New Orthogonal Designs from Weighing Matrices

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New orthogonal designs from weighing matrices

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Abstract

In this paper we show the existence of new orthogonal designs, based on a number of new weighing matrices of order $2n$ and weights $2n-5$ and $2n-9$ constructed from two circulants. These new weighing matrices were constructed recently by establishing various patterns on the locations of the zeros in a potential solution, in conjunction with the power spectral density criterion. We also demonstrate that some of these new orthogonal designs, the ones that are full, can be used to construct inequivalent Hadamard matrices.

1 Introduction

A weighing matrix $W = W(n, k)$ is a square matrix with entries $0, \pm 1$ having k non-zero entries per row and column and inner product of distinct rows equal to

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zero. Therefore W satisfies $WW^t = kI_n$. The number k is called the weight of W. Weighing matrices have been studied extensively, see [1], [6], [2] and references therein, for detailed information on known and unknown weighing matrices.

A well-known necessary condition for the existence of $W(2n, k)$ matrices states that if there exists a $W(2n, k)$ matrix with n odd, then $k < 2n$ and k is the sum of two squares. In this paper we are focusing on $W(2n, k)$ constructed from two circulants. The two circulants construction for weighing matrices is described in the theorem below, taken from [3].

Theorem 1 *If there exist two circulant matrices A_1, A_2 of order n, with $0, \pm 1$ elements, satisfying $A_1A_1^t + A_2A_2^t = fI_n$ and f is an integer, then there exists a $W(2n, f)$, given as*

$$W(2n, f) = \begin{pmatrix} A_1 & A_2 \\ -A_2^t & A_1^t \end{pmatrix} \quad or \quad W(2n, f) = \begin{pmatrix} A_1 & A_2R \\ -A_2R & A_1 \end{pmatrix}$$

where R is the square matrix of order n with $r_{ij} = 1$ if $i + j - 1 = n$ and 0 otherwise.

Recently, a number of new weighing matrices of order $2n$ and weights $2n - 5$, $2n - 9$ was constructed as per theorem 1 in [4, 5], by establishing patterns for the potential locations of zeros in the first rows of matrices A_1 and A_2. The existence of these weighing matrices has certain implications on the existence of orthogonal designs and this is what we want to exhibit in this paper. It is also important to point out that the construction patterns used to find new weighing matrices in [4, 5], are of crucial important for the orthogonal design constructions of this paper.

The orthogonal design constructions of this paper are based on formulations of a theorem of Goethals and Seidel which may be used in the following form:

Theorem 2 [3, Theorem 4.49] *If there exist four circulant matrices A_1, A_2, A_3, A_4 of order n satisfying*

$$\sum_{i=1}^{4} A_iA_i^T = fI$$

where f is the quadratic form $\sum_{j=1}^{u} s_jx_j^2$, then there is an orthogonal design $OD(4n; s_1, s_2, \ldots, s_u)$.

2 New Orthogonal designs from $W(2n, 2n - 5)$

The first lemma is stated for completeness only.

Lemma 1 *If there exists an $OD(2n; a, b)$ constructed using two circulant matrices then, using the two circulant $W(2n, 2n - 5)$ constructed here with the Goethals-Seidel array we have $OD(4n; a, b, 2n - 5)$.*

The next theorem relies heavily on the construction pattern used to find the $W(2n, 2n-5)$.

Theorem 3 *Suppose there exists a $W(2n, 2n-5)$ constructed using two circulants with first rows of the required pattern. Then there exist*

1. *$OD(4n; 4, 4, 4n-10)$ and*
2. *$OD(4n; 2, 8, 4n-10)$ constructed using four circulant matrices in the Goethals-Seidel array.*

Proof We multiply each of the elements of the two $\{0, \pm 1\}$ sequences by the commuting variable a. We rearrange the required sequences of the $W(2n, 2n-5)$, if necessary, to the form

$$\begin{array}{rccccccccc} A = & 0 & 0 & a_3 & \cdots & a_{\frac{n-1}{2}} & 0 & a_{\frac{n+3}{2}} & \cdots & a_n \\ B = & 0 & 0 & b_3 & \cdots & b_{\frac{n-1}{2}} & b_{\frac{n+1}{2}} & b_{\frac{n+3}{2}} & \cdots & b_n \end{array} \qquad (1)$$

where each element a_i and b_j is $\pm a$.

Now let p, q, x and y be commuting variables.

Then

$$\begin{array}{rccccccccc} P = & p & q & a_3 & \cdots & a_{\frac{n-1}{2}} & 0 & a_{\frac{n+3}{2}} & \cdots & a_n \\ Q = & -p & -q & a_3 & \cdots & a_{\frac{n-1}{2}} & 0 & a_{\frac{n+3}{2}} & \cdots & a_n \\ R = & p & -q & b_3 & \cdots & b_{\frac{n-1}{2}} & b_{\frac{n+1}{2}} & b_{\frac{n+3}{2}} & \cdots & b_n \\ S = & -p & q & b_3 & \cdots & b_{\frac{n-1}{2}} & b_{\frac{n+1}{2}} & b_{\frac{n+3}{2}} & \cdots & b_n \end{array}$$

give the required sequences for the $OD(4n; 4, 4, 4n-10)$ and

$$\begin{array}{rccccccccc} X = & y & -y & a_3 & \cdots & a_{\frac{n-1}{2}} & x & a_{\frac{n+3}{2}} & \cdots & a_n \\ Y = & -y & y & a_3 & \cdots & a_{\frac{n-1}{2}} & -x & a_{\frac{n+3}{2}} & \cdots & a_n \\ Z = & y & y & b_3 & \cdots & b_{\frac{n-1}{2}} & b_{\frac{n+1}{2}} & b_{\frac{n+3}{2}} & \cdots & b_n \\ W = & -y & -y & b_3 & \cdots & b_{\frac{n-1}{2}} & b_{\frac{n+1}{2}} & b_{\frac{n+3}{2}} & \cdots & b_n \end{array}$$

give the required sequences for the $OD(4n; 2, 8, 4n-10)$.

Remark. We note that these last orthogonal designs are full and so give rise to Hadamard matrices.

3 New Orthogonal designs from $W(2n, 2n-9)$

The first lemma is stated for completeness only.

Lemma 2 *If there exists an $OD(2n; a, b)$ constructed using two circulant matrices then, using the two circulant $W(2n, 2n-9)$ constructed here with the Goethals-Seidel array we have $OD(4n; a, b, 2n-9)$.*

The next theorem relies heavily on the construction pattern used to find the $W(2n, 2n-9)$.

3.1 The 1-4-4 Pattern

Theorem 4 *Suppose there exists a $W(2n, 2n-9)$ constructed using two circulants with first rows of the required pattern. Then there exist*

1. $OD(4n; 2, 4n-18)$,
2. $OD(4n; 4, 4n-18)$,
3. $OD(4n; 4, 4, 4n-18)$,
4. $OD(4n; 2, 8, 4n-18)$ *and*
5. $OD(4n; 8, 8, 4n-18)$ *constructed using four circulant matrices in the Goethals-Seidel array.*

Proof We multiply each of the elements of the two $\{0, \pm 1\}$ sequences by the commuting variable a. We rearrange the required sequences of the $W(2n, 2n-9)$, if necessary, to the form

$$\begin{array}{rcccccccccccc} A = & 0 & 0 & 0 & 0 & a_5 & \cdots & a_{\frac{n-1}{2}} & 0 & a_{\frac{n+3}{2}} & \cdots & a_n \\ B = & 0 & 0 & 0 & 0 & b_5 & \cdots & b_{\frac{n-1}{2}} & b_{\frac{n+1}{2}} & b_{\frac{n+3}{2}} & \cdots & b_n \end{array} \quad (2)$$

where each element a_i and b_j is $\pm a$.

Now let p, q, r, s, x, y, z and w be commuting variables. Cases 1) to 5) are constructed by forming the four circulant matrices with first rows which are then used in the Goethals-Seidel array

$$\begin{array}{rcccccccccccc} P = & p & q & r & s & a_5 & \cdots & a_{\frac{n-1}{2}} & 0 & a_{\frac{n+3}{2}} & \cdots & a_n \\ Q = & -p & -q & -r & -s & a_5 & \cdots & a_{\frac{n-1}{2}} & 0 & a_{\frac{n+3}{2}} & \cdots & a_n \\ R = & x & y & z & w & b_5 & \cdots & b_{\frac{n-1}{2}} & b_{\frac{n+1}{2}} & b_{\frac{n+3}{2}} & \cdots & b_n \\ S = & -x & -y & -z & -w & b_5 & \cdots & b_{\frac{n-1}{2}} & b_{\frac{n+1}{2}} & b_{\frac{n+3}{2}} & \cdots & b_n \end{array} \quad (3)$$

For the cases 1) to 5) use the values given in the following table or other suitable values:

$OD(4n; t, u, 4n-18)$	t, u	p	q	r	s	x	y	z	w
	2	g	0	0	0	0	0	0	0
	4	g	0	0	0	g	0	0	0
	4, 4	g	h	0	0	g	$-h$	0	0
	2, 8	g	h	$-g$	0	g	0	g	0
	8, 8	g	g	h	$-h$	g	$-g$	h	h

(4)

3.2 The 5-2-2 Pattern

We also get the same theorem using the 5-2-2 pattern. Of course the results will be inequivalent.

Theorem 5 *Suppose there exists a $W(2n, 2n-9)$ constructed using two circulants with first rows of the required pattern. Then there exist*

1. $OD(4n; 2, 4n-18)$,
2. $OD(4n; 4, 4n-18)$,
3. $OD(4n; 4, 4, 4n-18)$,
4. $OD(4n; 2, 8, 4n-18)$ *and*
5. $OD(4n; 8, 8, 4n-18)$ *constructed using four circulant matrices in the Goethals-Seidel array.*

Proof We multiply each of the elements of the two $\{0, \pm 1\}$ sequences by the commuting variable a. We rearrange the required sequences of the $W(2n, 2n-9)$, if necessary, to the form

$$\begin{array}{lcccccccc} A = & 0 & 0 & 0 & 0 & 0 & a_6 & \cdots & a_n \\ B = & 0 & 0 & \star & 0 & 0 & b_6 & \cdots & b_n \end{array} \tag{5}$$

where each element a_i and b_j is $\pm a$. The element $\star$ is $\pm a$.

Now let p, q, r, s, x, y, z and w be commuting variables. Cases 1) to 5) are constructed by forming the four circulant matrices with first rows which are then used in the Goethals-Seidel array

$$\begin{array}{lcccccccc} P = & p & q & 0 & r & s & a_6 & \cdots & a_n \\ Q = & -p & -q & 0 & -r & -s & a_6 & \cdots & a_n \\ R = & x & y & \star & z & w & b_6 & \cdots & b_n \\ S = & -x & -y & \star & -z & -w & b_6 & \cdots & b_n \end{array} \tag{6}$$

For the cases 1) to 5) use the values given in the following table or other suitable values:

$OD(4n; t, u, 4n-18)$	t, u	p	q	r	s	x	y	z	w
	2	g	0	0	0	0	0	0	0
	4	g	0	0	0	g	0	0	0
	4, 4	g	h	0	0	g	$-h$	0	0
	2, 8	g	h	$-g$	0	g	0	g	0
	8, 8	g	g	h	$-h$	g	$-g$	h	h

(7)

4 Inequivalent Hadamard matrices from the full $OD(4n; 2, 8, 4n-10)$

The full orthogonal design $OD(4n; 2, 8, 4n-10)$ constructed in theorem 3 can be used to construct Hadamard matrices of orders $4n$, for every n for which $W(2n, 2n-5)$ weighing matrices are available. We carried out these computations for the values of n that belong to the set $\{15, 17, 21, 23, 25, 27, 29, 33, 35, 39, 43, 45, 47\}$. The corresponding $W(2n, 2n-5)$ weighing matrices constructed from two circulants (many of which are new) have been constructed in [4] and are also available in the web page http://www.cargo.wlu.ca/weighing/

For each value of n in $\{15, 17, 21, 23, 25, 27, 29, 33, 35, 39, 43, 45, 47\}$, we set the variables x, a, y to ± 1 values and for each of the resulting 8 combinations, we obtained a Hadamard matrix of order $4n$. Then we used the 4-profile criterion as implemented in Magma, to decide whether these Hadamard matrices are inequivalent. For each one of the thirteen orders $\{60, 68, 84, 92, 100, 108, 116, 132,$ 140, 156, 172, 180, 188$\}$ we have located two inequivalent Hadamard matrices, having different 4-profiles. The results (the matrices and their 4-profiles) are available in the web page http://www.cargo.wlu.ca/HMfromWM/

In addition, one could check these Hadamard matrices for inequivalence, using the graph isomorphism criterion, which is more time consuming.

5 Conclusion

In this paper we have found new orthogonal designs using $W(2n, 2n-5)$ and $W(2n, 2n-9)$ weighing matrices constructed from two circulants. We also use the full $OD(4n; 2, 8, 4n-10)$ to construct inequivalent Hadamard matrices of 13 orders.

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