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HARISH-CHANDRA MODULES OVER THE $\mathbb{Q}$ HEISENBERG–VIRASORO ALGEBRA

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Abstract

In this paper, it is proved that all irreducible Harish-Chandra modules over the $\mathbb{Q}$ Heisenberg–Virasoro algebra are of the intermediate series (all weight spaces are at most one-dimensional).

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1. Introduction

The Heisenberg–Virasoro algebra is the universal central extension of the Lie algebra of differential operators on a circle of order at most one. It contains the classical Heisenberg algebra and the Virasoro algebra as subalgebras. The structure of their irreducible highest weight modules was studied in [1, 3]. Irreducible Harish-Chandra modules over the Heisenberg–Virasoro algebra were classified in [13]. They are either highest weight modules, lowest weight modules, or modules of the intermediate series. The representation theory of the Heisenberg–Virasoro algebra is closely related to those of other Lie algebras, such as the Virasoro algebra and toroidal Lie algebras, see [4, 5, 8]. For other results on Heisenberg–Virasoro algebras, see [11, 16] and the references therein.

Recently, some authors have introduced generalized Heisenberg–Virasoro algebras and started to study their representations (see [12, 17]). In this paper, we give the classification of irreducible Harish-Chandra modules over the $\mathbb{Q}$ Heisenberg–Virasoro algebra. As in the $\mathbb{Q}$ Virasoro algebra case [15], only modules of the intermediate series appear. The main ideas in our proof (Lemma 3.1) are similar to those of [15] and [6].

In this paper we denote by $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{C}$ the sets of integers, rational numbers and complex numbers, respectively. Now we give the definitions of the generalized Heisenberg–Virasoro algebras and the $\mathbb{Q}$ Heisenberg–Virasoro algebra.

DEFINITION 1.1. Suppose that G is an additive subgroup of $\mathbb{C}$. The generalized Heisenberg–Virasoro algebra $\mathrm{HVir}[G]$ is the Lie algebra over $\mathbb{C}$ with a basis

$$\{d_g, I(g), C_D, C_{DI}, C_I \mid g \in G\},$$

subject to the Lie brackets given by

$$\begin{gathered}[d_g, d_h] = (h-g)d_{g+h} + \delta_{g,-h}\frac{g^3-g}{12}C_D,\\ [d_g, I(h)] = hI(g+h) + \delta_{g,-h}(g^2+g)C_{DI},\\ [I(g), I(h)] = g\delta_{g,-h}C_I,\\ [\mathrm{HVir}[G], C_D] = [\mathrm{HVir}[G], C_{DI}] = [\mathrm{HVir}[G], C_I] = 0.\end{gathered}$$

It is easy to see that $\mathrm{HVir}[G] \simeq \mathrm{HVir}[G']$ if and only if there exists a nonzero $a \in \mathbb{C}$ such that $G = aG'$. If $G = \mathbb{Z}$, then $\mathrm{HVir}[\mathbb{Z}] = \mathrm{HVir}$ is the classical Heisenberg–Virasoro algebra. When $G = \mathbb{Q}$, we call $\mathrm{HVir}[\mathbb{Q}]$ the $\mathbb{Q}$ *Heisenberg–Virasoro algebra*; this is the main object of this paper. An $\mathrm{HVir}[G]$ module V is said to be trivial if $\mathrm{HVir}[G]V = 0$, and we denote the one-dimensional trivial module by T.

The modules of the intermediate series $V(\alpha, \beta; F)$ over $\mathrm{HVir}[G]$ are defined as follows, for all $\alpha, \beta, F \in \mathbb{C}$. As a vector space over $\mathbb{C}$, the space $V(\alpha, \beta; F)$ has a basis $\{v_g \mid g \in G\}$ and the actions are

$$d_g v_h = (\alpha + h + g\beta)v_{g+h}, \quad I(g)v_h = F v_{g+h}, \tag{1.1}$$

$$C_D v_g = 0, \quad C_I v_g = 0, \quad C_{DI} v_g = 0 \quad \forall g, h \in G. \tag{1.2}$$

It is well known that $V(\alpha, \beta; F) \cong V(\alpha + g, \beta; F)$ for all $\alpha, \beta, F \in \mathbb{C}$ and $g \in G$. So we always assume that $\alpha = 0$ when $\alpha \in G$. It is also easy to see that $V(\alpha, \beta; F)$ is reducible if and only if $F = 0$, $\alpha = 0$, and $\beta \in \{0, 1\}$. The module $V(0, 0; 0)$ has a one-dimensional submodule T and $V(0, 0; 0)/T$ is irreducible; $V(0, 1; 0)$ has a codimension one irreducible submodule. We denote the unique nontrivial irreducible subquotient of $V(\alpha, \beta; F)$ by $V'(\alpha, \beta; F)$. It is easy to see that $V'(0, 0; 0) \cong V'(0, 1; 0)$.

REMARK 1.2. When $F \neq 0$, it is not hard to verify that $V(\alpha, \beta; F) \cong V(\alpha', \beta'; F')$ if and only if $\alpha - \alpha' \in G$ and $(\beta, F) = (\beta', F')$.

Here is our main theorem.

THEOREM 1.3. *Suppose that V is an irreducible nontrivial Harish-Chandra module over* $\mathrm{HVir}[\mathbb{Q}]$. *Then V is isomorphic to $V'(\alpha, \beta; F)$ for suitable $\alpha, \beta, F \in \mathbb{C}$.*

This paper is organized as follows. In Section 2, we collect some known results on the Virasoro algebra and on the Heisenberg–Virasoro algebra for later use. In Section 3, we give the proof of the main theorem.

2. Preliminaries

Since generalized Heisenberg–Virasoro algebras are closely related to generalized Virasoro algebras, we first recall some results on generalized Virasoro algebras.

DEFINITION 2.1. Let G be a nonzero additive subgroup of $\mathbb{C}$. The generalized Virasoro algebra Vir[G] is a Lie algebra over $\mathbb{C}$ with a basis $\{d_g, C_D \mid g \in G\}$ and is subject to the Lie brackets given by

$$[d_g, d_h] = (h-g)d_{g+h} + \delta_{g,-h}\frac{g^3-g}{12}C_D, \quad [\mathrm{Vir}[G], C_D] = 0.$$

Notice that Vir[G] is a subalgebra of HVir[G]. If $G = \mathbb{Z}$, then Vir[$\mathbb{Z}$] = Vir is the classical Virasoro algebra and if $G = \mathbb{Q}$, then Vir[$\mathbb{Q}$] is called the $\mathbb{Q}$ Virasoro algebra.

A module of the intermediate series $V(\alpha, \beta)$ has a $\mathbb{C}$-basis $\{v_g \mid g \in G\}$ and the Vir-actions are

$$d_g v_h = (\alpha + h + g\beta)v_{g+h}, \quad C_D v_g = 0 \quad \forall g, h \in G.$$

It is well known that $V(\alpha, \beta)$ is reducible if and only if $\alpha \in G$ and $\beta \in \{0, 1\}$. The unique nontrivial irreducible subquotient of $V(\alpha, \beta)$ is denoted by $V'(\alpha, \beta)$. It is also known that $V(\alpha, \beta) \cong V(\alpha + g, \beta)$ for all $\alpha, \beta \in \mathbb{C}$ and all $g \in G$. So we always assume that $\alpha = 0$ if $\alpha \in G$.

Mazorchuk [15] classified irreducible Harish-Chandra modules over Vir[$\mathbb{Q}$].

THEOREM 2.2. *Every irreducible Harish-Chandra module over* Vir[$\mathbb{Q}$] *is isomorphic to* $V'(\alpha, \beta)$ *for suitable* $\alpha, \beta \in \mathbb{C}$.

Now we consider the $\mathbb{Q}$ Heisenberg–Virasoro algebra HVir[$\mathbb{Q}$], which contains the classical Heisenberg–Virasoro algebra HVir[$\mathbb{Z}$] = HVir. The classification of irreducible Harish-Chandra modules over HVir was given in [13].

THEOREM 2.3. *Any irreducible Harish-Chandra module over* HVir *is isomorphic to either a highest weight module, a lowest weight module, or* $V'(\alpha, \beta; F)$ *for suitable* $\alpha, \beta, F \in \mathbb{C}$.

3. Proof of Theorem 1.3

We decompose $\mathbb{Q}$ as the union of a series of rings $\mathbb{Q}_k = \{n/k! \mid n \in \mathbb{Z}\}$, where $k \in \mathbb{N}$. Then we can view each HVir[$\mathbb{Q}_k$] as a subalgebra of HVir[$\mathbb{Q}$] naturally, and thus HVir[$\mathbb{Q}$] $= \bigcup_{k\in\mathbb{N}}$ HVir[$\mathbb{Q}_k$]. Clearly, HVir[$\mathbb{Q}_k$] $\simeq$ HVir for all $k \in \mathbb{N}$. For convenience, we write $U(\mathbb{Q}) = U(\mathrm{HVir}[\mathbb{Q}])$ and $U(\mathbb{Q}_k) = U(\mathrm{HVir}[\mathbb{Q}_k])$. Denote $U(\mathbb{Q})_q = \{u \in U(\mathbb{Q}) \mid [d_0, u] = qu\}$ and $U(\mathbb{Q}_k)_q = \{u \in U(\mathbb{Q}_k) \mid [d_0, u] = qu\}$, for all $q \in \mathbb{Q}$.

LEMMA 3.1. *Suppose that* N *is a finite-dimensional irreducible* $U(\mathbb{Q})_0$*-module; then there exists* $k \in \mathbb{N}$ *such that* N *is an irreducible* $U(\mathbb{Q}_k)_0$*-module.*

PROOF. There is an associative algebra homomorphism $\Phi\colon U(\mathbb{Q})_0 \to \mathrm{gl}(N)$, where $\mathrm{gl}(N)$ is the general linear associative algebra of N.

Since N and hence $\mathrm{gl}(N)$ are both finite-dimensional, $U(\mathbb{Q})_0/\ker(\Phi)$ is finite-dimensional. Take $y_1, y_2, \dots, y_m \in U(\mathbb{Q})_0$ such that $\overline{y_1}, \overline{y_2}, \dots, \overline{y_m}$ form a basis of $U(\mathbb{Q})_0/\ker(\Phi)$. Then there exists $k \in \mathbb{N}$ such that $y_1, y_2, \dots, y_m$ are all in $U(\mathbb{Q}_k)_0$.

We claim that N is an irreducible $U(\mathbb{Q}_k)_0$-module. Let M be a proper $U(\mathbb{Q}_k)_0$-submodule of N. For all $y \in U(\mathbb{Q})_0$, there exist $y_0 \in \ker(\Phi)$ and $a_i \in \mathbb{C}$ such that $y = y_0 + \sum_{i=1}^m a_i y_i$. Thus

$$yM = \Big(y_0 + \sum_{i=1}^{m} a_i y_i\Big)M \subseteq \Big(\sum a_i y_i\Big)M \subseteq M,$$

that is, M is a $U(\mathbb{Q})_0$-submodule of N, which forces $M = 0$. Thus N is irreducible over $U(\mathbb{Q}_k)_0$. □

Now we fix a nontrivial irreducible Harish-Chandra module V over $\mathrm{HVir}[\mathbb{Q}]$. Then there exists $\alpha \in \mathbb{C}$ such that $V = \bigoplus_{q\in\mathbb{Q}} V_q$, where $V_q = \{v \in V \mid d_0 v = (\alpha + q)v\}$. We define the support of V as $\operatorname{supp} V = \{q \in \mathbb{Q} \mid V_q \neq 0\}$.

LEMMA 3.2. *$\operatorname{supp} V = \mathbb{Q}$ or $\mathbb{Q} \setminus \{-\alpha\}$ and $\dim V_q = 1$ for all $q \in \operatorname{supp} V$.*

PROOF. First view V as a $\mathrm{Vir}[\mathbb{Q}]$ module; by Theorem 2.2, $\dim V_p = \dim V_q$ for all $p, q \in \mathbb{Q} \setminus \{-\alpha\}$. In particular, V is a uniformly bounded module, that is, the dimensions of all weight spaces are bounded by a positive integer.

Suppose that $\dim V_q > 1$ for some $q \in \operatorname{supp} V$. Then it is easy to see that V_q is an irreducible $U[\mathbb{Q}]_0$-module. By Lemma 3.1, there is some $k \in \mathbb{N}$ such that V_q is an irreducible $U(Q_k)_0$-module.

We consider the $\mathrm{HVir}[\mathbb{Q}_k]$-module $W = U(\mathbb{Q}_k)V_q$, which is uniformly bounded. Then, by Theorem 2.3, there is a composition series of $\mathrm{HVir}[\mathbb{Q}_k]$-modules, namely,

$$0 = W^{(0)} \subset W^{(1)} \subset W^{(2)} \subset \cdots \subset W^{(m)} = W.$$

Each factor $W^{(i)}/W^{(i-1)}$ is either trivial or of the intermediate series, and in both cases all weight spaces are one-dimensional.

Take the first i such that $\dim W_q^{(i)} \neq 0$. Then $\dim W_q^{(i)} = \dim(W^{(i)}/W^{(i-1)})_q = 1$. But, on the other hand, $W_q^{(i)}$ is a $U(\mathbb{Q}_k)_0$-module, that is, a nontrivial proper $U(\mathbb{Q}_k)_0$-submodule of V_q. This is a contradiction, so $\dim V_q = 1$ for all $q \in \operatorname{supp} V$.

Since V is nontrivial, there exists $q \in \operatorname{supp} V \setminus \{-\alpha\}$. Our result follows since $\dim V_p = \dim V_q = 1$ for all $p \in \mathbb{Q} \setminus \{-\alpha\}$. □

Now we can give the proof of our main theorem.

PROOF OF THEOREM 1.3. By Lemma 3.2, we know that $\operatorname{supp} V = \mathbb{Q}$ for some $\alpha \in \mathbb{C}$ or $\operatorname{supp} V = \mathbb{Q} \setminus \{-\alpha\}$. Denote $V^{(0)} = 0$ and $V^{(k)} = \bigoplus_{q\in\mathbb{Q}_k} V_q$ for all $k \geq 1$. Then $\operatorname{supp} V^{(k)} = \{q \in \mathbb{Q} \mid V_q^{(k)} \neq 0\} = \operatorname{supp} V \cap \mathbb{Q}_k$ for all $k \geq 1$. There is a vector space filtration of V, namely,

$$0 = V^{(0)} \subset V^{(1)} \subset V^{(2)} \subset \cdots \subset V^{(k)} \subset \cdots \subset V \quad \text{and} \quad V = \bigcup_{k=0}^{\infty} V^{(k)}.$$

It is clear that each $V^{(k)}$ can be viewed as an HVir[$\mathbb{Q}_k$]-module, and each HVir[$\mathbb{Q}_k$] is isomorphic to HVir.

If $-\alpha \notin \operatorname{supp} V$, then each $V^{(k)}$ is irreducible over HVir[$\mathbb{Q}_k$] when $k \in \mathbb{Z}$, by Lemma 3.2 and Theorem 2.3.

If $-\alpha \in \operatorname{supp} V$, then $\alpha = 0$ by assumption. Since V is an irreducible HVir[$\mathbb{Q}$]-module, there exists $p \in \mathbb{Q} \setminus \{0\}$ such that $u_p \in U(\mathrm{HVir}[\mathbb{Q}])_p$ and $u_p V_0 = V_p$, as well as $q \in \mathbb{Q} \setminus \{0\}$ such that $u_q \in U(\mathrm{HVir}[\mathbb{Q}])_q$ and $u_q V_{-q} = V_0$. Then there exists k_0 such that $u_p, u_q \in U(\mathrm{HVir}[\mathbb{Q}_{k_0}])$, and thus $V^{(k_0)}$ is irreducible as an HVir[$\mathbb{Q}_{k_0}$]-module.

Now $V^{(k_0)}$ is irreducible over HVir[$\mathbb{Q}_{k_0}$], and so $V^{(k)}$ is irreducible over HVir[$\mathbb{Q}_k$] whenever $k \geq k_0$. By Theorem 2.3, we see that $C_D V = C_I V = C_{DI} V = 0$, that $I(0)$ acts as a scalar $F \in \mathbb{C}$, and that there exist $\alpha_k, \beta_k \in \mathbb{C}$ such that $V^{(k)} \cong V'(\alpha_k, \beta_k; F)$ as modules over HVir[$\mathbb{Q}_k$] (defined by (1.5) and (1.6) with $G = \mathbb{Q}_k$) when $k \geq k_0$. Write $\alpha = \alpha_{k_0}$ and $\beta = \beta_{k_0}$ for short.

If $F = 0$, V is an irreducible Vir($\mathbb{Q}$)-module and $I(p)V = 0$ for all $p \in \mathbb{Q}$. Theorem 1.3 follows from Theorem 2.2. Next we assume that $F \neq 0$.

Now we need to prove that $V \cong V'(\alpha, \beta; F)$ as modules over HVir[$\mathbb{Q}$], that is, we can choose a basis of V such that (1.1) and (1.2) hold. We proceed by choosing a basis of each $V^{(k)}$ inductively, such that (1.1) and (1.2) hold when one only considers the actions of HVir[$\mathbb{Q}_k$], and that the basis of each $V^{(k+1)}$ is the extension of the basis of $V^{(k)}$ for all $k \geq k_0$.

Naturally, there is such a basis for $V^{(k_0)}$. Now suppose that $m > k_0$ and that we have found such bases for $V^{(k)}$ whenever $k_0 \leq k \leq m-1$. In particular, there is a basis $\{v_q \in V_q \mid q \in \operatorname{supp} V^{(m-1)}\}$ with the property that $d_p v_q = (\alpha + q + p\beta) v_{p+q}$ and $I(p) v_q = F v_{p+q}$ for all $p, q \in \operatorname{supp} V^{(m-1)}$. Now we consider $V^{(m)}$.

Let Φ be the canonical isomorphism HVir $\to$ HVir[$\mathbb{Q}_m$] given by $\Phi(d_n) = m!\, d_{n/m!}$ and $\Phi(I(n)) = m!\, I(n/m!)$ for all $n \in \mathbb{Z}$. Then $V^{(m)}$ can be viewed as an HVir module via Φ and is isomorphic to $V(\alpha', \beta'; F')$ as modules over HVir for suitable $\alpha', \beta', F' \in \mathbb{C}$, by Theorem 2.3. Then there is a basis $\{v'_q \mid q \in \operatorname{supp} V^{(m)}\}$ of $V^{(m)}$ such that, for all $i/m!, k/m! \in \operatorname{supp} V^{(m)}$,

$$(m!\, d_{i/m!}) v'_{k/m!} = (\alpha' + k + i\beta') v'_{(k+i)/m!}, \quad (m!\, I(i/m!)) v'_{k/m!} = F' v'_{(k+i)/m!}.$$

That is,

$$d_{i/m!} v'_{k/m!} = \left(\frac{\alpha'}{m!} + \frac{k}{m!} + \frac{i}{m!}\beta'\right) v'_{(k+i)/m!}, \quad I(i/m!) v'_{k/m!} = \frac{F'}{m!} v'_{(k+i)/m!},$$

or equivalently,

$$d_p v'_q = \left(\frac{\alpha'}{m!} + q + p\beta'\right) v'_{p+q}, \quad I(p) v'_q = \frac{F'}{m!} v'_{p+q} \quad \forall p, q \in \operatorname{supp} V^{(m)}.$$

We take $p = 0$ and $q \in \operatorname{supp} V^{(m-1)} \subset \operatorname{supp} V^{(m)}$, and then $d_0 v'_q = (\alpha'/m! + q) v'_q$ and $I(0) v'_q = (F'/m!) v'_q$, which shows that $\alpha' = m!\, \alpha$ and $F' = m!\, F$.

Assume that $v'_q = c_q v_q$ for all $q \in \text{supp}\, V^{(m-1)}$. Compare $I(p)v'_q = F v'_{p+q}$ and $I(p)v_q = F v_{p+q}$. We see that c_p is independent of p. Then we may choose $v'_p = v_p$ for all $p \in \text{supp}\, V^{(m-1)}$. By the remark in Section 1 we see that $\beta' = \beta$. Denote $v_q = v'_q$ for all $q \in \text{supp}\, V^{(m)} \setminus \text{supp}\, V^{(m-1)}$. Thus the basis $\{v_q \mid q \in \text{supp}\, V^{(m)}\}$ is an extension of $\{v_p \mid p \in \text{supp}\, V^{(m-1)}\}$ such that $d_p v_q = (\alpha + q + p\beta)v_{p+q}$ and $I(p)v_q = F v_{p+q}$ for all $p, q \in \text{supp}\, V^{(m)}$.

By induction, we can produce a basis $\{v_q \mid q \in \text{supp}\, V\}$ for V such that $d_p v_q = (\alpha + q + p\beta)v_{p+q}$ and $I(p)v_q = F v_{p+q}$ for all $p, q \in \text{supp}\, V$ and further $C_D V = C_I V = C_{DI} V = 0$. That is, $V \cong V'(\alpha, \beta; F)$. This completes the proof. $\square$

Recall from [10] that the *rank* of an additive subgroup G of $\mathbb{C}$, denoted by $\text{rank}(G)$, is the maximal number r for which we can find $g_1, \ldots, g_r \in G \setminus \{0\}$ such that $\mathbb{Z}g_1 + \cdots + \mathbb{Z}g_r$ is a direct sum. If such an r does not exist, then we define $\text{rank}(G) = \infty$.

Now we can see that the proof of Theorem 1.3 is also valid when G is an infinitely generated additive subgroup of $\mathbb{C}$ of rank one. Thus we have the following result.

THEOREM 3.3. *Let G be an infinitely generated additive subgroup of $\mathbb{C}$ of rank one. Then any nontrivial irreducible Harish-Chandra module over* $\text{HVir}[G]$ *is isomorphic to $V'(\alpha, \beta; F)$ for suitable $\alpha, \beta, F \in \mathbb{C}$.*

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