark 5.2 In the case of a Stable process of index $\alpha \in (0, 2)$, $\nu(dx) = a|x|^{-1-\alpha} 1(x < 0)dx + bx^{-1-\alpha} 1(x > 0)dx$, $a, b \geq 0, a + b > 0$ and $\sigma(\epsilon) = (\frac{a+b}{2-\alpha})^{1/2} \epsilon^{1-\alpha/2}$. Therefore the normal approximation is valid.

Remark 5.3 The normal inverse Gaussian Lévy process is a Lévy process with $X(1)$ distributed according to the normal inverse Gaussian distribution (see [8]). The normal approximation is valid since $\sigma(\epsilon) \sim (2\delta/\pi)^{1/2} \epsilon^{1/2}$ as $\epsilon \to 0$.

By the result in Resnick [54], the approximation of the α-quantile z_α of $\varepsilon(\pi \hat{L})$ can be obtained:

$$\begin{aligned} z_\alpha &\approx z_\alpha^\epsilon(\pi) \\ &= \inf(z \in \mathbb{R} : P(\gamma_\pi^\epsilon T + \pi(\beta^2 + \sigma^2(\epsilon))^{1/2} W(T) + M_\pi^\epsilon(T) \leq \ln z) \geq \alpha) \end{aligned}$$

From this we calculate VaR as in [20]:

$$VaR(w, \pi, T) = w z_\alpha^\epsilon(\pi) \exp((r + \pi(b - r))T)$$

and hence

$$CaR(w,\pi,T) = w\exp(rT)(1 - z_\alpha^\epsilon(\pi)\exp(\pi(b-r)T)). \tag{5.23}$$

Let $f_g(x)$ be the general pdf of the random variable $X = \ln Z$ which follows the process given by $\gamma_\pi^\epsilon T + \pi(\beta^2+\sigma^2(\epsilon))^{1/2}W(T) + M_\pi^\epsilon(T)$, then $W^\pi(T) = aZ(T)$ where $a = w\exp(T(r+\pi(b-r)))$. Then we can get EaR which is given by

$$\begin{aligned} EaR(\pi,T) &= \int_{-\infty}^{\infty} ae^x f(x)dx - \frac{1}{\alpha}\int_{-\infty}^{\ln(az_\alpha)} ae^x f(x)dx \\ &= a[\int_{\ln(az_\alpha)}^{\infty} e^x f(x)dx - \int_{-\infty}^{\ln(az_\alpha)} (\tfrac{1-\alpha}{\alpha})e^x f(x)dx] \end{aligned} \tag{5.24}$$

5.3 Monte Carlo simulation of the risk measures

Since there is no difference between investment in our multi-stock market and a market consisting of the bond and just one stock with appropriate market coefficients b and σ, one simplification of the simulation work is to assume that we are in the one stock and one bond market. Therefore in the following result, π is a real number. Since, for the optimization problem we discuss here deals mainly with the risk, we give simulation results for three risks, i.e. variance (or more precisely, the standard deviation), CaR and EaR when they follow Gaussian and Compound Poisson processes respectively.

5.3.1 Computing risks that follow a Gaussian process

Since the expected return of the wealth process is an increasing function of π, (see 4.3), $E(W^\pi(T)) \geq C$ is equivalent to $\pi \geq C'$. Assume the confidence level is $1-\alpha$ and the variance of the Brownian motion is σ, then we have figures 5.1 to figure 5.6:

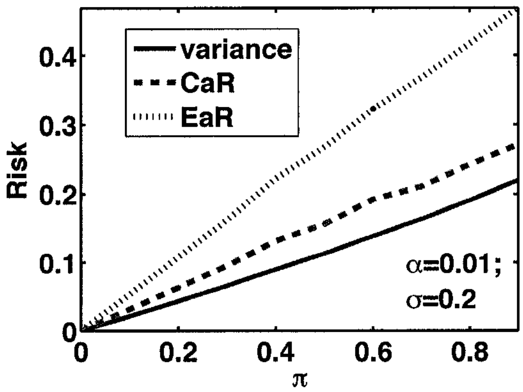

Figure 5.1

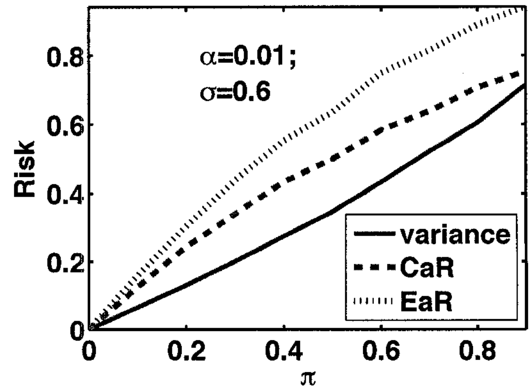

Figure 5.2

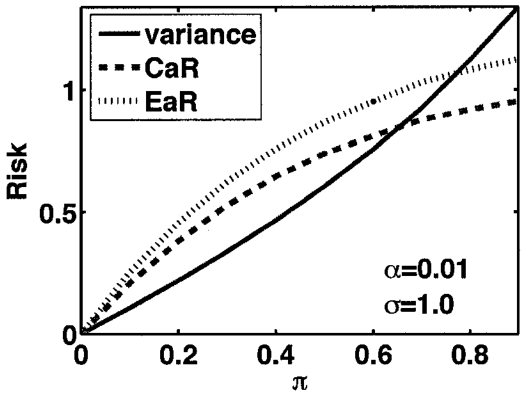

Figure 5.3

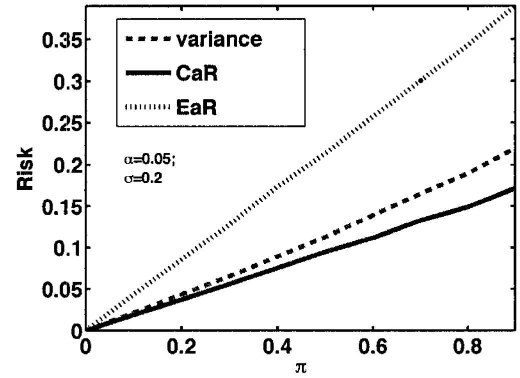

Figure 5.4

Figure 5.1: For any fixed π , standard deviation is smaller than CaR and EaR.

Figure 5.2: When the variance is bigger, with the growth of π, standard deviation grows faster than CaR and EaR.

Figure 5.3: When the variance is big enough, standard deviation becomes bigger than CaR and EaR from some big value of π, and also CaR and EaR are getting closer.

Figure 5.4: When the value of α grows, CaR decreases quicker than EaR and standard deviation for any fixed π . This is because CaR is decreasing function of α quantile . While standard deviation and EaR are not as sensitive as CaR.

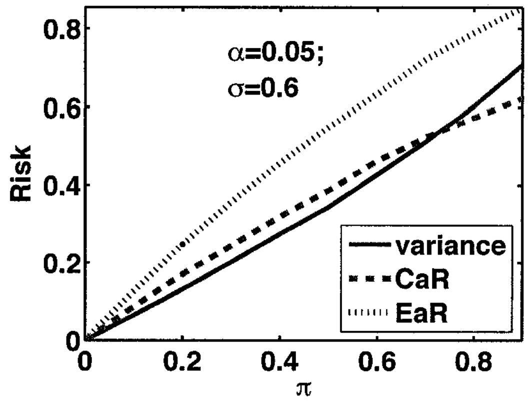

Figure 5.5

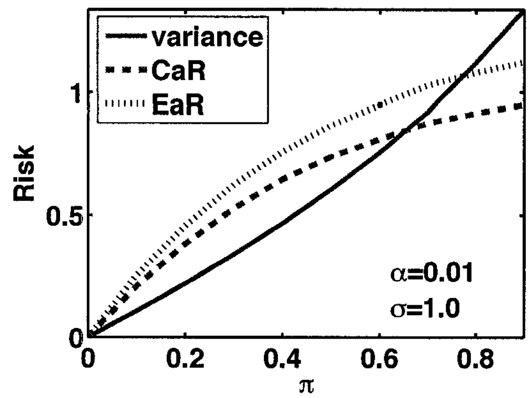

Figure 5.6

Figure 5.5: When the variance increases, the standard deviation grows faster than CaR and EaR and the reason is quite trivial.

Figure 5.6: Comparing with figure 5.3, the change of the value of α does not change the value of risk too much.

5.3.2 Computing risks that follow a Compound Poisson Process

Now we consider the process of Compound Poisson by given figure 5.7 to figure 5.12:

When a Compound Poisson process is assumed, from the above six figures we can see that, for fixed α, increasing σ will increase the values of all these three risks. But obviously, CaR increases more quickly than EaR and standard deviation. And the bigger σ is, the higher degree of the curve for each risk measure. For example, when σ is small, the standard deviation looks like a straight line. While, when σ is relative big, the standard deviation looks more curly. When σ is fixed, increasing α does not change the figures too much, except that the curves turns a bit smooth.

5.3.3 Results comparison between two processes

Now let's change the point of view of comparison. For fixed $\alpha = 0.05$ and $\sigma = 0.6$, we have the following results, see captions of figure 5.13 to figure 5.15:

If the Compound Poisson process is assumed, CaR and EaR show us close

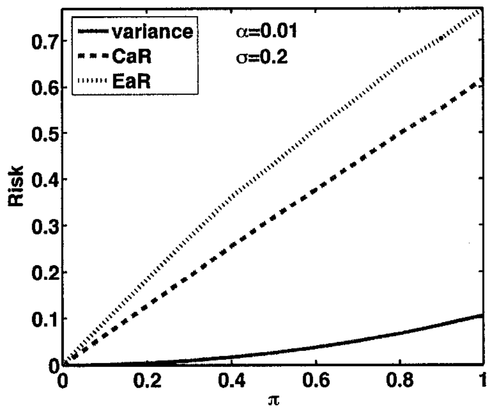

Figure 5.7

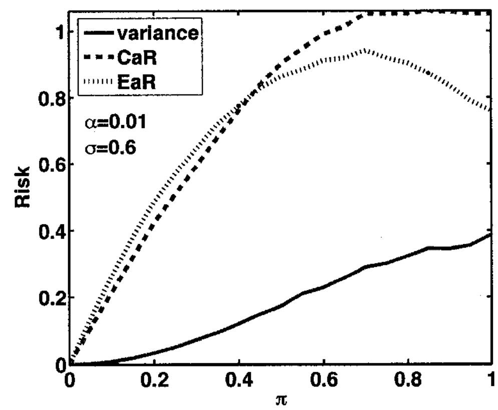

Figure 5.8

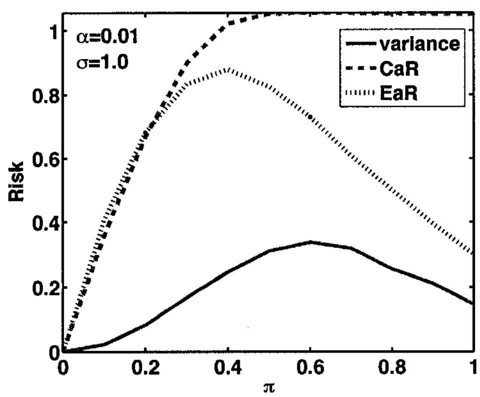

Figure 5.9

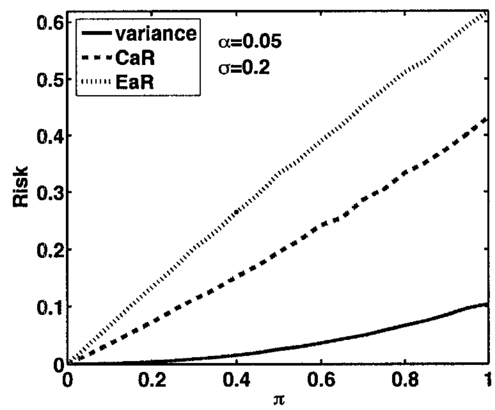

Figure 5.10

Figure 5.7: $\alpha = 0.01$ and $\sigma = 0.2$

Figure 5.8: $\alpha = 0.01$ and $\sigma = 0.6$

Figure 5.9: $\alpha = 0.01$ and $\sigma = 1.0$

Figure 5.10: $\alpha = 0.05$ and $\sigma = 0.2$

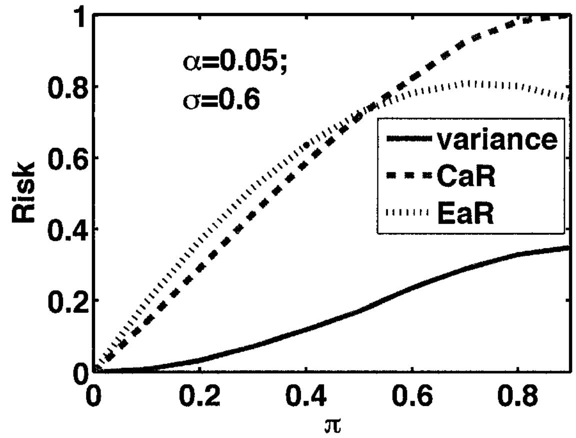

Figure 5.11

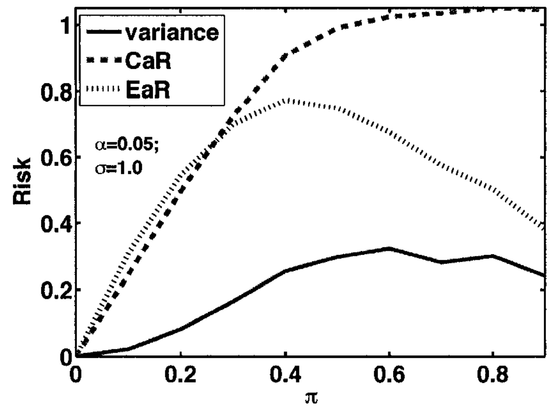

Figure 5.12

Figure 5.11: $\alpha = 0.05$ and $\sigma = 0.6$

Figure 5.12: $\alpha = 0.05$ and $\sigma = 1.0$

results. When σ is small CaR behaves better than EaR, while when σ is getting bigger, EaR goes gradually below CaR as π grows; at the same time, when α grows bigger, the three risks as a whole are smaller, see figure 5.7 to figure 5.12. If the Gaussian normal process is assumed, the increase of the value of σ changes the increment of EaR and CaR, but it does not change the camber. While, the curve of variance becomes more and more steep for which the reason is quite obvious. When α increases, the same performance is shown in the figures. From the above discussion we can see that there is no absolute one criteria to tell which risk measure is good and how good. As a matter of fact, it really depends on the individual investor.

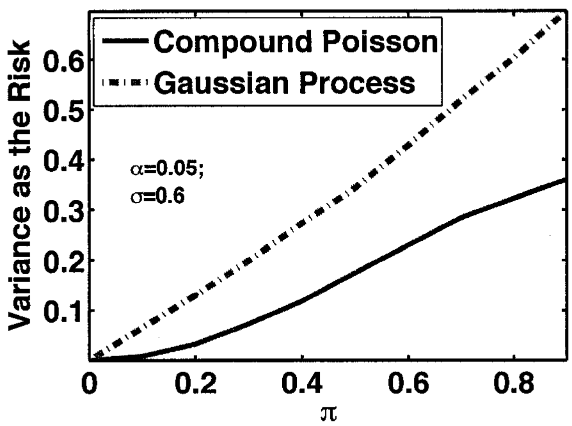

Figure 5.13

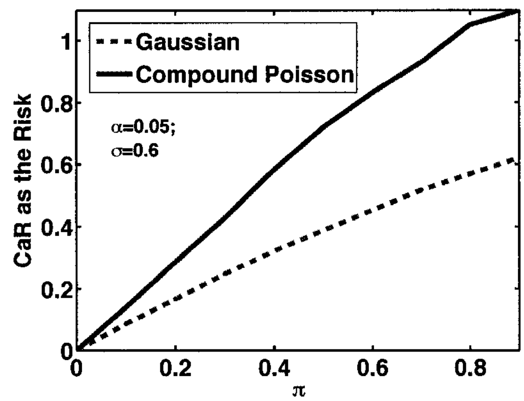

Figure 5.14

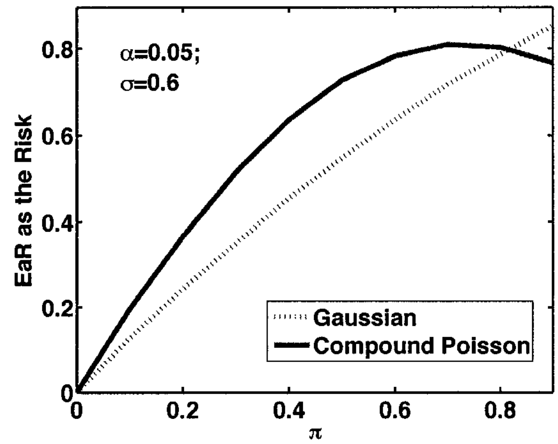

Figure 5.15

Figure 5.13: In the case variance as the risk measure, assuming that the risk asset follows Compound Poisson will give us a smaller risk for any value of π

Figure 5.14: When CaR is concerned , Gaussian process gives us better results compound poisson process

Figure 5.15: For risk measure EaR, Gaussian process is a better model for relative small value of π. But when π is very big Compound Poisson turns better.

Bibliography

[1] Acerbi, C., Tasche, D.(2002): On the Coherence of Expected Shortfall. Journal of Bandking and Finance 26, 1487-1503.

[2] Albanese, C., Campolieti, G.(2006):Advanced Derivatives Pricing ans Risk Management. Elsevier Academic Press.

[3] Alexander, J.G., Baptista, A.M.(2002): Economic Implications of Using a Mean-VaR Model for Portfolio Selection: A Comparison with Mean-Variance Analysis. Journal of Economic Dynamics and Control, 1159-1193.

[4] Artzner, P., Delbaen, F., Eber,J. and Heath, D.(1998): Coherence Measures of Risk, Mathematical Finance 9, 203-228.

[5] Asmussen, S. Rosinski, J.(2000): Approximation of small jumps of Lévy processes with a view towards simulation. Journal of Applied Probability 38, 482-493.

[6] Barndorff-Nielsen, O.E. and Halgreen,O.(1977): Infinite Divisibility of the Hyperbolic and Generalized Inverse Gaussian Distributions. Z. wahrscheinlinchkeitstheorie Verw.Geb. 38, 309-312.

[7] Barndorff-Nielsen, O.E.(1977): Exponentially decreasing distributions for the logarithm of particle size, Proceedings of the Royal Society London A 353, 401-419.

[8] Barndorff-Nielsen, O.E.(1977): Normal Inverse Gaussian Distributions and Stochastic Volatility Modelling, Scand. J. Statist, 24, 1-13.

[9] Barndorff-Nielsen, O.E.(1998): Processes of Normal Inverse Gaussian Type. Finance and Stochastics 2, 41-68.

[10] Bell, D.E.(1988): One-switch Utility Functions and a Measure of Risk. Management Science 34, 1416-1424.

[11] Bell, D.E.(1995): Risk, Return and Utility. Management Science 41, 23-30.

[12] Camerer, C.F.(1992): Recent Tests of Gereralized Utility Theories. In Utility Theories: Measurement and applications, ed. W.Edwards, Cambridge University Press.

[13] Campbell, R. Huisman, R., and Koedijk, K.(2001): Optimal Portfolio Selection in a Value-at-Risk Framework. Journal of Banking and Finance 25, 1789-1804.

[14] Cheridito, P., Delbaen, F. Kluppelberg, C.(2004): Coherent and Convex Monetary Risk Measures for Bounded Cadlag Process. Stochastic Processes and their Applications 112, 1-22.

[15] Cont, Rama and Tankov, Peter."Financial Modelling With Jump processes", Chapman and Hall/CRC, 2004.

[16] Duffie, D., Pan, J.(1997): An Overview of Value at Risk. Journal of derivatives 4, 7-49.

[17] Dybvig, P.H.(1984): Short Salies Restrictions and Kinks on the Mean Variance Frontier. The Journal of Finacne 39, 239-244.

[18] Eberlein, E. and Keller, U.(1995): Hyperbolic Distributions in Finance, Bernoulli 1, 281-299.

[19] Emmer,S. , Klüppelberg, C. and Korn, r.(2001): Optimal Portfolios with Bounded Capital-at-Risk. Math. Finance 11, 365-384.

[20] Emmer, S. Klüppelberg, C.(2004): Optimal portfolios when stock prices follow an exponential Lévy process. Finance Stochast. 8, 17-44.

[21] Fama, E.F.(1965):The Behavior of Stock Market Prices, Journal of Business 38, 34-105.

[22] Fishburn, P.C.(1970): Utility Theory for Decision Making. Robert E. Krieger Publishing Co.

[23] Föllmer, H., Schied, A.(2002): Convex Measures of Risk and Trading Constraints. Finance and Stochastics 6, 429-447.

[24] Frey, R., and McNeil, A.(2002):VaR and Expected Shortfall in Portfolios of Dependent Credit Risks: Conceptual and Practical Insights. Journal of Banking and Finance, vol 26, p1317-1334.

[25] Goll,T.,Kallsen,J.(2000):Optimal portfolios for logarithmic utility. Stoch. Proc. Appl. 89, 31-48.

[26] Hakansson, N.H.(1971): Multi-period Mean-variance Analysis: Toward a General Theory of Portfolio Choice. Journal of Finance 26, 857-884.

[27] Imen, B.(2006): Tail Conditional Expectation for Vector-valued Risks. SFB 649 Discussion Papers from Humboldt University, Collaborative Research Center 649.

[28] Jan, D., Vanduffel, S., Tang, Q., Goovaerts, M.J., Kass, R. and Vyncke, D.(2004): Capital requirements, risk measures and comonotonicity. Belgian Actuarial Bulletin 4, 53-61.

[29] Jaschke, S., Küchler, U.(2001): Coherent Risk Measures and Good-Deal Bounds. Finance and Stochastics 5, 181-200.

[30] Jia, J., Dyer, J.S.(1996): A Standard Measure of Risk and Risk-value Models. Management Science 42, 1961-1750.

[31] Jorion, P. (2000): Value at Risk: the New Benchmark for Managing Financial Risk. McGraw-Hill.

[32] Kaplanski, G.(2004): Analytical Portfolio Value-at-Risk. Journal of Risk 7(2).

[33] Kon, S.J. (1984): Models of Stock Returns - a comparison. Journal of Finance 39, 147-165.

[34] Korn, R.(1997): Optimal Portfolios: Stochastic Models for Optimal Investment and Risk Management in Continuous Time. Singapore: World Scientific.

[35] Korn, E., and Korn, R.(2000): Option Pricing and Portfolio Optimization-Modern Methods of Financial Mathematics. Providence, RI: American Mathematics Society.

[36] Kroll, Y., Levy, H. and Markowitz, H.(1984): Mean-variance versus direct utility maximization. Journal of Finance 39, 47-61.

[37] Li, Z.F., Yang, H. and Deng, X. T. (2007): Optimal Dynamic Portfolio Selection with Earning-at Risk. Journal of Optimization Theory and Applications 132, 459-473.

[38] Li, Z.F., Ng, K.W., Tan, K.S., Yang, H.(2006): Optimal Constant Rebalanced Portfolio investment Strategies for Dynamic Portfolio selection. International Journal of Theoretical and Applied Finance 9, No. 6, 951-966.

[39] Li, Z.F., Yang, H., Deng, X.T.(2003): An Exact Solution to a Continuous-Time Portfolio Selection Problem with EaR, Lecture Notes in Decision Sciences, vol 2, Financial Systems Engineering, 2003.

[40] Mandelbrot, B.B.(1963): The variation of certain speculative prices, Journal of Busimess 36, 392-417.

[41] Markowitz, H.M.(1952): Portfolio Selection. The Journal of Finance 7, 77-91.

[42] Markowitz, H.M.(1959): Portfolio selection, Efficient Diversification of Investments. New York, John wiley and sons, Inc.

[43] Markowitz, H.M.(1989): Mean-Variance Analysis in Portfolio Choice and Capital Markets, Cambridge, MA: Basil Blackwell.

[44] Markowitz, H.M.(1991): Foundations of Portfolio Theory. Journal of Finance, XLVI, 469-477.

[45] Merton, R.C.(1969): Lifetime Portfolio Seletion under Uncettainty: The continuous-time Model. Review of Economics ans Statistics 51, 247-256.

[46] Merton, R.C.(1971):Optimum Consumption ans Portfolio Rules in a Continuous-time Model. Journal of Economic Theory 3, 373-413.

[47] Merton, R.C., Samuelson, P.A.(1974): Fallacy of the Log-normal Approximation of Optimal Portfolio Decision-making Over Many Periods. Journal of Fiancial Economics 1, 67-94.

[48] Merton, R.(1976): Option Pricing when Underlying Stock Returns are Discontinuous. Journal of Financial Economics 3, 125-144.

[49] Merton, R.(1990): Continuous-Time Finance, Cambridge, MA: Basil Blackwell.

[50] Mossin, J.(1968): Optimal Multiperiod Portfolio Policies. Journal of Business 41, 215-229.

[51] Neumann, J.V., Morgenstern, O.(1944): Theory of Games and Economic Behavior. Princeton University Press.

[52] Protter, P.(1990): Stochastic Integration and differenctial Equations: A New Approach, Vol.21 of Applications of Matjematics, springer-Verlag, Berlin.

[53] Rachev, S., Mittnik, S.(2002): Stable Paretian Models in Finance. John Wiley and Sons, New York, 2002.

[54] Resnick, S.(1987): Extreme values, regular variation, and point processes. Berlin Heidelberg New York: Springer.

[55] Stulz, R.(1996): Rethinking Risk Management. Journal of Applied Corporate Fianace 9, 8-24.

[56] Rydberg,T.(1997): The Normal Inverse Gaussian Lêvy Process: Simulation and Approximation. Stochastic Models 13, 887-910.

[57] Samorodnitsky,G., Taqqu, M.S.(1994): Stable Non-Gaussian Random Processes, Chapman and Hall.

[58] Samuelson, P.A.(1969): Lifetime Portfolio Selection by Dynamic Stochastic Programming. The Review of Economics and statistics 51, 239-246.

[59] Sarin, R.K., Weber, M.(1993): Risk-value Moedls. European Journal of Operational Research 70, 135-149.

[60] Sato, K-I.: Lêvy Process and Infinitely Divisible Distributions. Cambridge: Cambridge University Press 1999.

[61] Sentana, E.(2001): Mean Variance Portfolio allocation with a Value at Risk Constraint. C.E.P.R., Discussion Papers, 2997.

[62] Sharpe, W.F.(1964): Capital Asset Prices: A Theory of Market Equilibrium under Conditions of Risk. Journal of Finance 19, 425-442.

[63] Sharpe, W.F.(1970): Portfolio Theory and Capital Markets, McGraw-Hill, New York.

[64] Smith, K.V.(1967): A Transition Model for Portfolio Revision. Journal of Finance 22, 425-439.

[65] Smithson, C.W.(1998): Managing Financial risk: A Guide to Derivative Products, financial Engineering, and Value Maximization, McGraw-Hill, New York.

[66] Stone, B.K.(1970): Risk, Return and Equilibrium, MIT Press, Cambridge, Mass.

[67] Tobin, J.(1958): Liquidity Preference as Behavior Towards Risk. Review of Economic Studies 67, 65-86.

[68] Weron, R.(1996): On the Chambers-Mallows-Stuck Method for Simulating Skewed Stable Random Variables, Statistics and Probability Letters 28, 165-171.

[69] Weron, R.(2004): Computationally intensive Value at Risk Calculations. In: Handbook of Computational Statistics, J.E. Gentle, W. Härdle, Y. Mori, Springer, Berlin, 911-950.

[70] Yiu, K.F.C.(2004): Optimal Portfolios under a Value-at-Risk Constraint. Journal of Economic Dynamics and Control 28(7), 1317-1334.