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ON THE CO-P_3-STRUCTURE OF PERFECT GRAPHS*

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Abstract. Let $\mathcal{F}$ be a family of graphs. Two graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ are said to have the same $\mathcal{F}$-structure if there is a bijection $f : V_1 \to V_2$ such that a subset S induces a graph belonging to $\mathcal{F}$ in G_1 if and only if its image $f(S)$ induces a graph belonging to $\mathcal{F}$ in G_2. We characterize those graphs which have the same $\{P_3, \overline{P}_3\}$-structure, or the same $\{K_3, \overline{K}_3\}$-structure. This characterization shows that graph H is perfect if and only if it has the $\{P_3, \overline{P}_3\}$-structure of some perfect graph G. In proving the main result, we need and prove the following result, which is of independent interest: If a graph J is claw-free and co-claw-free, then either (i) J has at most nine vertices, or (ii) every component of J is a path or a hole, or (iii) every component of $\overline{J}$ is a path or a hole.

Key words. graph coloring, perfect graph, claw-free graph

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1. Introduction.

Let $\mathcal{F}$ be a family of graphs. Two graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ are said to have the same $\mathcal{F}$-structure if there is a bijection $f : V_1 \to V_2$ such that a subset S induces a graph belonging to $\mathcal{F}$ in G_1 if and only if its image $f(S)$ induces a graph belonging to $\mathcal{F}$ in G_2. In [4], Chvátal discussed the $\mathcal{F}$-structure when $\mathcal{F} = \{P_4\}$ in the context of perfect graph theory. A graph G is *perfect* if for each induced subgraph H of G, the chromatic number of H equals the number of vertices in a largest clique of H. A conjecture of Berge, which was proved by Lovász [10], states that a graph is perfect if and only if its complement is. This result is nowadays known as the Perfect Graph Theorem (PGT). Berge [1] also made a stronger conjecture stating that a graph is perfect if and only if it does not contain as an induced subgraph the odd chordless cycle on at least five vertices or its complement. This conjecture was known as the Strong Perfect Graph Conjecture (SPGC). Graphs satisfying the hypothesis of the SPGC are called *Berge graphs*. Recently, Chudnovsky et al. [3] announced a proof of the SPGC. This result is now known as the Strong Perfect Graph Theorem (SPGT).

Chvátal [4] conjectured that a graph H is perfect if and only if it has the P_4-structure of some perfect graph G. This was first proved by Reed [12], and is now a consequence of [3]. Further research into $\mathcal{F}$-structures also centered around its relationship to the SPGC [7], [8], [9].

In this paper, we prove two theorems of this nature. Let K_t (respectively, P_t) denote the clique (respectively, induced path) on t vertices.

THEOREM 1. *A graph H is perfect if and only if it has the $\{K_3, \overline{K}_3\}$-structure of a perfect graph G.* □

THEOREM 2. *A graph H is perfect if and only if it has the $\{P_3, \overline{P}_3\}$-structure of a perfect graph G.*

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It is easy to see that two graphs have the same $\{K_3, \overline{K}_3\}$-structure if and only if they have the same $\{P_3, \overline{P}_3\}$-structure. Thus, Theorems 1 and 2 are equivalent. Furthermore, it is a tedious but routine matter to show that graphs with the $\{P_3, \overline{P}_3\}$-structure of a Berge graph are Berge. Thus, our two theorems are implied by the SPGT. These two theorems were proved before the proof of the SPGC was announced and the original motivation was that they might be a step towards a proof of the SPGC. We feel they are still interesting because (a) they might help in finding a shorter proof of the SPGC, and (b) we prove the following result of independent interest. A *triangle* is the clique on three vertices. A *claw* is the graph with vertices a, b, c, d and edges ab, ac, ad.

THEOREM 3. *If a graph J and its complement $\overline{J}$ are claw-free, then either* (i) *J has at most nine vertices, or* (ii) *J and $\overline{J}$ are triangle-free.*

In section 2, we will discuss background results needed to prove Theorems 2 and 3, the proofs of which will be given in section 3.

2. Definitions and background. Before proving Theorem 2, we need to introduce some definitions and background results. Let G be a graph. Let S be a set of vertices of G. $G[S]$ will denote the subgraph of G induced by S (for simplicity, we will write $G[a, b, \ldots]$ for $G[\{a, b, \ldots\}]$). Let x be a vertex of G. Then $N_G(x)$ denotes the set of vertices adjacent to x in G. If $xy \in E(G)$, then we say that x *sees* y in G, otherwise we say that x *misses* y in G. We use $v_1v_2 \ldots v_k$ to denote the chordless path with vertices $v_1, v_2, \ldots, v_k$ and edges v_iv_{i+1} for $i = 1, 2, \ldots, k-1$. A *hole* is a chordless cycle with at least four vertices. Two vertices x, y are *antitwins* of G if every vertex z, different from x and y, sees x or y but not both. Note that x, y are antitwins in G if and only if they are antitwins in $\overline{G}$. Two nonempty sets A, B of vertices of G are a *homogeneous pair* if (i) A or B has at least two vertices, (ii) there are at least two vertices outside $A \cup B$, and (iii) for any vertex x outside $A \cup B$, if x sees some vertex in A (respectively, B) then it sees all vertices of A (respectively, B). If A, B are a homogeneous pair in G, then they are a homogeneous pair in $\overline{G}$.

A graph is *minimal imperfect* if it is not perfect but each of its proper induced subgraphs is. The PGT implies that a graph is minimal imperfect if and only if its complement is. We will rely on the following two results on minimal imperfect graphs.

Olariu [11] proved that

(1) a minimal imperfect graph cannot contain antitwins.

Chvátal and Sbihi [5] proved that

(2) a minimal imperfect graph cannot contain a homogeneous pair.

If two graphs have the same $\{P_3, \overline{P}_3\}$-structure, then we will say that they have the same *co-P_3-structure*. If F is a graph, then *co-F* denotes the complement of F. For example, the cotriangle is the complement of a triangle. The *pyramid* is the graph obtained from taking a triangle with vertices a, b, c and a cotriangle with vertices x, y, z and adding edges xa, xb, yb, yc, za, zb; such a pyramid will be referred to as $P(a, b, c, x, y, z)$. A graph is *elementary* [6] if its edges can be colored by two colors in such a way that each edge receives a color and there is no monochromatic P_3. Pyramids and copyramids, claws, odd holes, and odd antiholes are not elementary.

3. The proofs. We are going to prove a theorem stronger than Theorem 2.

THEOREM 4. *If two graphs G and H have the same co-P_3-structure and H is not isomorphic to G or $\overline{G}$, then either* (i) *both G and H have antitwins or a homogeneous pair, or* (ii) *G and H have at most nine vertices.*

The key to Theorem 4 is Theorem 3.

Proof of Theorem 3. Let J be a claw-free, co-claw-free graph. Suppose that both (i) and (ii) fail. Thus, J contains a triangle K, and a cotriangle S. Let $K = \{a, b, c\}$ and $S = \{x, y, z\}$.

Note that

(3) every vertex u of $J - K$ must see at least one vertex of K,

for otherwise there is a coclaw induced by K and u.

Suppose that $K \cap S = \emptyset$. We claim that

(4) there cannot be a vertex of S that sees at least two vertices of K and another vertex of S that misses at least two vertices of K.

Suppose x sees vertices a, b of K, and without loss of generality, y misses at least two vertices of K. If y misses a, b then $\{a, b, x, y\}$ is a coclaw in J. So, without loss of generality we may assume y misses a and c. Now, y sees b (by (3)), and x misses c (for otherwise $\{a, c, x, y\}$ is a coclaw). Now, $\{b, x, y, c\}$ is a claw. So (4) holds. Next, we prove that

(5) there cannot be a vertex of S that sees three vertices of K.

Suppose x sees all three vertices of K. Then (4) implies that y sees (at least) two vertices, say a, b, of K. Both a and b must miss z, for otherwise there is a claw with vertices x, y, z and a or b. Now, there is a coclaw with z, a, b, x. So (5) holds. Next, we prove that

(6) if some vertex of S sees two vertices of K, then $K \cup S$ induces a pyramid.

If some vertex of S sees two vertices of K, then, by (3), (4), and (5), every vertex of S sees exactly two vertices of K. If two vertices, say x, y, of S see the same two vertices, say a, b, of K, then a and b miss z (for otherwise, there is a claw with S and a or b), a contradiction to (4). It is now easy to see that $K \cup S$ induces a pyramid. Next, we prove that

(7) if $K \cup S$ induces a pyramid, then J contains a claw or coclaw.

By assumption, J has a vertex t outside the pyramid $P(a, b, c, x, y, z)$. Vertex t cannot be adjacent to all vertices in S, for otherwise $\{t, x, y, z\}$ induces a claw. We may assume that t misses x.

Suppose t sees a. Then t sees z (for otherwise $\{a, x, t, z\}$ is a claw) and c (for otherwise $\{a, x, t, c\}$ is a claw). But now $\{x, t, z, c\}$ is a coclaw.

So t misses a, and by symmetry t misses b. But then $\{t, x, a, b\}$ induces a coclaw. We have proved (7).

By (6) and (7), we may assume that no vertex of S sees two vertices of K. It follows from (3) that every vertex of S sees exactly one vertex of K. If some two vertices, say x, y, of S see the same vertex, say a, in K, then there is a claw (induced by $\{a, x, y, b\}$). Now, it is easy to see that $K \cup S$ induces a copyramid. By (7), J contains a claw or coclaw. Thus,

(8) J cannot contain a triangle that is disjoint from a cotriangle.

Consider a vertex a that is the intersection of a triangle $K = \{a, b, c\}$ and a cotriangle S. Let N be the set of neighbors of a, and M be the set of nonneighbors of a (different from a). By (8), M has no cotriangle. Also, M has no triangle, for otherwise this triangle and a form a coclaw. Thus M has at most five vertices (this is a well-known case of Ramsey's theorem).

Suppose M has five vertices. Then M is the C_5. Enumerate the vertices of the C_5 as $v_1, v_2, \ldots, v_5$ in the cyclic order. By (3), each v_i sees b, or c, or both. We may suppose v_1 sees b. Then b has to miss v_3 and v_4, for otherwise $\{b, a, v_1, v_3\}$ or $\{b, a, v_1, v_4\}$ induces a claw. Now (3) implies that c sees v_3 and v_4, and therefore, c misses v_1, for otherwise $\{c, a, v_3, v_1\}$ induces a claw. But then $\{v_1, c, v_3, v_4\}$ induces a coclaw. (This argument actually implies that M is P_4-free.)

So, M has at most four vertices. A similar argument applied to $\overline{J}$ shows that N has at most four vertices. □

COROLLARY 5. *If a graph J is claw-free and co-claw-free, then either* (i) *J has at most nine vertices, or* (ii) *every component of J is a path or a hole, or* (iii) *every component of $\overline{J}$ is a path or a hole.*

Proof of Corollary 5. By Theorem 3, we may assume that J is triangle-free. Consider a component C of J. If C contains a hole F of length at least 4, then $F = C$, for otherwise some vertex in $C - F$ has a neighbor in F and it follows that C contains a claw. Thus C must be a tree and therefore a path since J is claw-free. □

Proof of Theorem 4. Let G and H be two graphs with the same co-P_3-structure. We may assume that G and H are defined on the same vertex-set in such a way that for any set X of vertices, X induces a P_3 or co-P_3 in G if and only if X does so in H. We may suppose H is not isomorphic to G or $\overline{G}$. A pair (x, y) of vertices will be called *variant* if x sees y in H but misses it in G, or vice versa. We may assume that

(9) H contains a variant pair (x, y),

for otherwise H is isomorphic to G or to $\overline{G}$, a contradiction. A pair of vertices is *invariant* if it is not variant. By the *variant graph* J, we mean a graph whose vertices are those of G and in which a pair of vertices is joined by an edge if and only if they are a variant pair. We say that J is the variant graph for G and H. We can two-color the edges of J so that xy is colored 1 if xy is an edge of G, and 2 if xy is an edge of H. We say that a set is *bad* if it induces a P_3 or co-P_3 in H but does not do so in G, or vice versa. The fact that G and H have the same co-P_3-structure implies that J contains no monochromatic P_3 (such a P_3 would be bad); thus

(10) J is elementary.

It is easy to see that

(11) an elementary graph is bipartite if and only if it is triangle-free.

Since $\overline{J}$ is the variant graph for G and $\overline{H}$, $\overline{J}$ is elementary. Thus J is claw-free and co-claw-free. By Corollary 5, we may assume that each component of J is a path or a hole. Since J is elementary,

(12) each component of J is a path or an even cycle.

OBSERVATION 1. *Let ab be an edge of J and let x be a vertex with $xa, xb \notin E(J)$. Then, in H and G, x sees exactly one vertex of $\{a, b\}$.*

Proof of Observation 1. In G and in H, x cannot see, or miss, both a and b, for otherwise the set $\{a, b, x\}$ is bad. So, in G and in H, x sees exactly one vertex in $\{a, b\}$. ◻

We know there is a component with at least two vertices. We will prove that

$$\text{if } J \text{ has a component, different from a } C_4, \text{ with at least two vertices, then } H \text{ has antitwins.} \tag{13}$$

Consider a component K of J with at least two vertices. Since K is a path or a cycle, we can enumerate the vertices of K as $v_1, v_2, \ldots, v_t$ such that $v_i v_{i+1} \in E(J)$ for $i = 1, 2, \ldots t-1$, and if K is a cycle, then $v_1 v_t \in E(J)$. We may assume that $v_1 v_2 \in E(H)$ (for otherwise, we can replace H by $\overline{H}$ and G by $\overline{G}$ in the following argument. Note that antitwins of H are antitwins of $\overline{H}$). It follows that $v_i v_{i+1} \in E(H)$ if and only if i is odd. We are going to show that if K is not a C_4, then either $\{v_1, v_2\}$ or $\{v_2, v_3\}$ is a pair of antitwins of H.

Suppose K is not a C_4 but $\{v_1, v_2\}$ is not a pair of antitwins of H. Then, in H there is a vertex u seeing, or missing, both vertices v_1 and v_2. Observation 1 implies $u = v_3$ or $v_t v_1 \in E(J)$ and $u = v_t$. Without loss of generality, we may assume $u = v_3$. In H, since v_3 misses v_2, v_3 misses v_1. Now, in H, v_4 sees v_2, for otherwise, $\{v_2, v_3\}$ is a pair of antitwins by Observation 1. The same observation, with $a = v_1, b = v_2, x = v_4$ implies $v_4 v_1 \notin E(H)$. But then the observation is contradicted with $a = v_3, b = v_4, x = v_1$. We have established (13).

Suppose some component K of J is a C_4. Then it is easy to see that H has a homogeneous pair (with $A = \{v_1, v_3\}, B = \{v_2, v_4\}$) whenever J has at least six vertices, and if J has five vertices, then (ii) holds. Theorem 4 is proved. ◻

Now, we prove Theorem 2.

Proof of Theorem 2. Let G and H be two graphs with the same co-P_3-structure. Suppose G is perfect but H is not. Every imperfect graph contains a minimal imperfect graph. So we may assume H is minimal imperfect. By Theorem 4, (1), (2), and the perfect graph theorem, H has at most nine vertices.

It is a tedious but routine matter to verify that all graphs on at most nine vertices with the co-P_3-structure of a perfect graph are Berge (indeed, one can show easily that graphs with the co-P_3-structure of Berge graphs are Berge but this is much more than is needed here). It is well known and easy to prove that Berge graphs with at most nine vertices are perfect. ◻

Note added in proof. After this paper was written, we learned that a slightly weaker version of Corollary 5 was previously proved by A. Brandstädt and S. Mahfud [2]. They proved that if G is claw-free and co-claw-free, then G or $\overline{G}$ is a chordless path or cycle, or G has a homogeneous set, or G has at most nine vertices.

REFERENCES

[1] C. Berge, *Les problèmes de coloration en théorie des graphes*, Publ. Inst. Stat. Univ. Paris, 9 (1960), pp. 123–160.

[2] A. Brandstädt and S. Mahfud, *Maximum weight stable set on graphs without claw and co-claw (and similar graph classes) can be solved in linear time*, Inform. Process. Lett., 84 (2002), pp. 251–259.

[3] M. Chudnovsky, N. Robertson, P. Seymour, and R. Thomas, *The Strong Perfect Graph Theorem*, manuscript, 2002.

[4] V. Chvátal, *A semi-strong perfect graph conjecture*, in Topics on Perfect Graphs, C. Berge and V. Chvátal, eds., North-Holland, Amsterdam, 1984, pp. 279–280.

[5] V. Chvátal and N. Sbihi, *Bull-free perfect graphs*, Graphs Combin., 3 (1987), pp. 127–140.
[6] V. Chvátal and N. Sbihi, *Recognizing claw-free berge graphs*, J. Combin. Theory Ser. B, 44 (1988), pp. 154–176.
[7] C. Hoàng, *On the disc-structure of perfect graphs* I. *The paw-structure*, Discrete Appl. Math., 94 (1999), pp. 247–262.
[8] C. Hoàng, *On the disc-structure of perfect graphs II. The co-C_4-structure*, Discrete Math., 252 (2002), pp. 141–159.
[9] S. Hougardy, *On the P_4-Structure of Perfect Graphs*, Shaker-Verlag, Aachen, 1995.
[10] L. Lovász, *Normal hypergraphs and the perfect graph conjecture*, Discrete Math., 2 (1972), pp. 253–267.
[11] S. Olariu, *No antitwins in minimal imperfect graphs*, J. Combin. Theory Ser. B, 45 (1988), pp. 255–257.
[12] B. Reed, *A semi-strong perfect graph theorem*, J. Combin. Theory Ser. B, 43 (1987), pp. 223–240.