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First passage time problem for multivariate jump-diffusion processes: Models, computation, and applications in finance

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First Passage Time Problem for

Multivariate Jump-Diffusion Processes:

Models, Computation, and Applications in Finance

by

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Abstract

The first passage time (FPT) problems are ubiquitous in many applications, from physics to finance. Mathematically, such problems are often reduced to the evaluation of the probability density of the time for a process to cross a certain level, a boundary, or to enter a certain region. While in other areas of applications the FPT problems can often be solved analytically, in finance we usually have to resort to the application of numerical procedures, in particular when we deal with jump-diffusion stochastic processes (JDP). The application of the conventional Monte-Carlo procedure is possible for the solution of the resulting model, but it becomes computationally inefficient which severely restricts its applicability in many practically interesting cases.

In this dissertation, we are interested in the development of efficient Monte-Carlo-based computational procedures for the estimation of the probability density of the time for a random process to cross a specified threshold level. Our main application is the credit risk analysis where we focus on a case of several "coupled" companies for which we attempt to evaluate their dependent defaults. In particular, we consider a situation where individual companies are linked together via certain economic conditions, so the default events of companies are correlated. This is usually the case, for example, when the companies are in the same industry or in supply chain management problems. In this dissertation, we have successfully developed such efficient computational procedures that can be carried out for multivariate (and correlated) jump-diffusion processes. We have also provided details of the implementation of the developed Monte-Carlo-based technique for a subclass of multidimensional Lévy processes with several compound Poisson shocks. Finally, we have demonstrated the applicability of the developed methodologies to the analysis of the default rates and default correlations of several different, but correlated firms via a set of empirical data.

***Keywords*:** Credit Risk, Default Correlation, First Passage Time, Multivariate Jump-Diffusion Processes, Monte-Carlo Simulation, Multivariate Uniform Sampling Method

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Chapter 1

Introduction

Many problems in finance require the information on the first passage time (FPT) of a stochastic process. Mathematically, such problems are often reduced to the evaluation of the probability density of the time for a process to cross a certain level, a boundary, or to enter a certain region. While in other areas of applications the FPT problem can often be solved analytically, in finance we usually have to resort to the application of numerical procedures, in particular when we deal with jump-diffusion processes (JDP). Recent research in finance theory has renewed the interest in JDPs, and the FPT problem for such processes is applicable to several finance problems. For example, Merton [1] has modeled the market value of assets as a jump-diffusion process. The jump part is needed to model significant events or shocks such as a surprise earnings release. Another application is in derivatives pricing, such as pricing barrier options which involves crossing certain levels [2], or pricing American options [3, 4] which entails evaluating the first passage time density for a time varying boundary. There are also a number of important applications in the credit risk analysis [5], some of which will be analyzed further in this dissertation.

Among numerical procedures, Monte-Carlo methods remain a primary candidate for applications. However, the conventional Monte-Carlo procedure becomes computationally inefficient when it is applied to the jump-diffusion processes. Many researchers have contributed to the field of enhancement of the efficiency of Monte-Carlo simulations. Atiya and Metwally [2, 6] have recently developed a fast Monte-Carlo-type numerical method to solve the FPT problem in the one-dimensional case.

Note that apart from the pricing and hedging of options on a single asset, practically all financial applications require a multivariate model with dependence between different assets. Examples include basket option pricing, portfolio optimization, simulation of risk scenarios for portfolios. In most of these applications, jumps in the price process must be taken into account [7, 8]. In our recent contribution, we have developed an efficient Monte-Carlo method for multivariate jump-diffusion processes and successfully applied it to analyze the default risk and default correlations of several correlated firms [9–11]. The developed methodology provides an efficient computational technique that is applicable in other areas of credit risk and pricing options.

Therefore, in this dissertation, we focus on the development of efficient Monte-Carlo-based computational procedures for the estimation of the FPT problem in the context of multivariate (and correlated) jump-diffusion processes, and their applications to credit risk analysis. We also discuss the implementation of the developed Monte-Carlo-based techniques for a subclass of multidimensional Lévy processes with several compound Poisson shocks. The dissertation is organized as follows, in Chapter 1, we present the background of credit risk analysis, the structural credit models, (multivariate) jump-diffusion processes and Monte-Carlo methods. Chapter 2 provides details of our mathematical model in the context of multivariate jump-diffusion processes. The methodology of the solution of the resulting problem is presented in Chapter 3. In Chapter 4, we demonstrate the applicability of the developed methodologies for simulating the multivariate jump-diffusion processes by using different parameters. In particular, by using the developed methodologies, we analyze the default rates and default correlations of differently rated firms via a set of empirical data. Conclusions are given in Chapter 5.

1.1 Credit Risk

Credit risk is risk due to uncertainty in a counterparty's (also called an obligor's or credit's) ability to meet its obligations. In assessing credit risk from a single counterparty, we must consider three issues [12]:

- **default probability**: What is the likelihood that the counterparty will default on its obligation either over the life of the obligation or over some specified horizon, such as a year? Calculated for a one-year horizon, this may be called the expected default frequency.
- **credit exposure**: In the event of a default, how large will the outstanding obligation be when the default occurs?
- **recovery rate**: In the event of a default, what fraction of the exposure may be recovered through bankruptcy proceedings or some other form of settlement?

When we speak of the credit quality of an obligation, this refers generally to the counterparty's ability to perform on that obligation. This encompasses both the obligation's default probability and anticipated recovery rate.

The term "credit analysis" is used to describe any process for assessing the credit quality of a counterparty. During the analysis, people usually assign the counterparty (or the specific obligation) a credit rating, which can be used for making credit decisions. There are three top agencies that deal with credit ratings for the investment world: Moody's, Standard and Poor's (S&P's) and Fitch IBCA. Each of these agencies aims to provide a rating system to help investors determine the risk associated with investing in a specific company, investing instrument or market. Table 1.1 shows the Standard & Poor's credit ratings. In Table 1.2, we give the equivalent ratings for different rating services. In Section 4.2 we will analyze the default events of differently rated firms which are defined according to the Moody's debt rating system in Table 1.2.

An important phenomenon that we account for in this dissertation lies with the fact that, in the market economy, individual companies are inevitably linked together via dynamically changing economic conditions [13]. Therefore, the default events of companies are often correlated, especially in the same industry. Take two firms i and j as an example, whose probabilities of default are P_i and P_j, respectively. Then the default correlation can be defined as

$$\rho_{ij} = \frac{P_{ij} - P_i P_j}{\sqrt{P_i(1-P_i)P_j(1-P_j)}}, \tag{1.1}$$

where P_{ij} is the probability of joint default.

From Eq. (1.1) we have $P_{ij} = P_i P_j + \rho_{ij}\sqrt{P_i(1-P_i)P_j(1-P_j)}$. Let us assume that $P_i = P_j = 0.05$. If these two firms are independent, i.e., the default correlation $\rho_{ij} = 0$, then the probability of joint default is $P_{ij} = 0.0025$. If the two firms are positively correlated, for example, $\rho_{ij} = 0.4$, then the probability of both firms default becomes $P_{ij} = 0.0215$ that is almost 10 times higher than in the former case. Thus, the default correlation ρ_{ij} plays a key role in the joint default with important implications in the field of credit analysis. Furthermore, default correlation analysis

Table 1.1: Standard & Poor's Credit Ratings.

AAA	Best credit quality – extremely reliable with regard to financial obligations.
AA	Very good credit quality – very reliable.
A	More susceptible to economic conditions – still good credit quality.
BBB	Lowest rating in investment grade.
BB	Caution is necessary – best sub-investment credit quality.
B	Vulnerable to changes in economic conditions – currently showing the ability to meet its financial obligations.
CCC	Currently vulnerable to nonpayment – dependent on favorable economic conditions.
CC	Highly vulnerable to a payment default.
C	Close to or already bankrupt – payment on the obligation currently continued.
D	Payment default on some financial obligation has actually occurred.

Source: Standard & Poor's.

This is the system of credit ratings Standard & Poor's applies to bonds. Ratings can be modified with + or - signs, so a AA- is a higher rating than is an A+ rating. With such modifications, BBB- is the lowest investment grade rating. Other credit rating systems are similar.

Table 1.2: Equivalent ratings for different rating services.

Equivalent Credit Ratings				
Credit Risk	Moody's*	Standard & Poor's*	Fitch IBCA†	Duff & Phelps†
Investment Grade				
Highest quality	Aaa	AAA	AAA	AAA
High quality (very strong)	Aa	AA	AA	AA
Upper medium grade (strong)	A	A	A	A
Medium grade	Baa	BBB	BBB	BBB
Not Investment Grade				
Lower medium grade (somewhat speculative)	Ba	BB	BB	BB
Low grade (speculative)	B	B	B	B
Poor quality (may default)	Caa	CCC	CCC	CCC
Most speculative	Ca	CC	CC	CC
No interest being paid or bankruptcy petition filed	C	C	C	C
In default	C	D	D	D

Source: The Bond Market Association.

*The ratings from Aa to Ca by Moody's may be modified by the addition of a 1, 2 or 3 to show relative standing within the category.

†The ratings from AA to CC by Standard & Poor's, Fitch IBCA and Duff & Phelps may be modified by the addition of a plus or minus sign to show relative standing within the category.

has many applications in asset pricing and risk management [13].

Credit risk modeling is a concept that broadly encompasses any algorithm-based methods of assessing credit risk. There are three main quantitative approaches to analyzing credit [5, 14]. In the structural approach, we make explicit assumptions about the dynamics of firm's assets, its capital structure, and its debt and share holders. A firm defaults if its assets are insufficient according to some measure. The reduced form approach is silent about why a firm defaults. Instead, the dynamics of default are exogenously given through a default rate, or intensity. The incomplete information approach [14, 15] combines the structural and reduced form models. While avoiding their difficulties, it picks the best features of both approaches: the economic and intuitive appeal of the structural approach and the tractability and empirical fit of the reduced form approach.

The structural approach gives better description of the dynamics of firm's assets after accounting for jumps. Therefore, in this dissertation, we focus mainly on the two most important components in the credit analysis: default risk and default correlation risk in the context of structural approach but incorporating with jumps. Next, we describe the structural credit models and then give the strategy of incorporating jumps into such models.

1.2 Structural Credit Models

1.2.1 Black-Merton-Scholes Model

The basis of the structural approach, which can be traced back to the influential works by Black, Scholes and Merton [16, 17], is that corporate liabilities are contingent claims on the assets of a firm. The market value of the firm is the fundamental source of uncertainty driving credit risk.

Consider a firm with market value V, which represents the expected discounted future cash flows of the firm. The firm is financed by equity and a zero coupon bond with face value K and maturity date T. The firm's contractual obligation is to repay the amount K to the bond investors at time T. Debt covenants grant bond investors absolute priority: if the firm cannot fulfil its payment obligation, then bond holders will immediately take over the firm, or in other words, the firm defaults. Hence the Merton default time τ is a discrete random variable given by

$$\tau = \begin{cases} T, & \text{if } V_T < K, \\ \infty, & \text{if else.} \end{cases} \tag{1.2}$$

To calculate the probability of default, we make assumptions about the distribution of assets at debt maturity under the physical probability P. The Black-Merton-Scholes (BMS) model for the evolution of asset prices over time is geometric Brownian motion (GBM) [14, 16, 18]:

$$\frac{dV_t}{V_t} = \mu dt + \sigma dW_t, \; V_0 > 0, \tag{1.3}$$

where $\mu \in \mathbb{R}$ is a drift parameter, $\sigma > 0$ is a volatility parameter, and W is a standard Brownian motion. Setting $m = \mu - \frac{1}{2}\sigma^2$, Ito's lemma implies that [14]

$$V_t = V_0 e^{mt + \sigma W_t}.$$

Since W_T is normally distributed with mean zero and variance T, default probabilities $p(T)$ are given by [14]

$$p(T) = P[V_T < K] = P[\sigma W_T < \log L - mT] = \Phi\left(\frac{\log L - mT}{\sigma\sqrt{T}}\right), \tag{1.4}$$

where $L = \frac{K}{V_0}$ is the initial leverage ratio and Φ is the standard normal distribution function.

The BMS model has some very strong points in its favor:

- it is consistent with stocks as limited liability securities (and so the prices never fall below zero);
- it has uncorrelated returns, which are a compelling consequence of highly efficient markets with strong statistical support over many time scales;
- it is very tractable computationally.

Until today, no model has been more widely applied in the financial world than BMS model. This model has experienced unprecedented success and eventually earned Scholes and Merton a Nobel Prize in 1997.

1.2.2 First Passage Time Approach: Black-Cox Model

In the BMS model, a firm defaults only at the maturity date of its bond, this is unfavorable to bondholders as noted by Black and Cox [19]. The indenture agreements often include safety covenants that give bond investors the right to reorganize a firm if its value falls below a given barrier. Hence, Black and Cox introduced the first passage time (FPT) approach to describe such situation (we called it as "Black-Cox model"). The first passage time is defined as the time required to cross the boundary level of specific processes. The FPT problem is of great importance in many areas, such as insurance, finance, and sequential analysis. In finance, it is a tool to compute options for which prices are depended on stopping times (first hitting times). Linetsky [20] discussed the lookback options by computing the hitting time.

In the first passage time approach, a firm defaults when its value first hits a default boundary. Suppose the default barrier D is a constant valued in $(0, V_0)$. Then the first passage default time τ

is a continuous random variable valued in $(0, \infty]$ given by

$$\tau = \inf\{t > 0 : V_t < D\}. \tag{1.5}$$

Reisz and Perlich [21] pointed out that if the barrier D is below the bond's face value K, then the above definition does not reflect economic reality anymore: it does not capture the situation when the firm is in default because $V_T < K$ although $M_T > D$ ($M_T = \min_{s \leq T} V_s$). Usually, there are two remedies to avoid this inconsistency [14].

Re-define Default

We re-define default as firm value falling below the barrier $D < K$ at any time before maturity or firm value falling below face value K at maturity. Formally, the default time is now given by

$$\tau = \min(\tau^1, \tau^2), \tag{1.6}$$

where τ^1 is the first passage of assets to the barrier D and τ^2 is the maturity time T if assets $V_T < K$ at T and ∞ otherwise. In other words, the default time is defined as the minimum of the first-passage default time (1.5) and Merton's default time (1.2). And the corresponding default probability is [14]

$$\begin{aligned} p(T) &= 1 - P[\min(\tau^1, \tau^2) > T] \\ &= \Phi\left(\frac{\log L - mT}{\sigma\sqrt{T}}\right) + \left(\frac{D}{V_0}\right)^{\frac{2m}{\sigma^2}} \Phi\left(\frac{\log(D^2/(KV_0)) + mT}{\sigma\sqrt{T}}\right). \end{aligned} \tag{1.7}$$

This default probability is obviously higher than the corresponding probability in the classical approach, which is obtained as the special case where $D = 0$.

Time-varying Default Barrier

The second way to avoid the inconsistency discussed above is to introduce a time-varying default barrier $D(t) = \kappa e^{-\gamma(T-t)}$ for all $t \leq T$ [19]. Where γ can be interpreted as the growth rate of firm's liabilities. Coefficient κ captures the liability structure of the firm and is usually defined as a firm's short-term liability plus 50% of the firm's long-term liability [13]. The firm defaults at

$$\tau = \inf\{t > 0 : V_t < D(t)\}. \tag{1.8}$$

And the default probability can be written as [14]

$$\begin{aligned} p(T) &= P[V_t < D(t) | 0 < t \leq T] \\ &= P[\min_{0<t\leq T}((m-\gamma)t + \sigma W_t) < \log L - \gamma T] \\ &= \Phi\left(\frac{\log L - mT}{\sigma\sqrt{T}}\right) + \left(Le^{-\gamma T}\right)^{\frac{2}{\sigma^2}(m-\gamma)} \Phi\left(\frac{\log L + (m-2\gamma)T}{\sigma\sqrt{T}}\right). \end{aligned} \tag{1.9}$$

1.2.3 Default Correlations for Black-Cox Model

As described in Section 1.1, in the market economy, individual companies are usually linked together via economic conditions, so default correlation, defined as the risk of multiple companies' default together, has been an important area of research in credit analysis with applications to joint default, credit derivatives, asset pricing and risk management.

Zhou [13] and Hull et al [22] were the first to incorporate default correlation into the Black-Cox first passage structural model. Zhou has proposed a first passage time model to describe default correlations of two firms under the "bivariate diffusion process":

$$\begin{bmatrix} d\ln(V_1) \\ d\ln(V_2) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} dt + \Omega \begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix}, \tag{1.10}$$

where μ_1 and μ_2 are constant drift terms, z_1 and z_2 are two independent standard Brownian motions, and Ω is a constant 2×2 matrix such that [13]

$$\Omega \cdot \Omega' = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \tag{1.11}$$

is the covariance matrix for $d\ln(V_1)$ and $d\ln(V_2)$. The coefficient ρ reflects the correlation between the movements in the asset values of the two firms. This correlation coefficient plays a critical role in determining the default correlation between the firms.

Zhou [13] and Iyengar [23] have obtained the closed form solutions for the joint probability of firm 1 to default before T_1 and firm 2 to default before T_2. However, their models cannot easily be extended to more than two assets. As for the multivariate correlated processes, the joint probability becomes very complicated and there is usually no analytical solutions for higher-dimensional processes [24, 25]. A first-order approximation is given in [24]. Furthermore, the developed models do not incorporate jump risk into the default processes.

1.3 Accounting for Jumps

1.3.1 Jump-diffusion Processes

As was pointed out in [18], the critical assumptions in the above structural models are that trading takes place continuously in time and that the price dynamics of the stock have a continuous sample path with probability one. However, these assumptions are usually not valid since continuous trading is not possible and because no empirical time series has a continuous sample path [1]. Hence, under the pure diffusion processes (only considering Brownian motion), because a sudden drop in the firm value is impossible, firms never default by surprise. The large credit spreads[1] of corporate bonds, especially those with short maturities, are unexplained in this context [5].

All of these shortfalls require much more sophisticated models to represent the market dynamics. As a matter of fact, many research efforts have been devoted to seeking for proper substitutes for the pure diffusion processes. They gave one an alternative way to look at the market and for particular markets those models may yield better explanations than the pure diffusion processes. Those research efforts can be roughly branched into developing three classes of models, namely jump-diffusion processes (JPDs) models, stochastic volatility models and local volatility models.

Jump-diffusion models are distinguished by their jump amplitude distributions, i.e., assuming that asset price changes do not follow a pure random process as described in (1.3) but also contain an unexpected "jump" component. Merton [1] was the first who solved the problem for option prices by including jumps in the price of the basic asset. According to the proposition of Merton – including jumps in the price of the basic asset, Eq. (1.3) then reads [5]

$$\begin{aligned} \frac{dV_t}{V_t} &= \mu dt + \sigma dW_t + J_t, \\ J_t &= \sum_{i=1}^{N_t} Y_i, \end{aligned} \tag{1.12}$$

where N_t follows the Poisson distribution with arrival rate λ, and $\{Y_1, Y_2, \ldots\}$ are independent and identically-distributed (i.i.d.) random variables.

The development of jump processes is focused on selecting the appropriate jump dynamics, which under Merton's proposal [1] is a normal distribution, i.e. the probability density function

[1] In finance, a credit spread, or net credit spread, is the difference in yield between different securities due to different credit quality.

(pdf) of Y is the Gaussian density

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(y-\mu)^2}{\sigma^2}}. \tag{1.13}$$

Another example is due to Kou [26] who conjectured that jumps follow a double exponential distribution, whose pdf reads

$$\begin{aligned} f_Y(y) &= p\eta_1 e^{-\eta_1 y} 1_{\{y<0\}} + q\eta_2 e^{-\eta_2 y} 1_{\{y\geq 0\}}, \\ p+q &= 1, \end{aligned} \tag{1.14}$$

where $1_{\{y<0\}}$ and $1_{\{y\geq 0\}}$ are the Lebesgue measures $\lambda(y<0) = \int_{y<0} dy$ and $\lambda(y \geq 0) = \int_{y\geq 0} dy$ on $\mathbb{R}$, respectively.

The jump-diffusion model has many nice features, such as [5]:

- The jump-diffusion model is consistent with the fact that bond prices often drop in a surprising manner at or around the time of default. Duffie et al [27] attribute this phenomenon to incomplete accounting information. That is, around the time of default, substantial accounting information about the issuer will be revealed to the market. Because of a jump in market information, bond prices jump accordingly. The jump caused by incomplete information is just a special case of many possible kinds of jumps, such as lawsuits and sudden financial turmoils.

- Jump risk can substantially raise credit spreads, especially the spreads of short-term bonds. A jump-diffusion model has the flexibility to generate a wide variety of the term structure of credit spreads, including upward sloping, flat, hump-shaped or downward sloping. In a diffusion model, some of these shapes (flat and downward-sloping) do not exist.

- In a jump-diffusion model, the remaining value of a firm at default is an endogenous random variable that is not necessarily equal to the default boundary. Thus, the model is able to endogenously generate random variations in recovery rates that are linked to a firm's capital structure and asset value at default.

1.3.2 Lévy Processes

In general, all of the jump-diffusion models fall into the family called Lévy processes, which provide the framework that accommodates all jump structures [7]. Lévy process, by definition, is a stochastic process X_t that has independent and stationary increments. As a consequence it is a Markov process and the marginal distribution of X_t is of infinite divisibility, which mathematically means that the characteristic function $\phi_{X_t}(z)$ of marginal random variable X_t can be expressed as follows [7]

$$\phi_{X_t}(z) = E[e^{izX_t}] = e^{t\psi_{X_1}(z)}, \tag{1.15}$$

where $\psi_{X_1}(z)$ is the characteristic exponent of the Lévy process at unit time. The property of infinite divisibility gives rise to the easiness of studying X_t by only looking at X_1.

The Lévy-Khinchin representation gives the mathematical expression for $\psi_{X_1}(z)$ as follows

$$\psi(z) = ibz - \frac{1}{2}\sigma^2 z^2 + \int_{-\infty}^{\infty} (e^{izx} - 1 - izx1_{|x|<1})\nu(dx), \tag{1.16}$$

where $b \in \mathbb{R}$, $\sigma \in \mathbb{R}^+$, $1_{|x|<1}$ is the Lebesgue measure $\lambda(|x| < 1) = \int_{|x|<1} dx$ on $\mathbb{R}$ and ν is a measure on $\mathbb{R}\backslash\{0\}$ with [7]

$$\int_{|x|\leq 1} |x|^2\nu(dx) < \infty, \int_{|x|\geq 1} \nu(dx) < \infty.$$

In the above representation, b is called the drift that dictates the deterministic movement along with time, σ is called the volatility that represents the continuous random walk, and ν is called the Lévy measure that describes at what arrival rate jumps may happen. Therefore, to characterize a Lévy process, one only needs to know the Lévy triplet (b, σ, ν) together.

Note that many well-known processes are special cases of Lévy process. For instance, the standard Brownian Motion can be expressed as $(0, \sigma, 0)$ where we set $b = 0$ and $\nu(x) = 0$ for all real x. And the Brownian motion with drift (Gaussian processes) can be characterized as $(b, \sigma, 0)$. If we set both b and σ to be zero and let $\nu = \lambda\delta(1)$, where $\delta(1)$ means the Dirac measure at 1, we have the Poisson process $(0, 0, \lambda\delta(1))$ with arrival rate λ.

A compound Poisson process X_t is such a Lévy process that can be considered as a Poisson process with intensity $\lambda > 0$ and random jump-sizes [7]

$$X_t = \sum_{i=1}^{N_t} Y_i, \tag{1.17}$$

where jump-sizes Y_i are i.i.d. with distribution f and N_t is a Poisson process with intensity λ, independent from Y_i. Therefore, the above jump-diffusion models can be considered as a Brownian motion plus a compound Poisson jumps [7, 28].

1.3.3 Multidimensional Models with Jumps

Apart from the pricing and hedging of options on a single asset, practically all financial applications require a multivariate model with dependence between different assets. Examples include basket option pricing, portfolio optimization, simulation of risk scenarios for portfolios. In most of these applications, jumps in the price process must be taken into account. A simple method to introduce jumps into a multidimensional model is to suppose that the stock prices do follow a multidimensional Brownian motion but time changing the Brownian motion [7, 8]. Another natural choice to introduce jumps into a multidimensional model is to utilize the compound Poisson shocks. Suppose that we want to improve a d-dimensional BMS model by allowing for "market crashes". The dates of market crashes can be modeled as arrival times of a Poisson process N_t. This leads us to the following model for the log-price processes of d assets [7]:

$$\begin{cases} X_i(t) = \mu_i t + B_i(t) + Z_i(t), \ i = 1, 2, \cdots, d, \\ Z_i(t) = \sum_{j=1}^{N_t} Y_{ij}, \end{cases} \tag{1.18}$$

where $X_i(t) = \ln[V_i(t)]$ ($V_i(t)$ is the value of asset i at time t), $B(t)$ is a d-dimensional Brownian motion with covariance matrix $\sigma = (\sigma_{ij})$ which can be written as

$$B_i(t) = \sum_{j=1}^{d} \sigma_{ij} W_j(t),$$

and $W_j(t)$ is the standard Brownian motion. For i-th asset, $\{Y_{ij}\}_{j=1}^{\infty}$ are i.i.d. $d-$dimensional random vectors which determine the sizes of jumps in individual assets during a market crash. At the j-th shock, the jump-sizes of different assets Y_{ij} may be correlated.

This model contains only one driving Poisson shock which stands for that the global market crash affecting all assets. Sometimes it is necessary to have several independent shocks to account for events that affect individual companies or individual sectors rather than the entire market. In this case we need to introduce several driving Poisson processes into the model, which now takes the following form [7]:

$$X_i(t) = \mu_i t + B_i(t) + \sum_{k=1}^{m} \sum_{j=1}^{N_t^k} Y_{ijk}, \ i = 1, 2, \cdots, d, \tag{1.19}$$

where $N_t^1, \cdots, N_t^m$ are Poisson processes driving m independent shocks and Y_{ijk} is the size of jump in i-th component after j-th shock of type k. The vectors $\{Y_{ijk}\}_{i=1}^{d}$ for different j and/or k

are independent.

1.4 Monte-Carlo Methods

In the above sections, we have described the structural credit models and how to incorporate jumps into these models. However, as soon as jumps are incorporated in the model, except for very basic applications where analytical solutions are available for the first passage time problems, for most practical cases we have to resort to numerical procedures. Examples of known analytical solutions include problems where the jump-sizes are doubly exponential or exponentially distributed [26] as well as some classes of problems where the jumps can have only nonnegative values (assuming that the crossing boundary is below the process starting value) [29]. For all other situations, numerical methods are unavoidable. Two prominent choices are Monte-Carlo methods and the numerical solution of partial integro-differential equations (PIDE) [30]. However, as the dimension of the problem grows, PIDE methods become less feasible as computational complexity for fixed precision grows exponentially with dimension. On the contrary, the complexity of Monte-Carlo methods for fixed precision grows only linearly with the dimension of the problem. Hence, in higher dimensions there is no alternative to simulation methods and it is very important to develop efficient algorithms for simulating Lévy processes [7].

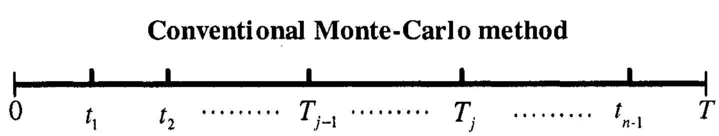

Figure 1.1: Schematic diagram of the conventional Monte-Carlo method.

In conventional Monte-Carlo methods, as shown in Fig. 1.1, we need to discretize the time horizon $[0, T]$ into small enough intervals in order to avoid discretization bias [31], and we need to evaluate the processes at each discretized time. This procedure exhibits substantial computational difficulties when applied to jump-diffusion processes. Indeed, for a typical jump-diffusion process, as shown in Fig. 1.1, let T_{j-1} and T_j be two successive jump instants. In the conventional Monte-Carlo method, even if there is no jump occurring in the interval $[T_{j-1}, T_j]$, we still need to evaluate the processes at each discretized time t in $[T_{j-1}, T_j]$. This very time-consuming procedure results in a serious shortcoming of the conventional Monte-Carlo methodology.

Therefore, many researchers have contributed to the field of enhancement of the efficiency of Monte-Carlo simulations. Among others, Kuchler et al [32] discussed the solution of stochastic differential equations (SDEs) in the framework of weak discrete time approximations and Liberati et al [33] considered the strong approximation where the SDE is driven by a high intensity Poisson process. The quasi-Monte-Carlo methods which are claimed to provide a faster convergence rate under appropriate conditions were also discussed extensively [34, 35]. Atiya and Metwally [2, 6] have recently developed a fast Monte Carlo-type numerical method to solve the FPT problem in the one-dimensional case. In our recent application, we have developed an efficient Monte-Carlo method for multivariate jump-diffusion processes and successfully applied it to analyze the default risk and default correlations of several correlated firms [9–11]. The developed methodology provides an efficient computational technique that is applicable in other areas of credit risk and pricing options.

Therefore, in this dissertation, we focus on the development of efficient Monte-Carlo-based computational procedures for the estimation of the FPT problem in the context of multivariate (and correlated) jump-diffusion processes, and their applications to credit risk analysis. We also discuss the implementation of the developed Monte-Carlo-based techniques for a subclass of multidimensional Lévy processes with several compound Poisson shocks.

1.5 Summary

In this chapter, we described the main idea of credit risk analysis, the structural credit models and how to incorporate jumps into the structural approach. However, as soon as jumps are incorporated in the model, except for very basic applications where analytical solutions are available for the first passage time problems, for most practical cases we have to resort to numerical procedures. Among numerical procedures, Monte-Carlo methods remain a primary candidate for applications, but the conventional Monte-Carlo procedure becomes computationally inefficient when it is applied to the jump-diffusion processes. Therefore, it is very important to develop efficient Monte-Carlo-type numerical methods to solve the FPT problem in the context of multivariate jump-diffusion processes. In next chapter, we focus on the default events and default correlations within the framework of multivariate jump-diffusion processes and we reduce the multidimensional formulas of default events to computable forms, which will be used to develop a fast Monte Carlo method.

Chapter 2

Mathematical Models

As mentioned in Chapter 1, when we deal with jump-diffusion stochastic processes, we usually have to resort to the application of numerical procedures. To develop efficient Monte-Carlo procedure is very important. However, the developed one-dimensional fast Monte-Carlo method [2, 6] cannot be directly generalized to the multivariate and correlated jump-diffusion case [9]. The difficulties arise from the fact that the multiple processes as well as their first passage times are indeed correlated, so the simulation must reflect the correlations of first passage times. In this dissertation, we propose a solution to circumvent these difficulties by combining the fast Monte-Carlo method of one-dimensional jump-diffusion process and the generation of correlated multidimensional variables [10, 11].

In this chapter, first we present a probabilistic description of default events and default correlations. Next, we describe the multivariate jump-diffusion processes and provide details on the first passage time distribution under the one-dimensional Brownian bridge (the sum-of-uniforms method which is used to generate correlated multidimensional variables will be described in Section 3.3.1).

Finally, we present kernel estimation in the context of our problem that can be used to represent the first passage time density function.

2.1 Default and Default Correlation

In a structural model, a firm i defaults when it can not meet its financial obligations, or in other words, when the firm assets value $V_i(t)$ falls below a threshold level $D_{V_i}(t)$. Generally speaking, finding the threshold level $D_{V_i}(t)$ is one of the challenges in using the structural methodology in the credit risk modeling, since in reality firms often rearrange their liability structure when they have credit problems. In this dissertation, we use an exponential form defining the threshold level $D_{V_i}(t) = \kappa_i \exp(\gamma_i t)$ as proposed by Black and Cox [19], where γ_i can be interpreted as the growth rate of firm's liabilities. Coefficient κ_i captures the liability structure of the firm and is usually defined as a firm's short-term liability plus 50% of the firm's long-term liability [13]. If we set $X_i(t) = \ln[V_i(t)]$, then the threshold of $X_i(t)$ is $D_i(t) = \gamma_i t + \ln(\kappa_i)$. Our main interest is in the process $X_i(t)$.

As emphasized in Section 1.1, the default correlation ρ_{ij} plays a key role in the joint default with important implications in the field of credit analysis. It measures the strength of the default relationship between different firms. Zhou [13] and Hull et al [22] were the first to incorporate default correlation into the Black-Cox first passage structural model. As it was already mentioned in Section 1.2.3, Zhou [13] has proposed a first passage time model to describe default correlations of two firms under the "bivariate diffusion process":

$$\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} dt + \Omega \begin{bmatrix} dz_1 \\ dz_2 \end{bmatrix}, \tag{2.1}$$

where μ_1 and μ_2 are constant drift terms, z_1 and z_2 are two independent standard Brownian motions, and Ω is a constant 2×2 matrix which satisfies Eq. (1.11).

The coefficient ρ reflects the correlation between the movements in the asset values of the two firms. Then the probability that firm i defaults at time t can be easily calculated as [13],

$$P_i(t) = 2 \cdot N\left(-\frac{X_i(0) - \ln(\kappa_i)}{\sigma_i \sqrt{t}}\right) = 2 \cdot N\left(-\frac{Z_i}{\sqrt{t}}\right), \tag{2.2}$$

where

$$Z_i \equiv \frac{X_i(0) - \ln(\kappa_i)}{\sigma_i}$$

is the standardized distance of firm i to its default point and $N(\cdot)$ denotes the cumulative probability distribution function for a standard normal variable.

Furthermore, if we assume that $\mu_i = \gamma_i$, then the probability that at least one firm defaults by time t can be written as [13],

$$P_{i\cup j}(t) = 1 - \frac{2r_0}{\sqrt{2\pi t}} \cdot e^{-\frac{r_0^2}{4t}} \sum_{n=1,3,\cdots} \frac{1}{n} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \left[I_{\frac{1}{2}(\frac{n\pi}{\alpha}+1)}\left(\frac{r_0^2}{4t}\right) + I_{\frac{1}{2}(\frac{n\pi}{\alpha}-1)}\left(\frac{r_0^2}{4t}\right)\right], \tag{2.3}$$

where $I_\nu(z)$ is the modified Bessel function I with order ν [1],

$$I_\nu(z) = i^{-\nu} J_\nu(iz) = i^{-\nu} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+\nu+1)} \left(\frac{iz}{2}\right)^{2m+\nu} \tag{2.4}$$

and

$$\alpha = \begin{cases} \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right), & \text{if } \rho < 0, \\ \pi + \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right), & \text{otherwise,} \end{cases}$$

[1]For large $z \gg |\nu^2 - 1/4|$, the function I becomes $I_\nu(z) \longrightarrow \frac{1}{\sqrt{2\pi z}} e^z$, while for small arguments $0 < z \ll \sqrt{\nu + 1}$, it becomes $I_\nu(z) \longrightarrow \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu$, where $\Gamma(\cdot)$ denotes the gamma function.

$$
\begin{aligned}
\theta_0 &= \begin{cases} \tan^{-1}\left(\frac{Z_2\sqrt{1-\rho^2}}{Z_1-\rho Z_2}\right), & \text{if } (\cdot) > 0, \\ \pi + \tan^{-1}\left(\frac{Z_2\sqrt{1-\rho^2}}{Z_1-\rho Z_2}\right), & \text{otherwise,} \end{cases} \\
r_0 &= Z_2/\sin(\theta_0).
\end{aligned}
$$

Then, the default correlation of these two firms is

$$
\rho_{ij}(t) = \frac{P_i(t) + P_j(t) - P_i(t)P_j(t) - P_{i\cup j}(t)}{\sqrt{P_i(t)[1-P_i(t)]P_j(t)[1-P_j(t)]}}. \tag{2.5}
$$

However, none of the above known models includes jumps in the processes. At the same time, it is well-known that jumps are a major factor in the credit risk analysis. With jumps included in such analysis, a firm can default instantaneously because of a sudden drop in its value which is impossible under a diffusion process. Zhou [5] has demonstrated the importance of jump risk in credit risk analysis of an obligor. He implemented a simulation method to show the effect of jump risk in the credit spread of defaultable bonds. He showed that the misspecification of stochastic processes governing the dynamics of firm value, i.e., falsely specifying a jump-diffusion process as a continuous Brownian motion process, can substantially understate the credit spreads of corporate bonds. Therefore, for multiple processes, considering the simultaneous jumps can be a better way to estimate the correlated default rates. The multivariate jump-diffusion processes can provide a convenient way to describe multivariate and correlated processes with jumps.

2.2 Multivariate Jump-diffusion Processes

A natural approach to introduce jumps into a multidimensional model is to utilize the compound Poisson shocks. If we want to improve a d-dimensional BMS model by allowing for "market crashes" (see Section 1.3.3), then the dates of market crashes can be modeled as arrival times of a standard Poisson process N_t. This leads us to the following model for the log-price processes of d assets as described in Eq. (1.18).

Eq. (1.18) contains only one driving Poisson shock which stands for that the global market crash affecting all assets. Sometimes it is necessary to have several independent shocks to account for events that affect individual companies or individual sectors rather than the entire market. In this case we need to introduce several driving Poisson processes into the model as described in Eq. (1.19).

2.3 First Passage Time Distribution of Brownian Bridge

Although for jump-diffusion processes, the closed form solutions are usually unavailable, yet between each two jumps the process is a Brownian bridge for one-dimensional jump-diffusion process. The authors of [6] have deduced one-dimensional first passage time distribution in time horizon $[0, T]$. In order to evaluate multiple processes, we obtain multidimensional formulas from Eq. (1.18) and reduce them to computable forms.

First, let us consider a firm i, as described by Eq. (1.18), such that its state vector X_i satisfies the following stochastic differential equation:

$$\begin{aligned} dX_i &= \mu_i dt + \sum_j \sigma_{ij} dW_j + dZ_i \\ &= \mu_i dt + \sigma_i dW_i + dZ_i, \end{aligned} \tag{2.6}$$

where W_i is a standard Brownian motion and σ_i is:

$$\sigma_i = \sqrt{\sum_j \sigma_{ij}^2}.$$

We assume that in the interval $[0, T]$, the total number of jumps for firm i is M_i. Let the jump instants be $T_1, T_2, \cdots, T_{M_i}$. Let $T_0 = 0$ and $T_{M_i+1} = T$. The quantities τ_j equal to interjump times, which are $T_j - T_{j-1}$. Following the notation of [6], let $X_i(T_j^-)$ be the process value immediately before the jth jump, and $X_i(T_j^+)$ be the process value immediately after the jth jump. The jump-size is $X_i(T_j^+) - X_i(T_j^-)$ and we can use such jump-sizes to generate $X_i(T_j^+)$ sequentially.

If we define $A_i(t)$ as the event consisting of process crossed the threshold level $D_i(t)$ for the first time in the interval $[t, t + dt]$. The conditional interjump first passage density is defined as

$$g_{ij}(t)dt = p(A_i(t) \in dt | X_i(T_{j-1}^+), X_i(T_j^-)). \tag{2.7}$$

If we only consider one interval $[T_{j-1}, T_j]$, we can obtain [6]

$$\begin{aligned} g_{ij}(t) \quad &= \quad \frac{X_i(T_{j-1}^+) - D_i(t)}{2y_i\pi\sigma_i^2}(t - T_{j-1})^{-\frac{3}{2}}(T_j - t)^{-\frac{1}{2}} \\ &* \exp\left(-\frac{[X_i(T_j^-) - D_i(t) - \mu_i(T_j - t)]^2}{2(T_j - t)\sigma_i^2}\right) \\ &* \exp\left(-\frac{[X_i(T_{j-1}^+) - D_i(t) + \mu_i(t - T_{j-1})]^2}{2(t - T_{j-1})\sigma_i^2}\right), \end{aligned} \tag{2.8}$$

where

$$y_i = \frac{1}{\sigma_i\sqrt{2\pi\tau_j}} \exp\left(-\frac{[X_i(T_{j-1}^+) - X_i(T_j^-) + \mu_i\tau_j]^2}{2\tau_j\sigma_i^2}\right).$$

After getting result in one interval, we combine the results to obtain the density for the whole interval $[0, T]$. Let $B(s)$ be a Brownian bridge in the interval $[T_{j-1}, T_j]$ with $B(T_{j-1}^+) = X_i(T_{j-1}^+)$ and $B(T_j^-) = X_i(T_j^-)$. Then the probability that the minimum of $B(s_i)$ is always above the boundary level is [6]

$$P_{ij} \quad = \quad P\left(\inf_{T_{j-1} \leq s_i \leq T_j} B(s_i) > D_i(t) | B(T_{j-1}^+) = X_i(T_{j-1}^+), B(T_j^-) = X_i(T_j^-)\right)$$

$$= \begin{cases} 1 - \exp\left(-\frac{2[X_i(T_{j-1}^+) - D_i(t)][X_i(T_j^-) - D_i(t)]}{\tau_j \sigma_i^2}\right), & \text{if } X_i(T_j^-) > D_i(t), \\ 0, & \text{otherwise.} \end{cases} \tag{2.9}$$

This implies that $B(s_i)$ is below the threshold level, which means the default happens or already happened, and its probability is $1 - P_{ij}$. Let $L(s_i) \equiv L_i$ denote the index of the interjump period in which the time s_i (first passage time) falls in $[T_{L_i - 1}, T_{L_i}]$. Also, let I_i represent the index of the first jump, which happened in the simulated jump instant [6],

$$\begin{aligned} I_i &= \min(j : X_i(T_k^-) > D_i(t); k = 1, \ldots, j, \text{ and} \\ &\qquad X_i(T_k^+) > D_i(t); k = 1, \ldots, j-1, \text{ and } X_i(T_j^+) \leq D_i(t)). \end{aligned} \tag{2.10}$$

If no such I_i exists, then we set $I_i = 0$.

By combining Eq. (2.8), (2.9) and (2.10), we get the probability of X_i crossing the boundary level in the whole interval $[0, T]$ as [6]

$$\begin{aligned} &P(A_i(s_i) \in ds | X_i(T_{j-1}^+), X_i(T_j^-), j = 1, \ldots, M_i + 1) \\ &= \begin{cases} g_{iL_i}(s_i) ds \prod_{k=1}^{L_i - 1} P_{ik} & \text{if } L_i < I_i \text{ or } I_i = 0, \\ g_{iL_i}(s_i) ds \prod_{k=1}^{L_i - 1} P_{ik} + \prod_{k=1}^{L_i} P_{ik} \delta(s_i - T_{I_i}) & \text{if } L_i = I_i, \\ 0 & \text{if } L_i > I_i, \end{cases} \end{aligned} \tag{2.11}$$

where δ is the Dirac's delta function.

2.4 The Kernel Estimator

For firm i, after generating a series of first passage times s_i, we use a kernel density estimator with Gaussian kernel to estimate the first passage time density (FPTD) f. The kernel density estimator is based on centering a kernel function of a bandwidth as follows:

$$\widehat{f} = \frac{1}{N}\sum_{i=1}^{N} K(h, t - s_i), \tag{2.12}$$

where

$$K(h, t - s_i) = \frac{1}{\sqrt{\pi/2}h} \exp\left(-\frac{(t - s_i)^2}{h^2/2}\right).$$

The optimal bandwidth in the kernel function K can be calculated as [36]:

$$h_{opt} = \left(2N\sqrt{\pi}\int_{-\infty}^{\infty}(f_t'')^2 dt\right)^{-0.2}, \tag{2.13}$$

where N is the number of generated points and f_t is the true density. Here we use the approximation for the distribution as a gamma distribution as proposed in [6]:

$$f_t = \frac{\alpha^\beta}{\Gamma(\beta)} t^{\beta-1}\exp(-\alpha t). \tag{2.14}$$

So the integral in Eq. (2.13) becomes:

$$\int_0^{\infty}(f_t'')^2 dt = \sum_{i=1}^{5}\frac{W_i\alpha_i\Gamma(2\beta - i)}{2^{(2\beta-i)}(\Gamma(\beta))^2}, \tag{2.15}$$

where

$$W_1 = A^2,\ \ W_2 = 2AB,\ \ W_3 = B^2 + 2AC,\ \ W_4 = 2BC,\ \ W_5 = C^2,$$

and

$$A = \alpha^2,\ \ B = -2\alpha(\beta - 1),\ \ C = (\beta - 1)(\beta - 2).$$

From Eq. (2.15), it follows that in order to get a nonzero bandwidth, we have to have constraint β to be at least equal to 3.

The kernel estimator can be generalized to the multivariate case. Suppose we consider multivariate processes $X = [X_1, X_2, \ldots, X_m]^\top$. Let $\overrightarrow{t} = [t_1, t_2, \ldots, t_m]$, $\overrightarrow{s_i} = [s_{1i}, s_{2i}, \ldots, s_{mi}]$, and s_{ji} $(j = 1, 2, \ldots, m)$ be the first passage time for X_j. Then, the multivariate kernel density estimator with kernel K and window width h is defined by [36]

$$\widehat{f}(\overrightarrow{t}) = \frac{1}{N} \sum_{i=1}^{N} K\left[h, (\overrightarrow{t} - \overrightarrow{s_i})\right], \tag{2.16}$$

where

$$K(\overrightarrow{t}) = (2\pi h^2)^{-m/2} \exp\left(-\frac{1}{2h^2} \overrightarrow{t}^{\top} \overrightarrow{t}\right). \tag{2.17}$$

If we approximate the true density f by a unit m-variate normal density, then the optimal bandwidth h_{opt} is [36]

$$h_{opt} = \frac{1}{N^{1/(m+4)}} \left[\frac{4}{2m+1}\right]^{1/(m+4)}. \tag{2.18}$$

2.5 Summary

In this chapter, we presented a probabilistic description of default events and default correlations. In Section 2.3, we reduced the multidimensional processes into computable one-dimensional forms. The default correlations and the probability of one firm defaults between each two jumps were obtained in Section 2.1 and 2.3, respectively. All the above results will be used to develop a fast Monte-Carlo method in next chapter. The kernel estimation can be used to represent the first passage time density function after successful simulations.

Chapter 3

The Methodology of FPT Problem Solution

In this chapter, first we analyze why the conventional Monte-Carlo method is inefficient when it is applied to jump-diffusion processes. Then we give a brief description of the (univariate) uniform sampling method in the context of recently made progress in this field [2, 6]. Next, we describe our computational procedures for multivariate (and correlated) jump-diffusion processes [10, 11]. Finally, we describe a procedure for the model calibration.

3.1 Conventional Monte-Carlo Method

Let us recall the conventional Monte-Carlo procedure in application to the analysis of the evolution of firm X_i within the time horizon $[0, T]$. We divide the time horizon into n small intervals $[0, t_1]$, $[t_1, t_2]$, $\cdots$, $[t_{n-1}, T]$ as shown in Fig. 3.1(a). In each Monte-Carlo run, we need to calculate the value of X_i at each discretized time t without explicitly distinguishing the effects of the jump and diffusion terms [34]. As usual, in order to reduce discretization bias, the number n must be large [31].

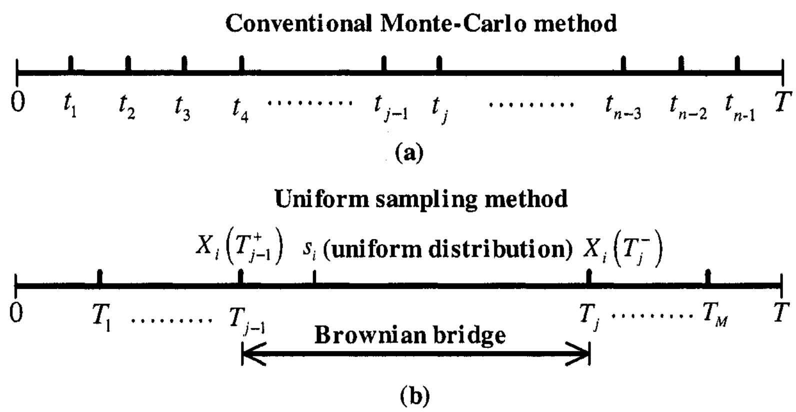

Figure 3.1: Schematic diagram of (a) the conventional Monte-Carlo and (b) the uniform sampling (UNIF) method.

If the jump-size satisfies normal distribution $N(\mu_Z, \sigma_Z)$, then the algorithms of conventional Monte-Carlo for multivariate jump-diffusion processes can be described as follows [34]

1. Generate $W_i \sim N(0,1)$, $i = 1, 2, \ldots, d$.

2. Generate $N \sim \text{Poisson}(\lambda(t_{j+1} - t_j))$.

3. Generate the jump-size $Z_i \sim N(\mu_{Z_i}, \sigma_{Z_i})$, $i = 1, 2, \ldots, d$.

4. For $i = 1, 2, \ldots, d$, set

$$X_i(t_{j+1}) = X_i(t_j) + \mu_i(t_{j+1} - t_j) + \sum_{j=1}^{d} \sigma_{ij} W_j \sqrt{t_{j+1} - t_j} + N Z_i. \tag{3.1}$$

If $X_i(t_{j+1}) < D_i(t_{j+1})$, then we get the first passage time t_{j+1}. The above algorithm is applied for every $j = 0, 1, 2, \ldots, n-1$ ($t_0 = 0$ and $t_n = T$).

3.2 Univariate Uniform Sampling Method

The conventional Monte-Carlo procedure exhibits substantial computational difficulties when applied to jump-diffusion processes. Indeed, for a typical jump-diffusion process, as shown in Fig. 3.1(b), let T_{j-1} and T_j be any successive jump instants, as described above. Then, in the conventional Monte-Carlo method, although there is no jump occurring in the interval $[T_{j-1}, T_j]$, yet we need to evaluate X_i at each discretized time t in $[T_{j-1}, T_j]$. This very time-consuming procedure results in a serious shortcoming of the conventional Monte-Carlo methodology.

To remedy the situation, two modifications of the conventional procedure were recently proposed [2, 6] that allow us a potential speed-up of the conventional methodology in 10-30 times. One of the modifications, the uniform sampling method (UNIF), involves sampling using uniform distribution. The other, inverse Gaussian density sampling is based on the inverse Gaussian density method for sampling. Both methodologies were developed for the univariate case.

The major improvement of the uniform sampling method is based on the fact that it only evaluates X_i at generated jump times, while between each two jumps the process is a Brownian bridge (see Fig. 3.1(b)). Hence, we just consider the probability of X_i crossing the threshold in (T_{j-1}, T_j) instead of evaluating X_i at each discretized time t. More precisely, in the uniform sampling method, we assume that the values of $X_i(T_{j-1}^+)$ and $X_i(T_j^-)$ are known as two end points of the Brownian bridge, the probability of firm i defaults in (T_{j-1}, T_j) is $1 - P_{ij}$ which can be computed according to Eq. (2.9). Then we generate a variable s_i from a distribution uniform in an interval $[T_{j-1}, T_{j-1} + \frac{T_j - T_{j-1}}{1 - P_{ij}}]$. If the generated point s_i falls in the interjump interval $[T_{j-1}, T_j]$, then we have successfully generated a first passage time s_i and can neglect the other intervals and perform another Monte-Carlo run. On the other hand, if the generated point s_i falls outside the interval $[T_{j-1}, T_j]$ (which happens with probability P_{ij}), then that point is "rejected". This means no boundary crossing has occurred in the interval, and we proceed to the next interval and repeat the whole process again.

Note that the generated s_i is not obtained according to the conditional boundary crossing density $g_{ij}(s_i)$ as described by Eq. (2.8). In order to obtain an appropriate density estimate, the authors of [6] proposed that the right hand side summation in Eq. (2.12) can be viewed as a finite sample estimate of the following:

$$\begin{aligned} E_{g_{ij}(s_i)}[K(h, t-s_i)] &\equiv \int_{T_{j-1}}^{T_j} g_{ij}(s_i)K(h, t-s_i)ds_i \\ &= \left(\frac{T_j - T_{j-1}}{1 - P_{ij}}\right) E_{U(s_i)}[g_{ij}(s_i)K(h, t-s_i)], \end{aligned} \tag{3.2}$$

where $E_{g_{ij}(s_i)}$ means the expectation of s_i, where s_i obeys the density $g_{ij}(s_i)$. $U(s_i)$ is the uniform density in $[T_{j-1}, T_{j-1} + \frac{T_j - T_{j-1}}{1-P_{ij}}]$ from which we sample the point s_i. Therefore, we should weight the kernel with $\left(\frac{T_j - T_{j-1}}{1-P_{ij}}\right) g_{ij}(s_i)$ to obtain an estimate for the true density.

3.3 Multivariate Methodology

In what follows, we focus on the further development of the uniform sampling method and extend it to multivariate and correlated jump-diffusion processes. In order to implement the UNIF method for our multivariate model as described in Eq. (1.18), we need to consider several points:

1. We assume that the arrival rate λ for the Poisson jump process and the distribution of $(T_j - T_{j-1})$ are the same for each firm. As for the jump-size, we generate them by a given distribution which can be different for different firms to reflect specifics of the jump process for each firm.

2. We exemplify our description by considering an exponential distribution (mean value μ_T) for $(T_j - T_{j-1})$ and a normal distribution (mean value μ_J and standard deviation σ_J) for the jump-size. We can use any other distribution when appropriate.

3. An array `IsDefault` (whose size is the number of firms denoted by N_{firm}) is used to indicate whether firm i has defaulted in this Monte-Carlo run. If the firm defaults, then we set $\texttt{IsDefault}(i) = 1$, and will not evaluate it during this Monte-Carlo run.

4. Most importantly, as we have mentioned before, the default events of firm i are inevitably correlated with other firms, for example firm $i+1$. The default correlation of firms i and $i+1$ is described by Eq. (2.5). Hence, firm i's first passage time s_i is indeed correlated with s_{i+1} – the first passage time of firm $i + 1$. We must generate several correlated s_i in each interval $[T_{j-1}, T_{j-1} + \frac{T_j - T_{j-1}}{1 - P_{ij}}]$ which is the key point for multivariate correlated processes.

Note that the assumption based on using the same arrival rate λ and distribution of $(T_j - T_{j-1})$ for different firms may seem to be quite idealized. One may argue that the arrival rate λ for the Poisson jump process should be different for different firms, which implies that different firms endure different jump rates. However, if we consider the real market economy, once a firm (called firm "A") encounters sudden economic hazard, its correlated firms may also endure the same hazard. Furthermore, it is common that other firms will help firm "A" to pull out, which may result in a simultaneous jump for them. Therefore, as a first step, it is reasonable to employ the simultaneous jumps' processes for all the different firms.

Next, we will give a brief description of the sum-of-uniforms (SOU) method which is used to generate correlated uniform random variables, followed by the description of the multivariate and correlated UNIF method and the model calibration.

3.3.1 Sum-of-uniforms Method

In the above sections, we have reduced the solution of the original problem to a series of one-dimensional jump-diffusion processes as described by Eq. (2.6). The first passage time distribution in an interval $[T_{j-1}, T_j]$ (between two successive jumps) was obtained in Section 2.3. As mentioned, the default events of firm i are inevitably correlated with other firms, for example firm $i+1$. We approximate the correlation of s_i and s_{i+1} as the default correlation of firms i and $i+1$ by the following formula:

$$\rho(s_i, s_{i+1}) \approx \rho_{i,i+1}(t) = \frac{P_i(t) + P_{i+1}(t) - P_i(t)P_{i+1}(t) - P_{i\cup i+1}(t)}{\sqrt{P_i(t)[1-P_i(t)]P_{i+1}(t)[1-P_{i+1}(t)]}}, \tag{3.3}$$

where t can be chosen as the midpoint of the interval $[T_{j-1}, T_j]$.

Note that each process is a Brownian motion in the interval $[T_{j-1}, T_j]$, so we can compute the correlation coefficient $\rho_{i,i+1}$ by using Zhou's model without jumps [13] and then use this value for modeling correlated s_i and s_{i+1}. More explicitly, in Eq. (3.3), $P_i(t)$ and $P_{i+1}(t)$ can be computed by using Eq. (2.2), and $P_{i\cup i+1}(t)$ can be obtained by using Eq. (2.3).

Therefore, we need to generate several correlated s_i in $[T_{j-1}, T_{j-1} + \frac{T_j - T_{j-1}}{1-P_{ij}}]$ whose correlations can be described by Eq. (3.3). Let us introduce a new variable $b_{ij} = \frac{T_j - T_{j-1}}{1-P_{ij}}$. Then we have $s_i = b_{ij}Y_i + T_{j-1}$, where Y_i are uniformly distributed in $[0,1]$. Moreover, the correlation of Y_i and Y_{i+1} is given by $\rho(s_i, s_{i+1}) \approx \rho_{i,i+1}(t)$.

Now we can generate the correlated uniform random variables $Y_1, Y_2, \cdots$ by using the sum-of-uniforms (SOU) method [37, 38] in the following steps:

1. Generate Y_1 from numbers uniformly distributed in $[0,1]$.

2. For $i = 2, 3, \cdots$, generate $W_i \sim U(0, c_{i-1,i})$, where $U(0, c_{i-1,i})$ denotes a uniform random number over range $(0, c_{i-1,i})$. Chen [37] has obtained the relationship of parameter $c_{i-1,i}$ and the correlation $\rho(s_{i-1}, s_i)$ (abbreviated as $\rho_{i-1,i}$) as follows:

Correlation	$c_{i-1,i} \geq 1$	$c_{i-1,i} < 1$
$\rho_{i-1,i} \geq 0$	$\frac{1}{c_{i-1,i}} - \frac{0.3}{c_{i-1,i}^2}(\rho_{i-1,i} \leq 0.7)$	$1 - 0.5c_{i-1,i}^2 + 0.2c_{i-1,i}^3(\rho_{i-1,i} \geq 0.7)$
$\rho_{i-1,i} \leq 0$	$-\frac{1}{c_{i-1,i}} + \frac{0.3}{c_{i-1,i}^2}(\rho_{i-1,i} \geq -0.7)$	$-1 + 0.5c_{i-1,i}^2 - 0.2c_{i-1,i}^3(\rho_{i-1,i} \leq -0.7)$

If Y_{i-1} and Y_i are positively correlated, then let

$$Z_i = Y_{i-1} + W_i.$$

If Y_{i-1} and Y_i are negatively correlated, then let

$$Z_i = 1 - Y_{i-1} + W_i.$$

Let $Y_i = F(Z_i)$, where for $c_{i-1,i} \geq 1$,

$$F(Z) = \begin{cases} Z^2/(2c_{i-1,i}), & 0 \leq Z \leq 1, \\ (2Z-1)/(2c_{i-1,i}), & 1 \leq Z \leq c_{i-1,i}, \\ 1 - (1 + c_{i-1,i} - Z)^2/(2c_{i-1,i}), & c_{i-1,i} \leq Z \leq 1 + c_{i-1,i}, \end{cases}$$

and for $0 < c_{i-1,i} \leq 1$,

$$F(Z) = \begin{cases} Z^2/(2c_{i-1,i}), & 0 \leq Z \leq c_{i-1,i}, \\ (2Z - c_{i-1,i})/2, & c_{i-1,i} \leq Z \leq 1, \\ 1 - (1 + c_{i-1,i} - Z)^2/(2c_{i-1,i}), & 1 \leq Z \leq 1 + c_{i-1,i}. \end{cases}$$

3.3.2 Multivariate Uniform Sampling Method

In this subsection, we will describe our algorithm for multivariate jump-diffusion processes, which is an extension of the one-dimensional case developed earlier by other authors (e.g. [2, 6]).

Consider N_{firm} firms in the given time horizon $[0, T]$. First, we generate the jump instant T_j by generating interjump times $(T_j - T_{j-1})$ and set all the $\texttt{IsDefault}(i) = 0 (i = 1, 2, \cdots, N_{\text{firm}})$ to indicate that no firm defaults at first.

From Fig. 3.1(b) and Eq. (2.6), we can conclude that for each process X_i we can make the following observations:

1. If no jump occurs, as described by Eq. (2.6), the interjump size $(X_i(T_j^-) - X_i(T_{j-1}^+))$ follows a normal distribution of mean $\mu_i(T_j - T_{j-1})$ and standard deviation $\sigma_i\sqrt{T_j - T_{j-1}}$. We get

$$\begin{aligned} X_i(T_j^-) &\sim X_i(T_{j-1}^+) + \mu_i(T_j - T_{j-1}) + \sigma_i N(0, T_j - T_{j-1}) \\ &\sim X_i(T_{j-1}^+) + \mu_i(T_j - T_{j-1}) + \sum_{k=1}^{N_{\text{firm}}} \sigma_{ik} N(0, T_j - T_{j-1}), \end{aligned}$$

   where the initial state is $X_i(0) = X_i(T_0^+)$.

2. If jump occurs, we simulate the jump-size by a normal distribution or another distribution when appropriate, and compute the postjump value:

$$X_i(T_j^+) = X_i(T_j^-) + Z_i(T_j).$$

This completes the procedure for generating beforejump and postjump values $X_i(T_j^-)$ and $X_i(T_j^+)$. As before, $j = 1, \cdots, M$ where M is the total number of jumps for all the firms. We compute P_{ij} according to Eq. (2.9). To recur the first passage time density (FPTD) $f_i(t)$, we have to consider three possible cases that may occur for each non-default firm i:

1. **First passage happens inside the interval.** We know that if $X_i(T_{j-1}^+) > D_i(T_{j-1})$ and $X_i(T_j^-) < D_i(T_j)$, then the first passage happened in the time interval $[T_{j-1}, T_j]$. To evaluate when the first passage happened, we introduce a new viable b_{ij} as $b_{ij} = \frac{T_j - T_{j-1}}{1-P_{ij}}$. We generate several correlated uniform numbers Y_i by using the SOU method as described in Section 3.3.1, then compute $s_i = b_{ij}Y_i + T_{j-1}$. If s_i belongs to interval $[T_{j-1}, T_j]$, then the first passage time occurred in this interval. We set $\texttt{IsDefault}(i) = 1$ to indicate firm i has defaulted and compute the conditional boundary crossing density $g_{ij}(s_i)$ according to Eq. (2.8). To get the density for the entire interval $[0, T]$, we use $\widehat{f}_{i,n}(t) = \left(\frac{T_j - T_{j-1}}{1-P_{ij}}\right) g_{ij}(s_i) * K(h_{opt}, t - s_i)$, where n is the iteration number of the Monte-Carlo cycle.

2. **First passage does not happen in this interval.** If s_i does not belong to interval $[T_{j-1}, T_j]$, then the first passage time has not yet occurred in this interval.

3. **First passage happens at the right boundary of the interval.** If $X_i(T_j^+) < D_i(T_j)$ and $X_i(T_j^-) > D_i(T_j)$ (see Eq. (2.10)), then T_{I_i} is the first passage time and $I_i = j$, we evaluate the density function using kernel function $\widehat{f}_{i,n}(t) = K(h_{opt}, t - T_{I_i})$, and set $\texttt{IsDefault}(i) = 1$.

Next, we increase j and examine the next interval and analyze the above three cases for each non-default firm again. After running N times Monte-Carlo cycle, we get the FPTD of firm i as $\widehat{f}_i(t) = \frac{1}{N}\sum_{n=1}^{N} \widehat{f}_{i,n}(t)$.

In the UNIF method, the multidimensional density estimate involves the evaluation of joint conditional interjump first passage time density. This problem can be divided into several one-dimensional density estimates if the processes are non-correlated [9],

$$\widehat{f}_n(\overrightarrow{t}) = \Pi_{i=1}^{m} \left(\frac{T_j - T_{j-1}}{1 - P_{ij}} \right) g_{ij}(s_i) * K(h_{opt}, \overrightarrow{t} - \overrightarrow{s}).$$

As for the multivariate correlated processes, the joint conditional interjump first passage time density becomes very complicated and there are usually no analytical solutions for higher-dimensional processes [24, 25]. We will not consider this problem in the current dissertation.

3.4 Generalizations

In the above sections, we assume that the arrival rate λ for the Poisson jump process and the distribution of $(T_j - T_{j-1})$ are the same for each firm which means there is only one driving Poisson shock and the global market crash affecting all firms. Sometimes it is necessary to have several independent shocks to account for events that affect individual companies or individual sectors rather than the entire market as described in Eq. (1.19). Hence, our next goal is to generalize the developed multivariate uniform sampling method (see Section 3.3.2) to the case of several independent Poisson shocks.

Suppose we have d firms which are driven by m independent Poisson shocks $N_t^1, \cdots, N_t^m$. The jump components are

$$Z_i(t) = \sum_{k=1}^{m} \sum_{j=1}^{N_t^k} Y_{ijk},$$

where Y_{ijk} is the size of jump for i-th firm after j-th shock of type k. The vectors $\{Y_{ijk}\}_{i=1}^{d}$ for different j and/or k are independent.

Let M_k be the number of jumps for each Poisson shock N_t^k, $M = \sum_{k=1}^{m} M_k$ is the total number of jumps. We generate the jump instant $T_{j,k}$ by generating the interjump times $(T_{j,k} -$

$T_{j-1,k}$) for each Poisson shock. Then we sort the jump instant $T_{j,k}$ by the relevant ascending order and still denote them as T_j ($j = 1, 2, \cdots, M$). Furthermore, an array ShockType (whose size is M) is used to record the type of shock at T_j. Then we can carry out multivariate uniform sampling method for this case as same as in Section 3.3.2, but the postjump value should be calculated as:

$$X_i(T_j^+) = X_i(T_j^-) + Y_i(T_j),$$

where $Y_i(T_j)$ is the size of jump for i-th firm at T_j, the type of shock is determined by the array ShockType. Besides, we may generate correlated $Y_i(T_j)$ for different firms.

3.5 Model Calibration

In order to apply our developed procedure, first we need to calibrate the developed model, in other words, to numerically choose or optimize the parameters, such as drift, volatility and jumps to fit the most liquid market data. This can be done by applying the least-square method, minimizing the root mean square error (*rmse*) given by:

$$rmse = \sqrt{\sum_{\text{derivatives}} \frac{(\text{Market price} - \text{Model price})^2}{\text{Number of derivatives}}}.$$

Luciano et al [39] have used a set of European call options $C(k, T)$ as their model price to calibrate their model parameters.

However, it will be demonstrated in Section 4.2, for a number of practically interesting cases, there is no option value that can be used to calibrate our model, so we have to use the historical default data to optimize the parameters in the model. As mentioned in Sections 2.4 and 3.3.2, after Monte-Carlo simulation we obtain the estimated density $\widehat{f}_i(t)$ by using the kernel estimator method.

The cumulative default rates for firm i in our model is defined as,

$$P_i(t) = \int_0^t \widehat{f}_i(\tau) d\tau. \tag{3.4}$$

Then we minimize the difference between our model and historical default data $\widetilde{A}_i(t)$ to obtain the optimized parameters in the model (such as σ_{ij}, arrival intensity λ in Eq. (2.6)):

$$\operatorname{argmin} \left(\sum_i \sqrt{\sum_{t_j} \left(\frac{P_i(t_j) - \widetilde{A}_i(t_j)}{t_j} \right)^2} \right). \tag{3.5}$$

3.6 Summary

Based on the computable formulas obtained in Chapter 2 and the sum-of-uniforms method (see Section 3.3.1), we developed a fast Monte-Carlo method for multivariate jump-diffusion processes in Section 3.3.2. Furthermore, we also discussed the implementation of the developed Monte-Carlo-based technique for a subclass of multidimensional Lévy processes with several compound Poisson shocks in Section 3.4. The model calibration procedure was explained in Section 3.5. In the next chapter, we provide examples of the applications of the developed Monte-Carlo-based technique, focusing on the credit risk analysis.

Chapter 4

Applications

In the above chapters, we have constructed the multivariate jump-diffusion processes model and developed an efficient Monte-Carlo-based computational procedure for multivariate cases. In this chapter, first we demonstrate the applicability of the developed methodology for simulating the multivariate jump-diffusion processes by using different parameters. The comparison between the developed methodology and the conventional Monte-Carlo method (see Section 3.1) confirms the validity and efficiency of the developed fast method. Next, by using the developed methodologies, we analyze the default rates and default correlations of differently rated firms via historical data. The successful applications indicate that the developed technique provides an efficient tool for a number of other applications, including credit risk and option pricing.

4.1 Non-correlated Processes

In this section, we will provide results of numerical simulations based on the developed multivariate UNIF method applied to the two-dimensional case. We use three representative examples with different arrival rates $\lambda = 1, 3, 8$ for the Poisson jump process to judge the efficiency of our algorithms. In order to investigate the dynamic behavior of different quantity processes, we choose the parameters as follows in which the second process is easier to cross the boundary level compared with the first one,

$$X_0 = [0, 0]^\top,\ D(t) = [\ln(0.9) - 0.002t, \ln(0.95) - 0.012t]^\top$$

$$\mu = [-0.002, -0.012]^\top,\ \sigma = \begin{bmatrix} 0.2 & 0.0 \\ 0.0 & 0.2 \end{bmatrix},$$

$$\mu_Z = [0, 0]^\top,\ \sigma_Z = [0.2, 0.12]^\top,$$

where X_0 is the starting value for the process, $D(t)$ is the threshold, μ is the constant instantaneous drift, σ represents the Brownian motion, and μ_Z and σ_Z are the mean and standard deviations of the jump-sizes, respectively.

The simulation was carried out with total Monte-Carlo runs $N = 100,000$ in horizon $[0, 1]$. Moreover, we have also carried out a conventional Monte-Carlo simulation (see algorithms in Section 3.1) with the same parameters and the discretization size of time horizon $\Delta = 0.0002$. All the simulations were carried out by using Scilab software [40] on SHARCNET (Shared Hierarchical Academic Research Computing Network). The estimated first passage time density functions are given in Fig. 4.1 to 4.3 for different arrival rates λ.

In every figure, the top two plots are one-dimensional first passage time density functions of

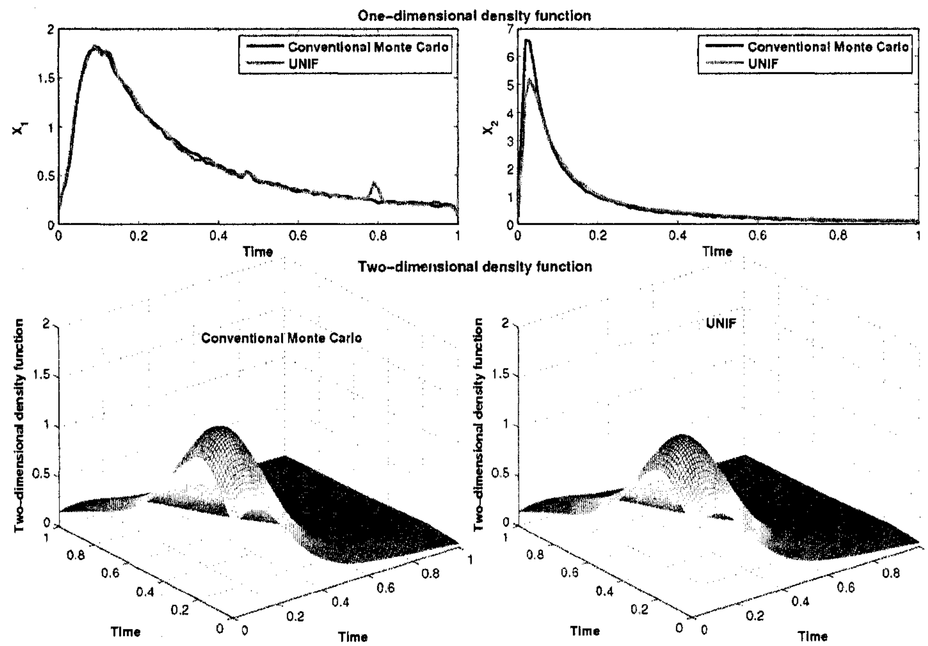

Figure 4.1: One-dimensional (top) and two-dimensional (bottom) density functions with $\lambda = 1$. The simulations were performed with Monte-Carlo runs $N = 100,000$, for the conventional Monte Carlo method, the discretization size of time horizon was $\Delta = 0.0002$.

two processes X_1 and X_2, obtained by using the conventional Monte-Carlo method and the UNIF method. The bottom two plots are two-dimensional first passage time density functions of processes X_1 and X_2. We observe that the multivariate UNIF method gives practically identical one- and two-dimensional density functions to those obtained with the conventional Monte-Carlo procedure which confirms the validity the developed multivariate UNIF method.

Furthermore, to compare results more accurately, we can obtain the mean first passage time of each process X_i in the interval $[0, T]$ as

$$\langle \tau \rangle_i = \int_0^T t\widehat{f}_i(t)dt. \tag{4.1}$$

The simulated mean first passage time of each process X_i is provided in Table 4.1.

Table 4.1: The calculated mean first passage time of processes X_1 and X_2. The simulations were performed with Monte-Carlo runs $N = 100,000$, for the conventional Monte-Carlo method, the discretization size of time horizon was $\Delta = 0.0002$.

		X_1	X_2
Example 1	CMC	0.201385	0.146695
	UNIF	0.204265	0.157412
Example 2	CMC	0.194684	0.142038
	UNIF	0.194943	0.147486
Example 3	CMC	0.174958	0.132100
	UNIF	0.176625	0.132649

From the results shown in Table 4.1, we notice that when the arrival rate λ increases, which indicates more sudden shocks happened in real market for each process, the mean first passage time will shift towards zero, as well as the peak of the estimated density function. This means that the intensity of jumps will affect the probability of crossing threshold for each process and this will be useful to describe the unexpected event, such as a sharp increase in oil price or a financial storm.

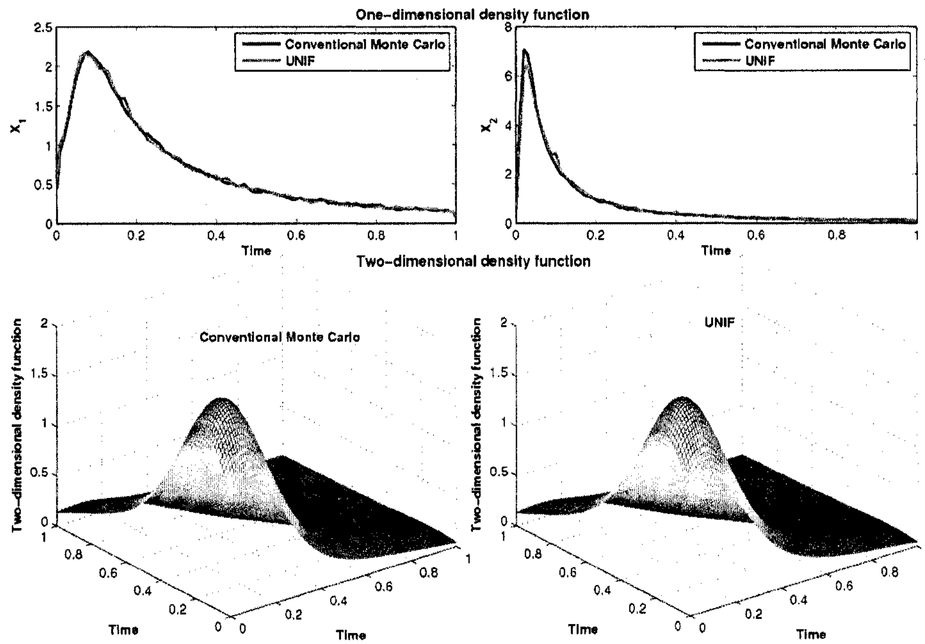

Figure 4.2: One-dimensional (top) and two-dimensional (bottom) density functions with $\lambda = 3$. The simulations were performed with Monte-Carlo runs $N = 100,000$, for the conventional Monte Carlo method, the discretization size of time horizon was $\Delta = 0.0002$.

We also draw another important conclusion from Table 4.1 when we compare the results between X_1 and X_2. The increase of λ will affect the first passage time density function of X_1 much more than that of X_2. For example, when λ increases from 1 to 8, the mean of first passage time of X_1 has changed 3%, from 0.20 to 0.17, but the mean of first passage time of X_2 has only changed 1%, from 0.14 to 0.13. Since X_1 is a high quantity process, the probability of X_1 crossing the

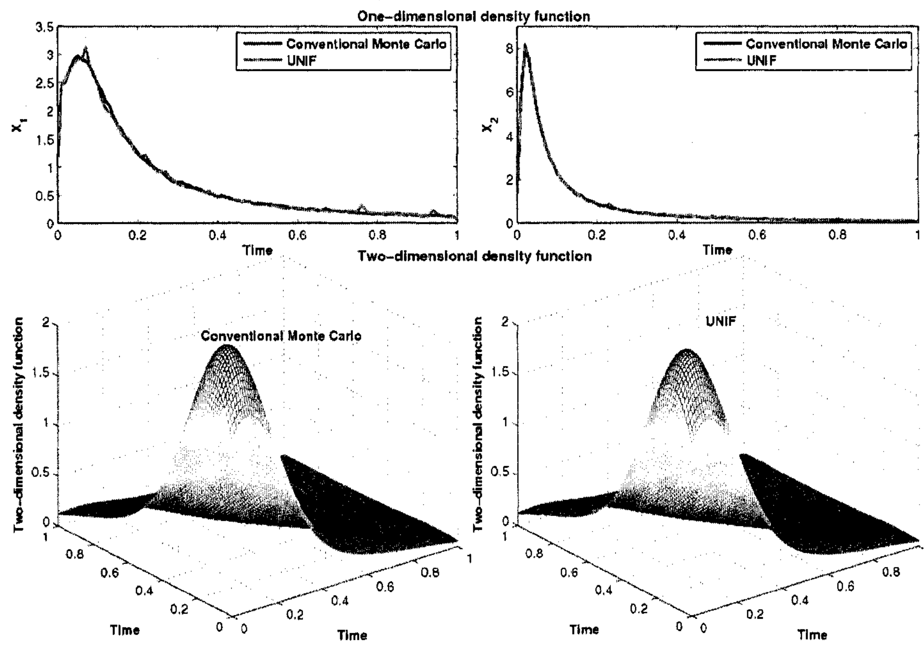

Figure 4.3: One-dimensional (top) and two-dimensional (bottom) density functions with $\lambda = 8$. The simulations were performed with Monte-Carlo runs $N = 100,000$, for the conventional Monte Carlo method, the discretization size of time horizon was $\Delta = 0.0002$.

threshold is low. Since X_2 is a low quality process, we can see that the arrival rate λ has much more influence on the high quality process and increases its probability of crossing threshold.

As for two correlated processes, one important issue is the probability or density function $\widehat{f}(t,t)$ of crossing the critical level together at the same time t. We have also plotted $\widehat{f}(t,t)$ in Fig. 4.4. Observe that the UNIF method gives profile of $\widehat{f}(t,t)$ identical to the conventional methodology. Moreover, from Fig. 4.4, we can deduce that the peak of $\widehat{f}(t,t)$ will become larger with increasing λ. This indicates that the increasing arrival rate λ also affects $\widehat{f}(t,t)$ quite substantially, and results in higher joint probabilities.

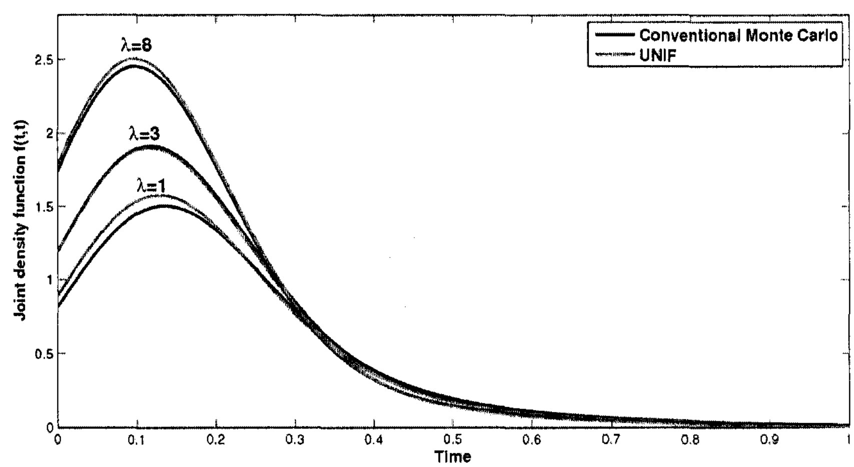

Figure 4.4: Estimated density function $\widehat{f}(t,t)$ for different λ. The simulations were performed with Monte-Carlo runs $N = 100,000$, for the conventional Monte-Carlo method, the discretization size of time horizon was $\Delta = 0.0002$.

In Table 4.2, we provide the calculated optimal bandwidths and the corresponding CPU times. As seen from Table 4.2, the multivariate UNIF approach is much more efficient compared to the conventional Monte-Carlo method.

Table 4.2: The optimal bandwidth h_{opt} and CPU time per Monte-Carlo run of the simulations. The first h_{opt} is for X_1 and the second for X_2. The simulations were performed with Monte-Carlo runs $N = 100,000$, for the conventional Monte-Carlo (CMC) method, the discretization size of time horizon was $\Delta = 0.0002$.

		Optimal bandwidth		CPU time
		X_1	X_2	
Example 1	CMC	0.012664	0.006943	0.286642
	UNIF	0.016030	0.013880	0.000527
Example 2	CMC	0.011157	0.006582	0.284554
	UNIF	0.012249	0.009443	0.000731
Example 3	CMC	0.008894	0.005921	0.299156
	UNIF	0.009117	0.006542	0.001222

4.2 Controlling Credit Risk

In Section 4.1, we have confirmed the validity and efficiency of the developed multivariate UNIF method. In this section, we demonstrate the developed methodology at work for analyzing the default events of multiple correlated firms via a set of historical default data.

4.2.1 Density Function and Default Rate

First, for completeness, let us consider a set of historical default data of differently rated firms given in Table 4.3. A, Baa, Ba and B stand for differently rated firms following the Moody's Investors Service's definition (see Section 1.1). Our first task is to describe the first passage time density functions and default rates of these firms.

Since there is no option value that can be used, we will employ Eq.(3.5) to optimize the parameters in our model. For convenience, we reduce the number of optimizing parameters by:

1. Setting $X(0) = 2$ and $\ln(\kappa) = 0$.

2. Setting the growth rate γ of debt value equivalent to the growth rate μ of the firm's value [13], so the default of firm is non-sensitive to μ. In our computations, we set $\mu = -0.001$.

3. The interjump times $(T_j - T_{j-1})$ satisfy an exponential distribution with mean value equals to 1.

4. The arrival rate for jumps satisfies the Poisson distribution with intensity parameter λ, where the jump-size is a normal distribution $Z_t \sim N(\mu_Z, \sigma_Z)$.

As a result of these assumptions, we only need to optimize σ, λ, μ_Z, σ_Z for each firm. This is done by minimizing the differences between our simulated default rates and historical data. More-

Table 4.3: Historical cumulative default rates of differently rated firms (%), 1970-93.

Year	A	Baa	Ba	B
1	0.01	0.16	1.79	8.31
2	0.09	0.51	4.38	14.85
3	0.28	0.91	6.92	20.38
4	0.46	1.46	9.41	24.78
5	0.62	1.97	11.85	28.38
6	0.83	2.46	13.78	31.88
7	1.06	3.09	15.33	34.32
8	1.31	3.75	16.75	36.71
9	1.61	4.39	18.14	38.38
10	1.96	4.96	19.48	39.96
11	2.30	5.56	20.84	41.08
12	2.65	6.19	22.22	41.74
13	2.99	6.77	23.54	42.45
14	3.29	7.44	24.52	43.04
15	3.62	8.16	25.46	43.70
16	3.95	8.91	26.43	44.43
17	4.26	9.69	27.29	45.27
18	4.58	10.45	28.06	45.58
19	4.96	11.07	28.88	45.58
20	5.23	11.70	29.76	45.58

over, as mentioned above, we will use the same arrival rate λ and distribution of $(T_j - T_{j-1})$ for differently rated firms, so we first optimize four parameters for, e.g., the A-rated firm, and then set the parameter λ of other three firms the same as A's.

The minimization was performed by using the quasi-Newton procedure implemented as a Scilab program. The optimized parameters for each firm are provided in Table 4.4.

Table 4.4: Optimized parameters for differently rated firms by using the UNIF method. The optimization was performed by using the quasi-Newton procedure implemented as a Scilab program. In each step of the optimization, we choose the Monte-Carlo runs $N = 50,000$.

	σ	λ	μ_Z	σ_Z
A	0.0900	0.1000	-0.2000	0.5000
Baa	0.0894	0.1000	-0.2960	0.6039
Ba	0.1587	0.1000	-0.5515	1.6412
B	0.4500	0.1000	-0.8000	1.5000

By using these optimized parameters, we carried out the final simulation with Monte-Carlo runs $N = 500,000$. The estimated first passage time density functions of these four firms are shown in Fig. 4.5. The simulated cumulative default rates (lines) together with historical data (squares) are given in Fig. 4.6. The theoretical data, denoted by circles in Fig. 4.6, were computed by using Eq. (2.2) where Z_i were evaluated in [13] as 8.06, 6.46, 3.73 and 2.10 for A-, Baa-, Ba- and B-rated firms, respectively. In Table 4.5, we give the optimal bandwidth and parameters α, β for the true density estimate.

Based on these results, we conclude that:

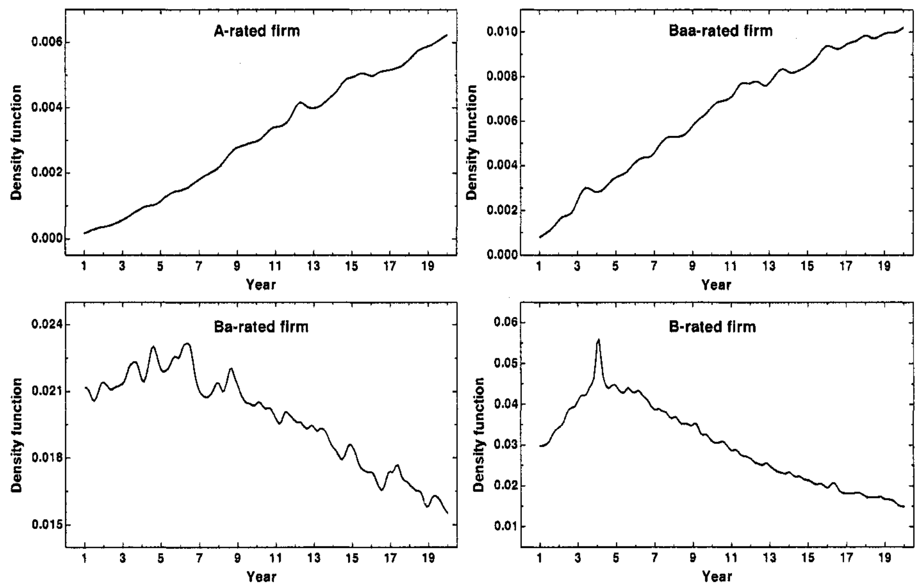

Figure 4.5: Estimated density functions for differently rated firms. All the simulations were performed with Monte-Carlo runs $N = 500,000$.

Table 4.5: The optimal bandwidth h_{opt}, parameters α, β for the true density estimate of differently rated firms. All the simulations were performed with Monte-Carlo runs $N = 500,000$.

	α	β	Optimal bandwidth
A	0.206699	3	0.655522
Baa	0.219790	3	0.537277
Ba	0.252318	3	0.382729
B	0.327753	3	0.264402

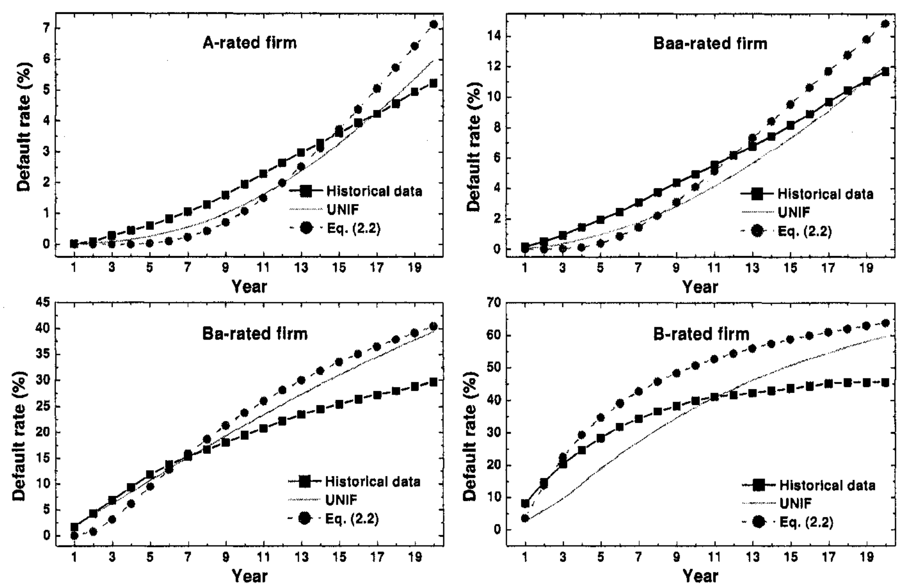

Figure 4.6: Historical (squares), theoretical (circles) and simulated (lines) cumulative default rates for differently rated firms. All the simulations were performed with Monte-Carlo runs $N = 500,000$.

1. From Fig. 4.6, we can see that the simulations give similar or better results to the analytical results predicted by Eq. (2.2). Note that the difference between our simulations and Eq. (2.2) is that Eq. (2.2) does not consider jumps. Hence including jumps may improve the models to describe the practical market.

2. A- and Baa-rated firms have a smaller Brownian motion part. Their parameters σ are much smaller than those of Ba- and B-rated firms.

3. The optimized parameters σ of A- and Baa-rated firms are similar, but the jump parts (μ_Z, σ_Z) are different, which explains their different cumulative default rates and density functions. Indeed, Baa-rated firm may encounter more severe economic hazard (large jump-size) than A-rated firm.

4. As for Ba- and B-rated firms, except for the large σ, both of them have large μ_Z and especially large σ_Z, which indicate that the loss due to sudden economic hazard may fluctuate a lot for these firms. Hence, the large σ, μ_Z and σ_Z account for their high default rates and low credit qualities.

5. From Fig. 4.5, we can conclude that the density functions of A- and Baa-rated firms still have the trend to increase, which means the default rates of A- and Baa-rated firms may increase little faster in future. As for Ba- and B-rated firms, their density functions have decreased, so their default rates may increase very slowly or be kept at a constant level. Mathematically speaking, the cumulative default rates of A- and Baa-rated firms are convex function, while the cumulative default rates of Ba- and B-rated firms are concave.

4.2.2 Correlated default

In the market economy, individual companies are inevitably linked together via dynamically changing economic conditions [13]. Therefore, the default events of companies are often correlated, especially in the same industry. Our final example concerns with the correlated defaults of two firms[1], whose state vectors can be described as

$$\begin{bmatrix} X_1(t) \\ X_2(t) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} dt + \sigma \begin{bmatrix} dW_1 \\ dW_2 \end{bmatrix} + \begin{bmatrix} dZ_1 \\ dZ_2 \end{bmatrix}, \tag{4.2}$$

where

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix} \tag{4.3}$$

and

$$\sigma\sigma^{\top} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \tag{4.4}$$

is the covariance matrix. In Eq. (4.4), ρ reflects the correlation of diffusion parts of the state vectors of the two firms.

The default correlation of these two firms is,

$$\rho_{12}(t) = \frac{P_1(t) + P_2(t) - P_1(t)P_2(t) - P_{1\cup 2}(t)}{\sqrt{P_1(t)[1 - P_1(t)]P_2(t)[1 - P_2(t)]}}, \tag{4.5}$$

where $P_i(t)$ $(i = 1, 2)$ is the probability that firm i defaults at time t, $P_{1\cup 2}(t)$ is the probability that at least one firm defaults by time t.

[1]The two firms may belong to the same rated category, e.g., both of them are A-rated firms. Therefore, from Tables 4.6 through 4.9, the default correlation of (A,A) stands for two different firms but both of them belong to the A-rated category, the same as (Baa,Baa), (Ba,Ba) and (B,B).

The default correlation $\rho_{12}(t)$ plays a key role in the joint default with important implications in the field of credit analysis. Furthermore, default correlation analysis has many applications in asset pricing and risk management [13]. In what follows, we focus on how to estimate the default correlation of two firms.

If we do not consider jumps in Eq. (4.2), we can obtain $P_i(t)$ $(i = 1, 2)$ and $P_{1\cup 2}(t)$ by using Eq. (2.2) and (2.3), respectively. Then the default correlation can be calculated by using Eq. (4.5). Zhou [13] has chosen $\rho = 0.4$ during the calculations and obtained the default correlations of two firms over different time horizons as presented in Tables 4.6 through 4.9.

Table 4.6: One year default correlations (%) of two different firms. All the simulations were performed with the Monte-Carlo runs $N = 500,000$

	UNIF				Zhou [13]			
	A	Baa	Ba	B	A	Baa	Ba	B
A	-0.01				0.00			
Baa	-0.02	3.69			0.00	0.00		
Ba	2.37	4.95	19.75		0.00	0.01	1.32	
B	2.80	6.63	22.57	26.40	0.00	0.00	2.47	12.46

Next, let us consider the default correlations with jumps which can be investigated by using the developed multivariate UNIF method. We use the following conditions in the simulations:

1. Setting $X(0) = 2$ and $\ln(\kappa) = 0$ for all firms.

2. Setting $\gamma = \mu$ and $\mu = -0.001$ for all firms.

Table 4.7: Two year default correlations (%) of two different firms. All the simulations were performed with the Monte-Carlo runs $N = 500,000$

	UNIF				Zhou [13]			
	A	Baa	Ba	B	A	Baa	Ba	B
A	2.35				0.02			
Baa	2.32	4.25			0.05	0.25		
Ba	4.17	7.17	20.28		0.05	0.63	6.96	
B	4.73	8.23	23.99	29.00	0.02	0.41	9.24	19.61

Table 4.8: Five year default correlations (%) of two different firms. All the simulations were performed with the Monte-Carlo runs $N = 500,000$

	UNIF				Zhou [13]			
	A	Baa	Ba	B	A	Baa	Ba	B
A	6.45				1.65			
Baa	6.71	9.24			2.60	5.01		
Ba	7.29	10.88	22.91		2.74	7.20	17.56	
B	6.77	10.93	22.97	27.93	1.88	5.67	18.43	24.01

Table 4.9: Ten year default correlations (%) of two different firms. All the simulations were performed with the Monte-Carlo runs $N = 500,000$

	UNIF				Zhou [13]			
	A	Baa	Ba	B	A	Baa	Ba	B
A	8.79				7.75			
Baa	10.51	13.80			9.63	13.12		
Ba	9.87	14.23	22.50		9.48	14.98	22.51	
B	8.50	12.54	20.49	24.98	7.21	12.28	21.80	24.37

3. From Eq. (4.4) we obtain,

$$\begin{cases} \sigma_1^2 = \sigma_{11}^2 + \sigma_{12}^2, \\ \sigma_2^2 = \sigma_{21}^2 + \sigma_{22}^2, \\ \rho = \dfrac{\sigma_{11}\sigma_{21} + \sigma_{12}\sigma_{22}}{\sigma_1\sigma_2}. \end{cases} \tag{4.6}$$

In order to compare with the results obtained from standard Brownian motion, we set $\rho = 0.4$ as in [13]. Furthermore, we use the optimized σ_1 and σ_2 in Table 4.4 for firms 1 and 2, respectively. Assuming $\sigma_{12} = 0$, we get,

$$\begin{cases} \sigma_{11} = \sigma_1, \\ \sigma_{12} = 0, \\ \sigma_{21} = \rho\sigma_2, \\ \sigma_{22} = \sqrt{1-\rho^2}\sigma_2, \end{cases} \Rightarrow \sigma = \begin{bmatrix} \sigma_1 & 0 \\ \rho\sigma_2 & \sqrt{1-\rho^2}\sigma_2 \end{bmatrix}.$$

4. The arrival rate for jumps satisfies the Poisson distribution with intensity parameter $\lambda = 0.1$ for all firms. The jump-size is a normal distribution $Z_t \sim N(\mu_{Z_i}, \sigma_{Z_i})$, where μ_{Z_i} and σ_{Z_i} can be different for different firms to reflect specifics of the jump process for each firm. We

adopt the optimized parameters given in Table 4.4.

5. As before, we generate the same interjump times $(T_j - T_{j-1})$ that satisfy an exponential distribution with mean value equals to 1 for each two firms.

We carry out the UNIF method to evaluate the default correlations via the following formula:

$$\rho_{12}(t) = \frac{1}{N}\sum_{n=1}^{N} \frac{P_{12,n}(t) - P_{1,n}(t)P_{2,n}(t)}{\sqrt{P_{1,n}(t)(1-P_{1,n}(t))P_{2,n}(t)(1-P_{2,n}(t))}}, \tag{4.7}$$

where $P_{12,n}(t)$ is the probability of joint default for firms 1 and 2 in each Monte-Carlo cycle, $P_{1,n}(t)$ and $P_{2,n}(t)$ are the cumulative default rates of firms 1 and 2, respectively, in each Monte-Carlo cycle.

The simulated default correlations for one-, two-, five- and ten-years are given in Tables 4.6-4.9. All the simulations were performed with the Monte-Carlo runs $N = 500,000$. Comparing those simulated default correlations with the theoretical data for standard Brownian motions, we can conclude that

1. Similarly to conclusions of [13], the default correlations of same rated firms are usually large compared to differently rated firms. Furthermore, the default correlations tend to increase over long horizons and may converge to a stable value.

2. In our simulations, the one year default correlations of (A,A) and (A,Baa) are negative. Note that the default correlation is defined as $\rho_{12} = \frac{P_{12}-P_1P_2}{\sqrt{P_1(1-P_1)P_2(1-P_2)}}$ (P_{12} is the probability of joint default). $\rho_{12} \leq 0$ ($P_{12} \leq P_1P_2$) indicates that these firms seldom default jointly during one year. Hence the negative numbers may arise from the fact that these firms seldom default jointly in the beginning. Besides, numerical errors may also contribute to the negative numbers which can be improved by increasing the Monte-Carlo runs N.

3. For two and five years, the default correlations of different firms increase. This can be explained by the fact that their individual first passage time density functions increase during these time horizon, hence the probability of joint default increases.

4. As for ten year default correlations, our simulated results are almost identical to the theoretical data for standard Brownian motions. The differences are that the default correlations of (Ba,Ba), (Ba,B) and (B,B) decrease from the fifth year to tenth year in our simulations. The reason is that the first passage time density function of Ba- and B-rated firms begin to decrease from the fifth year, hence the probability of joint default may increase slowly.

5. A remarkable phenomenon is that our implied default correlations deviate (generally higher) from Zhou's results [13] (without jumps, in other words, pure diffusion processes) until we hit 10 default correlations. In order to resolve these differences, we also carried out simulations based on the conventional Monte-Carlo method to investigate the default correlations of firms (A,A) and (B,B) which have the highest and lowest credit qualities, respectively. The Monte-Carlo runs for the conventional Monte-Carlo method is $N = 100,000$, and the discretization size of time horizon is $\Delta = 0.005$. The results are described in Table 4.10. Undoubtedly, our multivariate UNIF method gives almost identical results compared with the conventional Monte-Carlo method, which confirms the validity of our methodology and justifies the results in Tables 4.6-4.9. The reasons which account for the differences between our methodology and theoretical data are as follows,

   (a) During the simulations based on multivariate UNIF method, we have used only one Poisson shock for both two firms, so the jump times are exactly the same for both two firms which will raise the default correlations of those firms. A finer strategy may use several independent Poisson shocks as described in Section 3.4.

(b) The optimized parameters in our model are obtained by minimizing the difference of the cumulative default rates between our model and historical data as described in Eq.(3.5). Therefore, the optimized parameters are sensitively dependent on the historical aggregate data on default rates. Unreliable historical data may lead to biased estimates on implied default correlations. Hence we may improve the default correlations' estimates by using a finer data, for example, by ratings-maturity buckets.

6. Meanwhile, from Table 4.10, we can also conclude that our developed methodology is much more consistent with the conventional Monte-Carlo method for long time horizon. The deviation arises from that we have used the approximated correlations to generate first passage time s_i (see Section 3.3.1). Therefore, our developed methodology will be much better to describe the long term maturity bonds.

Table 4.10: Simulated default correlations (%) of firms (A,A) and (B,B). The simulations were carried out based on the UNIF method and the conventional Monte-Carlo method (CMC).

Firms	Method	One year	Two year	Five year	Ten year
(A,A)	CMC	-0.01	2.48	7.41	8.94
	UNIF	-0.01	2.35	6.45	8.79
(B,B)	CMC	27.32	30.72	29.44	25.11
	UNIF	26.40	29.00	27.93	24.98

4.3 Summary

In this chapter, we demonstrated the applicability of the developed efficient Monte-Carlo-based procedure for simulating the multivariate jump-diffusion processes by using different parameters. Next, by using the developed methodologies, we analyzed the default rates and default correlations of differently rated firms via historical data. The successful applications indicate that the developed methodology provides an efficient computational technique that is applicable in other areas of credit risk and pricing options.

Chapter 5

Conclusions

The FPT problems are ubiquitous in many applications, from physics to finance. While in other areas of applications the FPT problems can often be solved analytically, in finance we usually have to resort to the application of numerical procedures, in particular when we deal with jump-diffusion stochastic processes. The application of the conventional Monte-Carlo procedure is possible for the solution of the resulting model, but it becomes computationally inefficient which severely restricts its applicability in many practically interesting cases.

In this dissertation, we developed efficient Monte-Carlo-based computational procedures for the solution of FPT problem in the context of multivariate (and correlated) jump-diffusion processes. This was achieved by combining a fast Monte-Carlo method for one-dimensional jump-diffusion process and the generation of correlated multidimensional variables. The developed procedures were applied to the analysis of multivariate and correlated jump-diffusion processes. We have also discussed the implementation of the developed Monte-Carlo-based technique for a subclass of multidimensional Lévy processes with several compound Poisson shocks.

Finally, we demonstrated the applicability of the developed methodologies to the analysis of the default rates and default correlations of several different, but correlated firms via a set of empirical data, in which we incorporated jumps to reflect the external shocks or other unpredicted events. The developed methodology provides an efficient computational technique that is applicable in other areas of credit risk and pricing options.

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