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## The Kepler problem with anisotropic perturbations

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We study a two-body problem given by the sum of the Newtonian potential and an anisotropic perturbation that is a homogeneous function of degree  $-\beta$ ,  $\beta \geq 2$ . For  $\beta > 2$ , the sets of initial conditions leading to collisions/ejections and the one leading to escapes/captures have positive measure. For  $\beta > 2$  and  $\beta \neq 3$ , the flow on the zero-energy manifold is chaotic. For  $\beta = 2$ , a case we prove integrable, the infinity manifold of the zero-energy level has heteroclinic connections with the collision manifold. © 2005 American Institute of Physics. [DOI: 10.1063/1.1952580]

### I. INTRODUCTION

In the past three centuries, celestial mechanics has stimulated the development of many branches of mathematics.<sup>11,13</sup> This trend continues, and even its most basic question, known as the *two-body problem* or the *Kepler problem*, still attracts the vivid interest of mathematicians and physicists, both in its classical form<sup>1</sup> and in more recent versions.

Among the latter are the problems raised by *quasihomogeneous* potentials, given by the sum of homogeneous functions, and problems set in *anisotropic* spaces, for which the interaction law is different in each direction of the space. For quasihomogeneous problems the terminology and the first qualitative results were introduced in the mid-1990s;<sup>9,12,21</sup> this potential unifies several dynamical laws, including those of Newton, Coulomb, Manev, Schwarzschild, Lennard-Jones, Birkhoff and others. The anisotropic case is more related to physics, and was initiated by Gutzwiller in the 1970s<sup>17,18</sup> for the quantization of classical ergodic systems. Among Gutzwiller's goals was also that of finding connections between classical and quantum mechanics. A combination of the quasihomogeneous and anisotropic aspects shows up in the anisotropic Manev problem, whose dynamics contains classical, quantum and relativistic features.<sup>6,10,14,15</sup>

In the present paper we consider a version of the Kepler problem, which combines two of the above characteristics, isotropy and anisotropy. The potential [see formula (3)] is the sum of the classical Keplerian potential and an anisotropic perturbation, the latter being a homogeneous function of degree  $-\beta$ ,  $\beta \geq 2$  that depends on a parameter  $\mu > 1$  measuring the strength of the anisotropy. This is the first analysis of a *quasihomogeneous* potential that mixes isotropic and anisotropic components. For previously studied problems, all terms have been either isotropic or anisotropic. As we will see, this case has some surprising dynamical properties, often very different from the ones that characterize potentials whose terms are not mixed.

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Such mixed potentials can be used to understand the dynamics of satellites around oblate planets or the motion of stars around black holes. Here, however, we are not interested in applications. Our endeavours are restricted to mathematical results.

In Sec. II we introduce the basic notations, the equations of motion, and set into evidence the symmetries of the problem. In Sec. III we begin the study of the case  $\beta > 2$ , define the collision manifold, which is an essential qualitative tool, and perform a geometric study of the flow in the neighborhood of collisions. We classify all collision-ejection orbits and prove that the set of initial conditions leading to them has positive measure. We achieve this while studying the flow on and near the collision manifold in terms of McGehee-type coordinates.

In Sec. IV we investigate the existence of heteroclinic orbits on the collision manifold for potentials with  $\mu > 1$  and  $\beta = 2 + [2/(1+2k)]$  or  $\beta = 2 + [1/(1+k)]$ . The main result of this section is that for an open and dense set of  $\mu$  values, saddle-saddle connections do not exist on the collision manifold. Section V deals with capture and escape orbits in the zero-energy case. We show that the infinity manifold has two circles of normally hyperbolic equilibria, one attractive and one repelling. This proves the existence of infinitely many capture and escape orbits.

In Sec. VI we consider the case  $\beta = 2$ , which we show to be integrable. Apparently this is quite surprising since the anisotropic Manev problem, which resembles this case except for the anisotropy of the Newtonian term, is nonintegrable.<sup>3,15</sup> But as we will show, the surprise element vanishes once we look at the problem in the larger context of the Hamilton-Jacobi theory. In Sec. VII we study the flow on and near the collision manifold and see that its qualitative behavior is similar to that of the anisotropic Manev problem.

In Sec. VIII we prove the existence of heteroclinic orbits connecting the collision and infinity manifolds in the zero-energy case for  $\beta = 2$ .

In Sec. IX we consider a perturbative approach of the problem. The perturbation function of the Hamiltonian is a homogeneous function of degree  $-\beta$  with  $\beta > 3/2$ . We end the paper with Sec. X, in which we apply the Melnikov method to show that for every  $\beta \neq 2, 3$ , the flow on the zero-energy manifold is chaotic.

## II. EQUATIONS OF MOTION AND SYMMETRIES

Consider the Hamiltonian

$$H_\beta(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\mathbf{p}^2 + U_\beta(\mathbf{q}), \quad (1)$$

where  $\mathbf{q} = (x, y)$  and  $\mathbf{p} = (p_x, p_y)$ . The equations of motion are

$$\dot{\mathbf{q}} = \mathbf{p},$$

$$\dot{\mathbf{p}} = -\nabla U_\beta(\mathbf{q}), \quad (2)$$

where  $U_\beta$  is a potential of the form

$$U_\beta(x, y) = -\frac{1}{\sqrt{x^2 + y^2}} - \frac{b}{(\mu x^2 + y^2)^{\beta/2}}, \quad (3)$$

with the constants  $\beta \geq 2$ ,  $\mu \geq 1$ , and  $b > 0$ . The symmetries of (2) are given by the following analytic diffeomorphisms in the extended phase space:

$$\text{Id: } (x, y, p_x, p_y, t) \rightarrow (x, y, p_x, p_y, t),$$

$$S_0: (x, y, p_x, p_y, t) \rightarrow (x, y, -p_x, -p_y, -t),$$

$$S_1: (x, y, p_x, p_y, t) \rightarrow (x, -y, -p_x, p_y, -t),$$

$$\begin{aligned}
S_2: \quad & (x, y, p_x, p_y, t) \rightarrow (-x, y, p_x, -p_y, -t), \\
S_3: \quad & (x, y, p_x, p_y, t) \rightarrow (-x, -y, -p_x, -p_y, t), \\
S_4: \quad & (x, y, p_x, p_y, t) \rightarrow (-x, y, -p_x, p_y, t), \\
S_5: \quad & (x, y, p_x, p_y, t) \rightarrow (x, -y, p_x, -p_y, t), \\
S_6: \quad & (x, y, p_x, p_y, t) \rightarrow (-x, -y, p_x, p_y, -t),
\end{aligned} \tag{4}$$

where Id is the identity. These diffeomorphisms form a group that is isomorphic to  $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ .<sup>14,22</sup> Invariance under these symmetries implies that if  $\gamma(t)$  is a solution of (2), then also  $S_i(\gamma(t))$  is a solution for  $i \in \{0, 1, 2, 3, 4, 5, 6\}$ .

### III. THE COLLISION MANIFOLD FOR $\beta > 2$

We will further express the equations of motion in McGehee-type coordinates,<sup>8,20</sup> which are suitable for understanding the motion near collision. The transformations are given step by step as follows. Take

$$\begin{aligned}
r &= \sqrt{x^2 + y^2}, \\
\theta &= \arctan \frac{y}{x}, \\
\tilde{v} &= r\dot{r} = (xp_x + yp_y),
\end{aligned} \tag{5}$$

$$\tilde{u} = r^2\dot{\theta} = (xp_y - yp_x),$$

rescale  $\tilde{v}$  and  $\tilde{u}$  by

$$v = r^{(\beta-2)/2}\tilde{v}, \quad u = r^{(\beta-2)/2}\tilde{u},$$

and then rescale the time variable using the transformation

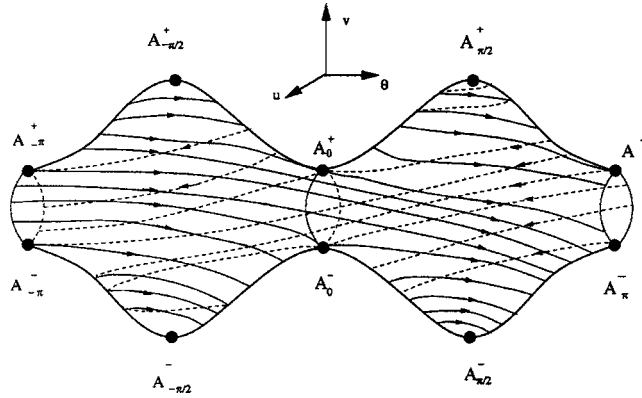
$$\frac{dt}{d\tau} = r^{\beta/2+1}.$$

The equations of motion take the form

$$\begin{aligned}
r' &= rv, \\
v' &= \frac{\beta-2}{2}v^2 + r^{\beta-1} + 2hr^\beta - \frac{b(\beta-2)}{\Delta^{\beta/2}}, \\
\theta' &= u, \\
u' &= \frac{\beta-2}{2}uv + \frac{b\beta(\mu-1)\sin 2\theta}{2\Delta^{(\beta+2)/2}},
\end{aligned} \tag{6}$$

where  $\Delta = \mu \cos^2 \theta + \sin^2 \theta$  and the prime denotes differentiation with respect to the independent time variable  $\tau$ . For simplicity, we keep the same notation for the dependent variables.

In these coordinates, the energy integral  $H_\beta = h$  [see Eq. (1)], takes the form

FIG. 1. The collision manifold  $C$  for  $\beta > 2$ .

$$u^2 + v^2 - 2r^{\beta-1} - \frac{2b}{\Delta^{\beta/2}} = 2hr^{\beta}. \quad (7)$$

We define the collision manifold (see Fig. 1) as

$$C = \left\{ (r, v, \theta, u) \mid r = 0, u^2 + v^2 = \frac{2b}{\Delta^{\beta/2}} \right\}. \quad (8)$$

Notice that  $C$  is homeomorphic to a torus. The flow on this manifold is given by the system

$$v' = \frac{\beta-2}{2}(-u^2),$$

$$\theta' = u,$$

$$u' = \frac{\beta-2}{2}uv + \frac{b\beta(\mu-1)\sin 2\theta}{2\Delta^{(\beta+2)/2}}. \quad (9)$$

We can now prove the following results.

*Proposition 3.1:* All the equilibrium points of system (6) lie on the collision manifold  $C$  and are given by

$$r = 0, \quad v = \pm \sqrt{\frac{2b}{\Delta^{\beta/2}}}, \quad \theta = 0, \pi/2, 3\pi/2, \quad u = 0.$$

*Proof:* It is obvious that the above points are equilibria for the flow defined by (6). To check that there are no equilibria outside the collision manifold, from the first equation in (6) with  $r \neq 0$  we see that  $v=0$  and from the third equation in (6) that  $u=0$ . So, if there are equilibrium points with  $r \neq 0$ , they must be on the zero velocity curve (ZVC); substituting these values in (7), we obtain that the ZVC exists just for  $h < 0$ , it is given by

$$r^{\beta-1} + \frac{b}{\Delta^{\beta/2}} = -hr^{\beta}. \quad (10)$$

Solving the equation  $v'=0$  in (6), we are led to

$$r^{\beta-1} - \frac{b(\beta-2)}{\Delta^{\beta/2}} = -2hr^\beta. \tag{11}$$

But (10) and (11) have no common solutions for  $h < 0$ , therefore there are no equilibria outside the collision manifold.  $\square$

*Proposition 3.2:* The flow on the collision manifold  $C$  is gradientlike (i.e., increasing) with respect to the  $(-v)$ -coordinate.

*Proof:* Since  $\beta > 2$ , we see from the first equation in (9) that  $v' < 0$  except at equilibria, therefore the flow on  $C$  increases with respect to  $-v$ , so is gradientlike relative to it.  $\square$

To match the sign of  $v$  and the value of  $\theta$ , we denote the equilibria on  $C$  by  $A_0^\pm, A_{\pi/2}^\pm, A_\pi^\pm$  and  $A_{3\pi/2}^\pm$ , respectively. Observe that  $\Delta(0) = \Delta(\pi) = \mu$  and  $\Delta(\pi/2) = \Delta(3\pi/2) = 1$ . With this notation, we can describe the following properties of the flow.

**Theorem 3.1:** On the collision manifold  $C$ , the equilibria  $A_0^\pm$  and  $A_\pi^\pm$  are saddles,  $A_{\pi/2}^+$  and  $A_{3\pi/2}^+$  are sources and  $A_{\pi/2}^-$  and  $A_{3\pi/2}^-$  are sinks. Outside  $C$ , the equilibria  $A_0^+, A_\pi^+, A_{\pi/2}^+$  and  $A_{3\pi/2}^+$  have a local one-dimensional unstable analytic manifold, whereas  $A_0^-, A_\pi^-, A_{\pi/2}^-$  and  $A_{3\pi/2}^-$  have a local one-dimensional stable analytic manifold. All these equilibrium points are hyperbolic.

*Proof:* Consider the function

$$F(r, v, \theta, u) = u^2 + v^2 - 2r^{\beta-1} - \frac{2b}{\Delta^{\beta/2}} - 2hr^\beta = 0.$$

According to Eq. (7), the surface of constant energy  $M_h$  defined by the equation

$$F(r, \theta, v, u) = 0$$

is a three-dimensional manifold. At every point  $B$  of  $M_h$ , the tangent space is given by

$$T_B F = \{ (r, v, \theta, u) \mid \nabla F(B) \cdot (r, v, \theta, u) = 0 \}.$$

At any equilibrium point  $A$ , the tangent space is defined by

$$T_A F = \{ (r, \theta, v, u) \mid v = 0 \}.$$

A straightforward computation shows that at the equilibria  $A_0^\pm$  and  $A_\pi^\pm$  the linearized system corresponding to (6) has the matrix

$$\begin{bmatrix} \pm \sqrt{\frac{2b}{\mu^{\beta/2}}} & 0 & 0 & 0 \\ 0 & (\beta-2) \pm \left( \sqrt{\frac{2b}{\mu^{\beta/2}}} \right) & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{b\beta(\mu-1)}{\mu^{\beta+2/2}} & \pm \frac{(\beta-2)}{2} \left( \sqrt{\frac{2b}{\mu^{\beta/2}}} \right) \end{bmatrix},$$

therefore the linear part of the vector field (6) restricted to  $T_{A_0, \pi}^{+,-}$  is given by

$$L = \begin{pmatrix} \pm \sqrt{\frac{2b}{\mu^{\beta/2}}} r \\ 0 \\ u \\ \frac{b\beta(\mu-1)}{\mu^{\beta+2/2}} \theta + \pm \frac{(\beta-2)}{2} \left( \sqrt{\frac{2b}{\mu^{\beta/2}}} \right) u \end{pmatrix}.$$

As a basis for the tangent space  $T_{A_0, \pi}^{+,-}$ , we take the vectors

$$\xi_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \quad (12)$$

The representation of the linear part  $L$  relative to this basis is given by the matrix

$$J^* = \begin{pmatrix} \pm \sqrt{\frac{2b}{\mu^{\beta/2}}} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & \frac{b\beta(\mu-1)}{\mu^{\beta+2/2}} & \pm \frac{(\beta-2)}{2} \left( \sqrt{\frac{2b}{\mu^{\beta/2}}} \right) \end{pmatrix}.$$

The characteristic polynomial shows that all eigenvalues are real and that the equilibrium is a saddle in each case.

At the equilibria  $A_{\pi/2}^\pm$  and  $A_{3\pi/2}^\pm$ , the same linearized system has the matrix

$$\begin{bmatrix} \pm \sqrt{2b} & 0 & 0 & 0 \\ 0 & (\beta-2) \pm (\sqrt{2b}) & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -b\beta(\mu-1) & \pm \frac{(\beta-2)}{2} (\sqrt{2b}) \end{bmatrix}.$$

Using the vectors given in (12) as a basis for the tangent space  $T_{A_{\pi/2, 3\pi/2}}^{+,-}$ , the linear part is given by the matrix

$$J^* = \begin{pmatrix} \pm \sqrt{2b} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & b\beta(\mu-1) & \pm \frac{(\beta-2)}{2} (\sqrt{2b}) \end{pmatrix}.$$

The eigenvalues at  $A_{\pi/2}^+$  and at  $A_{3\pi/2}^+$  are

$$\sqrt{2b}, \quad \frac{(\beta-2)}{2} (\sqrt{2b}) + \sqrt{\frac{b}{2} [(\beta-2)^2 - 8\beta(\mu-1)]},$$

and

$$\frac{(\beta-2)}{2} (\sqrt{2b}) - \sqrt{\frac{b}{2} [(\beta-2)^2 - 8\beta(\mu-1)]}.$$

This means that all of them are positive or have positive real part. At  $A_{\pi/2}^-$  and  $A_{3\pi/2}^-$  the eigenvalues are all negative or have negative real part.  $\square$

*Corollary 3.1:* The set of initial conditions leading to collisions or ejections has positive measure.

*Proof:* The equilibrium points  $A_{\pi/2}^-$  ( $A_{\pi/2}^+$ ) and  $A_{3\pi/2}^-$  ( $A_{3\pi/2}^+$ ) are sinks (sources) for the global flow, therefore their basin of attraction (repulsion) is a three-dimensional set of collision (ejection) orbits.  $\square$

*Remark 3.1.* If  $0 < \mu < (\beta+2)^2/8\beta$ , all the eigenvalues are real and positive. If  $\mu > (\beta+2)^2/8\beta$ , there are two complex eigenvalues with positive real part, so some orbits have the spiraling property, i.e., engage into an infinite spin. For example, to have spiraling orbits in the

case  $\beta=3$ , it is necessary that  $\mu > \frac{25}{24}$ . Therefore when  $\mu$  is sufficiently close to 1, no spiraling orbits exist.

*Remark 3.2:* The inequalities in Remark 3.1 do not depend on the parameter  $b$ .

*Remark 3.3:* For  $\mu$  close enough to 1, though the set of collision orbits has positive measure, there are no spiraling orbits.

#### IV. SADDLE-SADDLE CONNECTIONS ON $C$

Using ideas similar to those found in Ref. 7, we will further study the existence of saddle-saddle connections on  $C$  for  $\mu > 1$  and  $\beta$  of the form

$$\beta = 2 + \frac{2}{1+2k} \quad \text{and} \quad \beta = 2 + \frac{1}{1+k},$$

$k$  integer,  $k \neq -1$ . The reason for choosing these values of  $\beta$  will be clarified below. In the proof we restrict ourselves to the cases  $\beta=3,4$ . This is because the method requires the computation of an integral for each value of  $k$ , and each integral must be computed separately. But the principle is the same for every such  $k$ .

To show that there are no heteroclinic connections is sufficient to prove that the stable and unstable manifolds of corresponding fixed points miss each other. We will show that this holds true for most values of  $\mu > 1$ .

It is now convenient to introduce different coordinates on  $C$ . Since  $C$  is homeomorphic to a torus, we can describe the flow using angle variables. With the transformations

$$u = \frac{\sqrt{2b}}{\Delta^{\beta/4}} \sin \psi,$$

$$v = \frac{\sqrt{2b}}{\Delta^{\beta/4}} \cos \psi, \tag{13}$$

we can rewrite the flow on the equations of motion (9) on  $C$  as

$$\theta' = \frac{\sqrt{2b}}{\Delta^{\beta/4}} \sin \psi,$$

$$\psi' = \frac{\beta-2}{2} \frac{\sqrt{2b}}{\Delta^{\beta/4}} \sin \psi + \frac{d}{d\tau} \left( \frac{\sqrt{2b}}{\Delta^{\beta/4}} \right) \frac{\Delta^{\beta/4}}{\sqrt{2b}} \cot \psi, \tag{14}$$

where

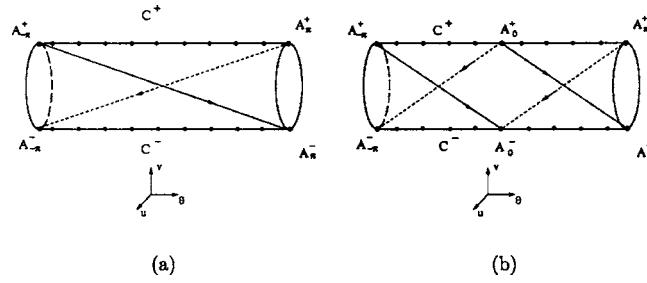
$$\frac{d}{d\tau} \left( \frac{\sqrt{2b}}{\Delta^{\beta/4}} \right) = \frac{\beta(\mu-1)\sqrt{2b}}{4\Delta^{(\beta+4)/4}} \sin 2\theta. \tag{15}$$

The equilibrium points in the variables  $(\theta, \psi)$  are  $A_{\pm\pi/2}^- = (\pm\pi/2, 0)$ ,  $A_{\pm\pi/2}^+ = (\pm\pi/2, \pi)$ ,  $A_0^- = (0, 0)$ ,  $A_0^+ = (0, \pi)$ ,  $A_\pi^- = (\pi, 0)$ , and  $A_\pi^+ = (\pi, \pi)$ .

Notice that if  $\mu=1$  (the isotropic case), the collision manifold is a torus for which the upper and lower circles  $C^+$  and  $C^-$  consist of fixed points. The equations (14) take the form

$$\theta' = \sqrt{2b} \sin \psi,$$



FIG. 2. The collision manifold and the heteroclinic connections for (a)  $\mu=1$ ,  $\beta=3$  and for (b)  $\mu=1$ ,  $\beta=4$ .

$$\psi' = \frac{\beta-2}{2} \sqrt{2b} \sin \psi. \quad (16)$$

It is easy to see that in this case there are heteroclinic orbits that connect the critical points  $A_0^+$  to  $A_\pi^-$  and  $A_\pi^+$  to  $A_0^-$  when

$$\beta = 2 + \frac{2}{1+2k},$$

$k$  integer,  $k \neq -1$ , and connect the critical points  $A_{-\pi}^+$  to  $A_\pi^-$  and  $A_\pi^+$  to  $A_{-\pi}^-$  when

$$\beta = 2 + \frac{1}{1+k},$$

$k$  integer,  $k \neq -1$ .

Figure 2(a) depicts the collision manifold for  $\mu=1$  and  $\beta=3$  and the heteroclinic connections on it, while Fig. 2(b) does the same for  $\mu=1$  and  $\beta=4$ .

In the following we will show that if  $\mu-1=\epsilon>0$  and small, such saddle-saddle connections are broken and the same result holds for an open dense set of  $\mu-1=\epsilon>0$ .

**Theorem 4.2:** For  $\beta=3$  and for an open and dense set of real numbers  $\mu>1$ , the unstable manifolds at  $A_\pi^-$  and  $A_\pi^+$  miss the stable manifolds at  $A_\pi^+$  and  $A_\pi^-$ . For  $\beta=4$  and for an open and dense set of real numbers  $\mu>1$ , the unstable manifolds at  $A_0^+$  and  $A_\pi^+$  miss the stable manifolds at  $A_\pi^-$  and  $A_0^-$ .

*Proof:* Dividing the first of equations (14) by the second one we have

$$\frac{d\psi}{d\theta} = \frac{d}{d\tau} \left( \frac{\sqrt{2b}}{\Delta^{\beta/4}} \right) \frac{\Delta^{\beta/2} \cos \psi}{2b \sin^2 \psi} + \left( \frac{\beta-2}{2} \right) = F_\beta(\theta, \psi, \epsilon), \quad (17)$$

where  $\epsilon=\mu-1$  and  $\Delta=1+\epsilon \cos^2 \theta$ .

First consider  $\beta=3$ , and the unstable manifolds  $W_3^u(-\pi, 0)=W_3^u(\pi, 0)$ . When  $\epsilon=0$ ,  $W_3^s(\pi, \pi)$  matches  $W_3^s(-\pi, 0)$ . Consider the branch of  $W_3^u(-\pi, 0)$  which contains  $(0, \pi/2)$ . This curve lies along the line

$$-2\psi + \theta = -\pi. \quad (18)$$

When  $\epsilon$  varies, this branch of the unstable manifold  $W_3^u(-\pi, 0)$  varies smoothly on  $C$ . Let  $\zeta^3(\theta, \epsilon)$  denote the  $\psi$ -coordinate of this curve, satisfying  $\zeta^3(-\pi, \epsilon)=0$ .

Now let  $\beta=4$  and  $W_4^u(-\pi, 0)=W_4^u(\pi, 0)$ . When  $\epsilon=0$ ,  $W_4^s(-\pi, 0)$  matches  $W_4^s(0, \pi)$ . Consider the branch of  $W_4^u(-\pi, 0)$  that contains  $(-\pi/2, \pi/2)$ . This curve lies along the line

$$-2\psi + 2\theta = -2\pi. \tag{19}$$

As  $\epsilon$  varies, this branch of  $W_4^u(-\pi, 0)$  varies smoothly on  $C$ . Let  $\zeta^4(\theta, \epsilon)$  denote the  $\psi$ -coordinate of this curve, satisfying  $\zeta^4(-\pi, \epsilon) = 0$ . Now we need the following result, which we will prove at the end of this demonstration.

*Lemma 4.1:* With the above notations,  $(\partial/\partial\epsilon)\zeta^3(0, 0) = 3/4\pi > 0$  and  $(\partial/\partial\epsilon)\zeta^4(-\pi/2, 0) = \pi/2 > 0$ .

From this lemma it follows that  $\zeta^3(0, \epsilon) > 0$  and  $\zeta^4(0, \epsilon) > 0$  for  $\epsilon > 0$  small. Thus, it is easy to show that  $v_{1+\epsilon}(0, \zeta^l(0, \epsilon)) > 0$ , where  $l = 3, 4$ . Equations (17) are reversed by the transformation

$$(\theta, \psi) \rightarrow (-\theta, \pi - \psi). \tag{20}$$

If  $\beta = 3$ , the unstable manifold through  $(-\pi, 0)$  is mapped onto the stable manifold through  $(\pi, \pi)$ . Hence the stable manifold intersects the line  $\theta = 0$  at some point  $(0, \psi_0)$  such that  $v_{1+\epsilon}(0, \psi_0) < 0$ . Consequently the stable manifold misses the unstable one for  $\epsilon > 0$  small.

Moreover the stable and unstable manifolds intersect only for a discrete set of  $\epsilon$ , since they vary analytically with  $\epsilon$ .

Furthermore equations (17) are reversed by the transformation

$$(\theta, \psi) \rightarrow (-\theta - \pi, \pi - \psi). \tag{21}$$

If  $\beta = 4$ , the unstable manifold through  $(-\pi, 0)$  is mapped onto the stable manifold through  $(0, \pi)$ . Hence the stable manifold intersects the line  $\theta = 0$  at some point  $(0, \psi_0)$  such that  $v_{1+\epsilon}(0, \psi_0) < 0$  and the stable manifold misses the unstable one for  $\epsilon > 0$  small.

Moreover the stable and unstable manifolds intersect only for a discrete set of  $\epsilon$ , since they vary analytically with  $\epsilon$ .

Similar arguments can be applied to the remaining stable and unstable manifolds. This concludes the proof of Proposition 4.1. □

*Proof of Lemma 4.1:* Observe that  $\zeta^\beta$  satisfies the equation

$$\zeta^\beta(\theta, \epsilon) = \int_{-\pi}^\theta F_\beta(\eta, \zeta^\beta(\eta), \epsilon) d\eta, \tag{22}$$

where  $F_\beta$  is given by (17). For  $\epsilon$  small we can write

$$\zeta^\beta = \zeta_0^\beta(\theta) + \epsilon \zeta_1^\beta(\theta) + O(\epsilon^2). \tag{23}$$

We also have

$$\begin{aligned} \zeta_0^3(\theta) &= (1/2)\theta + \pi/2, \\ \zeta_0^4(\theta) &= \theta + \pi. \end{aligned} \tag{24}$$

To compute  $\zeta_1(\theta)$ , we can use the Taylor expansion of (22) with respect to  $\epsilon$  and find

$$\zeta_1^\beta(\theta) = \int_{-\pi}^\theta \left( \frac{\partial}{\partial\epsilon} F_\beta(\eta, \zeta_0^\beta(\eta), 0) + \frac{\partial}{\partial\psi} F_\beta(\eta, \zeta_0^\beta(\eta), 0) \zeta_1^\beta(\eta) \right) d\eta. \tag{25}$$

Standard computations show that

$$\frac{\partial}{\partial\epsilon} F_\beta(\eta, \zeta_0^\beta(\eta), \epsilon) = \frac{\beta \cos(\eta) \sin(\eta) \cos(\zeta_0^\beta(\eta))}{2 \sin(\zeta_0^\beta(\eta))} + O(\epsilon) \tag{26}$$

and that

$$\frac{\partial}{\partial \psi} F_{\beta}(\eta, \zeta_0^{\beta}(\eta), \epsilon) = O(\epsilon). \quad (27)$$

We can now compute  $\zeta_1^{\beta}(\theta)$  as follows:

$$\zeta_1^{\beta}(\theta) = \int_{-\pi}^{\theta} \left( \frac{\partial}{\partial \epsilon} F_{\beta} + \frac{\partial}{\partial \psi} F_{\beta} \zeta_1 \right) d\eta = \frac{\beta}{2} \int_{-\pi}^{\theta} \frac{\cos(\eta) \sin(\eta) \cos(\zeta_0^{\beta}(\eta))}{\sin(\zeta_0^{\beta}(\eta))} d\eta. \quad (28)$$

When  $\beta=3$ ,

$$\begin{aligned} \zeta_1^3(\theta) &= \frac{3}{2} \int_{-\pi}^{\theta} \frac{\cos(\eta) \sin(\eta) \cos(\eta/2 + \pi/2)}{\sin(\eta/2 + \pi/2)} d\eta = -\frac{9}{2} \cos\left(\frac{1}{2}\theta\right) \sin\left(\frac{1}{2}\theta\right) + \frac{3}{4}\theta \\ &+ 3 \left( \cos\left(\frac{1}{2}\theta\right) \right)^3 \sin\left(\frac{1}{2}\theta\right) + \frac{3}{4}\pi \end{aligned} \quad (29)$$

and, in particular, for  $\theta=0$ , we have

$$\zeta_1^3(0) = \frac{3}{4}\pi = \frac{\partial}{\partial \epsilon} \zeta^3(0,0). \quad (30)$$

When  $\beta=4$ ,

$$\zeta_1^4(\theta) = \frac{3}{2} \int_{-\pi}^{\theta} \frac{\cos(\eta) \sin(\eta) \cos(\eta + \pi)}{\sin(\eta + \pi)} d\eta = \cos(\theta) \sin(\theta) + \theta + \pi \quad (31)$$

and, in particular, for  $\theta=-\pi/2$ , we have

$$\zeta_1^4\left(-\frac{\pi}{2}\right) = \frac{\pi}{2} = \frac{\partial}{\partial \epsilon} \zeta^4\left(-\frac{\pi}{2}, 0\right). \quad (32)$$

This concludes the proof of Lemma 4.1.  $\square$

## V. ESCAPE AND CAPTURE SOLUTIONS FOR $h=0$

We will further study escape (capture) solutions, i.e., the ones for which  $r \rightarrow \infty$  when  $t \rightarrow \infty$  ( $t \rightarrow -\infty$ ). From the energy relation (7), we can see that for  $h < 0$ , the radial coordinate  $r$  is bounded by the zero velocity curve ( $u=0, v=0$ ), so escapes exist only for  $h \geq 0$ . We restrict our analysis to the case  $h=0$ , in which the energy relation (7) takes the form

$$u^2 + v^2 = 2r^{\beta-1} + \frac{2b}{\Delta^{\beta/2}}. \quad (33)$$

With the transformations

$$\rho = r^{-1}, \quad \bar{v} = \rho^{(\beta-1)/2} v, \quad \bar{u} = \rho^{(\beta-1)/2} u,$$

the energy relation becomes

$$\bar{u}^2 + \bar{v}^2 = 2 + \frac{2b}{\Delta^{\beta/2}} \rho^{\beta-1}. \quad (34)$$

We define the *infinity manifold*  $I_0$  as

$$I_0 = \{ (\rho, \bar{v}, \theta, \bar{u}) \mid \rho = 0, \bar{u}^2 + \bar{v}^2 = 2 \}. \quad (35)$$

Since the variable  $\theta \in S^1$ , the infinity manifold  $I_0$  is a torus.

*Remark 5.4:* The infinity manifold  $I_0$  is independent of the parameter  $\beta$ .

After rescaling the time  $\tau$  with the transformation  $d\tau = \rho^{(\beta-1)/2} ds$ , the equations (6) take the form

$$\frac{d\rho}{ds} = -\rho\bar{v},$$

$$\frac{d\bar{v}}{ds} = -\frac{1}{2}\bar{v}^2 = \frac{b(\beta-2)}{\Delta^{\beta/2}}\rho^{\beta-1} + 1,$$

$$\frac{d\theta}{ds} = \bar{u},$$

$$\frac{d\bar{u}}{ds} = -\frac{1}{2}\bar{u}\bar{v} + \frac{b\beta(\mu-1)\sin 2\theta}{2\Delta^{\beta+2/2}}\rho^{\beta-1}. \quad (36)$$

Equations (34) and (36) are well defined on the boundary  $\rho=0$ . Consequently, the phase space of the coordinates  $(\rho, \bar{v}, \theta, \bar{u})$  can be analytically extended to contain the manifold  $I_0$ . Since  $d\rho/ds=0$  for  $\rho=0$ , this manifold is invariant under the flow.

*Proposition 5.3:* All the equilibrium solutions of the flow given by (36) lie on the infinity manifold  $I_0$  and they form two circles of equilibria given by

$$\rho = 0, \quad \bar{v} = \pm\sqrt{2}, \quad \theta \in S^1, \quad \bar{u} = 0.$$

*Proof:* It is obvious that any point of the above circles is an equilibrium orbit. If  $\bar{v}=0$  in (36), by the third equation we have that  $\bar{u}=0$ , but this is a contradiction with the energy relation given by (34). This proves the result.  $\square$

On the infinity manifold  $I_0$ , the equations of motion take the form

$$\frac{d\bar{v}}{ds} = -\frac{1}{2}\bar{v}^2 + 1 = \bar{u}^2/2,$$

$$\frac{d\theta}{ds} = \bar{u},$$

$$\frac{d\bar{u}}{ds} = -\frac{1}{2}\bar{u}\bar{v}. \quad (37)$$

We can now prove the following properties.

*Proposition 5.4:* The flow on  $I_0$  is gradientlike with respect to the  $\bar{v}$ -coordinate.

*Proof:* From the first equation in (37), we obtain that  $\bar{v}' > 0$  except at equilibria, which proves the gradientlike property.  $\square$

In agreement with the sign of  $\bar{v}$ , and by similarity with the collision manifold for  $\mu=1$  studied in Sec. IV, we also denote the equilibria on  $I_0$  as  $C^\pm$ , respectively.

**Theorem 5.3:** On the infinity manifold  $I_0$ , the two circles of equilibria  $C^+$  and  $C^-$  are normally hyperbolic.  $C^+$  corresponds to a sink, whereas  $C^-$  corresponds to a source. The escape orbits are the ones having  $C^+$  as an  $\omega$ -limit, whereas capture orbits are the ones having  $C^-$  as an  $\alpha$ -limit.

*Proof:* The proof is similar to the one of Theorem 3.1, so we will only sketch the main steps. From Eq. (34), we define

$$G(\rho, \bar{v}, \theta, \bar{u}) = \bar{u}^2 + \bar{v}^2 - 2 - \frac{2b}{\Delta^{\beta/2}} \rho^{\beta-1} = 0.$$

Then  $I_\infty$  is the three-dimensional manifold given by  $G^{-1}(0)$ . To study the tangent spaces to this manifold at the equilibria, we use as a basis the same vectors (12). Then the linear representation of the vector field (36) at any equilibrium  $C^+$  and  $C^-$  is given by the matrix

$$\begin{pmatrix} -v_0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & \frac{-v_0}{2} \end{pmatrix}.$$

Notice that for  $C^+$ ,  $v_0 = \sqrt{2}$  and for  $C^-$ ,  $v_0 = -\sqrt{2}$ . In the former case two eigenvalues are negative and one is zero, whereas in the latter case two eigenvalues are positive and one is zero. This completes the proof of the normal hyperbolicity and shows the existence of infinitely many escape orbits and capture orbits.  $\square$

The next result, which is a direct consequence of Theorem 5.3, characterizes the flow on the infinity manifold.

*Corollary 5.2:* For  $h=0$ , the infinity manifold  $I_0$  is foliated by heteroclinic orbits between  $C^-$  to  $C^+$ .

The flow on  $I_0$  given by Eqs. (37) is easy to draw. Because on  $I_0$ ,  $\bar{u}^2 + \bar{v}^2 = 2$ , we can introduce the angular variable  $\psi$  with the transformation

$$\bar{u} = \sqrt{2} \cos \psi, \quad \bar{v} = \sqrt{2} \sin \psi.$$

On  $I_0$ , the equations of motion take the form

$$\dot{\psi} = -2\sqrt{2} \cos \psi, \quad \dot{\theta} = \sqrt{2} \cos \psi.$$

From here we obtain that

$$\frac{d\psi}{d\theta} = -2.$$

On  $I_0$  we can also study the projection of the flow on the  $v-\theta$  plane, which is given by

$$\frac{dv}{d\theta} = \frac{\sqrt{2-v^2}}{2},$$

whose solution is  $v(\theta) = \sqrt{2} \sin[(\theta+k)/2]$ , where  $k$  is a constant determined by the initial condition.

## VI. INTEGRABILITY FOR $\beta=2$

We will further study the problem for  $\beta=2$  and deal with a Hamiltonian (1) of the form

$$H_2 = \frac{1}{2} \mathbf{p}^2 - \frac{1}{\sqrt{x^2 + y^2}} - \frac{b}{\mu x^2 + y^2}. \quad (38)$$

With the notation  $\epsilon = \mu - 1$ , the Hamiltonian expressed in polar coordinates becomes

$$H_2 = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} - \frac{1}{r} - \frac{b}{r^2(1 + \epsilon \cos^2(\theta))}. \quad (39)$$

The corresponding system is integrable since it admits another first integral independent of the Hamiltonian, namely

$$G = \frac{p_\theta^2}{2} - \frac{b}{1 + \epsilon \cos^2(\theta)}. \quad (40)$$

Indeed,

$$\{H_2, G\} = \frac{\partial H_2}{\partial \theta} \frac{\partial G}{\partial p_\theta} - \frac{\partial H_2}{\partial p_\theta} \frac{\partial G}{\partial \theta}, \quad (41)$$

and since

$$\frac{\partial H_2}{\partial \theta} = -\frac{\epsilon b \sin(2\theta)}{r^2(1 + \epsilon \cos^2(\theta))^2}, \quad \frac{\partial H_2}{\partial p_\theta} = \frac{p_\theta}{r^2} \quad (42)$$

and

$$\frac{\partial G}{\partial \theta} = -\frac{\epsilon b \sin(2\theta)}{(1 + \epsilon \cos^2(\theta))^2}, \quad \frac{\partial G}{\partial p_\theta} = p_\theta \quad (43)$$

the Poisson's bracket is  $\{H_2, G\}=0$ , also  $G$  and  $H_2$  are linearly independent.

The existence of the integral  $G$  is not surprising. Indeed it is well known<sup>19</sup> that given the Hamiltonian

$$\bar{H} = \frac{p_r^2}{2} + \frac{p_\theta^2}{2r^2} + U(r, \theta), \quad (44)$$

the corresponding Hamilton-Jacobi equation

$$\frac{\partial S}{\partial t} + \bar{H}\left(r, \theta; \frac{\partial S}{\partial r}, \frac{\partial S}{\partial \theta}\right) = 0 \quad (45)$$

[where  $S=S(r, \theta, t)$  is the action expressed as function of the coordinates and time,  $\partial S/\partial r=p_r$  and  $\partial S/\partial \theta=p_\theta$ ] can be solved by separation of variables if the potential energy is of the form

$$U = a(r) + \frac{b(\theta)}{r^2}. \quad (46)$$

Since the Hamiltonian is time independent we take  $S(r, \theta, t)=S_0(r, \theta)-Et$  (where  $E$  is a constant), and the Hamilton-Jacobi equation for  $S_0$  becomes

$$\frac{1}{2}\left(\frac{\partial S}{\partial r}\right)^2 + a(r) + \frac{1}{2r^2}\left[\left(\frac{\partial S}{\partial \theta}\right)^2 + 2b(\theta)\right] = E. \quad (47)$$

Looking for a solution of the form

$$S_0 = S_1(r) + S_2(\theta), \quad (48)$$

we find for  $S_1$  and  $S_2$  the equations

$$\left(\frac{dS_2}{d\theta}\right)^2 + 2b(\theta) = 2G,$$

$$\frac{1}{2}\left(\frac{dS_1}{dr}\right)^2 + a(r) + \frac{2G}{2r^2} = E, \quad (49)$$

which define two independent integrals. Solving these equations leads to a solution of the Hamilton-Jacobi equations and thus to a general solution of the equations of motion. The above technique applies to the Hamiltonian  $H_2$  and to the additional integral  $G$ .

This approach also shows that the Hamiltonian system given by  $H_2$  is integrable by quadratures. The same conclusion can be reached by directly applying the Liouville-Arnold theorem.<sup>2</sup>

## VII. THE COLLISION MANIFOLD FOR $\beta=2$

For  $\beta=2$  the equation of motion (6) in McGehee coordinates can be written as

$$r' = rv,$$

$$v' = 2r^2h + r,$$

$$\theta' = u,$$

$$u' = \epsilon b \sin(2\theta)\Delta^{-2}, \quad (50)$$

where the prime denotes differentiation with respect to  $\tau$  and  $\Delta = 1 + \epsilon \cos^2(\theta)$ . In McGehee coordinates, the energy relation takes the form

$$u^2 + v^2 - 2r - 2b\Delta^{-1} = 2r^2h, \quad (51)$$

where  $h$  is the energy constant. The first integral  $G$  can be written as

$$g = \frac{1}{2}(u^2 - 2b\Delta^{-1}), \quad (52)$$

where  $g$  is also constant along orbits.

The vector field (50) is analytic on the boundary  $r=0$ , since  $r$  no longer occurs in the denominators of the vector field. The collision manifold reduces to

$$C = \{(r, \theta, v, u) : r=0, u^2 + v^2 = 2b\Delta^{-1}\}. \quad (53)$$

This shows that  $C$  is homeomorphic to a torus. The restriction of equations (50) to  $C$  yields the system

$$v' = 0,$$

$$\theta' = u,$$

$$u' = \epsilon b \Delta^{-2} \sin 2\theta. \quad (54)$$

All nonequilibrium orbits on  $C$  are periodic. Comparing the collision manifold and vector field above with the corresponding ones in Ref. 6, we see that the collision manifold and the flow for  $\beta=2$  are identical to the ones of the anisotropic Manev problem.

## VIII. HETEROCLINIC ORBITS FOR $\beta=2$ AND $h=0$

The main goal of this section is to study the infinity manifold for  $h=0$  and the heteroclinic orbits that connect the collision and infinity manifold. First notice that for  $h<0$  the motion is bounded and therefore there is no infinity manifold. More precisely, we have the following result.

*Proposition 8.5: If  $h<0$  the motion is bounded by the zero velocity curve*

$$r_0 = \frac{-1 + \sqrt{1 - 4hb\Delta^{-1}}}{2h}, \quad v=0, \theta, u=0. \quad (55)$$

*Proof:* Obviously  $u=0$  and  $v=0$  if and only if  $u^2+v^2=0$ . Also  $v=0$  implies  $r'=0$ . Using the energy relation we can draw the conclusion that  $u^2+v^2=0$  if and only if  $2r^2h+2r+2b\Delta^{-1}=0$ . This quadratic equation has the solutions

$$r = \frac{-1 \pm \sqrt{1 - 4hb\Delta^{-1}}}{2h}. \quad (56)$$

Since  $r \geq 0$  and  $h < 0$  the only valid solution  $r_0$  is the one with the minus sign. This shows that (55) is the zero velocity curve. The fact that the motion is bounded by this curve follows from the fact that if  $r > r_0$  then  $u^2 + v^2 < 0$ .  $\square$

To describe the behavior of the solution at infinity we need to study the equations (36) with  $\beta=2$  and  $h=0$ , that is

$$\begin{aligned} \dot{\rho} &= -\rho\bar{v}, \\ \dot{\bar{v}} &= -\frac{1}{2}\bar{v}^2 + 1, \\ \dot{\theta} &= \bar{u}, \\ \dot{\bar{u}} &= -\frac{1}{2}\bar{v}\bar{u} + \epsilon b\rho \sin 2\theta\Delta^{-2}, \end{aligned} \quad (57)$$

where the dot denotes differentiation with respect to the time variable  $s$ . The energy relation is

$$\bar{u}^2 + \bar{v}^2 - 2 - 2b\rho\Delta^{-1} = 0 \quad (58)$$

and the other first integral can be written as

$$\bar{u}^2 - 2b\rho\Delta^{-1} = 2\rho g. \quad (59)$$

The infinity manifold is a two-manifold embedded in  $\mathbb{R}^3 \times S^1$  given by the equations

$$I_0 = \{(\rho, \bar{v}, \theta, \bar{u}) | \rho = 0, \bar{u}^2 + \bar{v}^2 = 2\}, \quad (60)$$

i.e., the points in phase space that satisfy the condition  $\rho=0$  and the energy relation. This shows that  $I_0$  is a torus  $S^1 \times S^1$ .

The first two equations of the system (57) are independent from the others, and we would like to determine  $\rho$  and  $\bar{v}$ . If we set  $\bar{v} = \pm\sqrt{2}$ , then  $\rho = \exp(\mp\sqrt{2}(s-s_0))$  is a solution of the two equations mentioned above. If  $\bar{v} = \pm\sqrt{2}$  the energy integral (58) gives the condition

$$\bar{u} = \pm\sqrt{2b\rho\Delta^{-1}}. \quad (61)$$

Moreover the previous condition and Eq. (59) impose  $g=0$ . Differentiating (61) with respect to  $s$  we obtain

$$\dot{\bar{u}} = \pm\sqrt{2b}\left(\frac{\Delta^{-1/2}}{2\rho^{1/2}}\dot{\rho} + \frac{\epsilon}{2}\bar{u}\sqrt{\rho\Delta}\frac{\sin 2\theta}{\Delta^2}\dot{\theta}\right) \quad (62)$$

and using the first equation in (57) and Eq. (61) it follows that

$$\dot{\bar{u}} = -\frac{1}{2}\bar{v}\bar{u} + \epsilon b\rho \sin 2\theta\Delta^{-2}. \quad (63)$$

This shows that system (57) admits solutions with  $\bar{v} = \pm\sqrt{2}$ .

The other solutions of the first two equations of the system (57) can be found by dividing the second equation by the first. This leads to the equation

$$\frac{\dot{\bar{v}}}{\dot{\rho}} = \frac{1}{2}\frac{\bar{v}}{\rho} - \frac{1}{\rho\bar{v}}, \quad (64)$$

which can be solved by separating the variables. This leads to



$$\int_{\rho_0}^{\rho} \frac{d\xi}{\xi} = \int_{\bar{v}_0}^z \frac{z dz}{\frac{1}{2}z^2 - 1} \quad (65)$$

and consequently yields

$$\rho = \left( \frac{\rho_0}{\bar{v}_0^2 - 2} \right) (\bar{v}^2 - 2), \quad (66)$$

where  $\rho_0$  and  $v_0$  are initial conditions and  $\rho \geq 0$ , since  $\rho < 0$  has no physical meaning. Now we can prove the following result relative to the heteroclinic solutions connecting the infinity and the collision manifolds.

**Theorem 8.4:** *The solutions whose  $\omega$ -limit set belongs to the infinity manifold have the  $\alpha$ -limit set contained in the collision manifold. In particular, the following properties take place:*

- (1) *If  $\bar{v} = \sqrt{2}$  ( $\bar{v} = -\sqrt{2}$ ), the above solutions belong to the unstable (stable) manifold of one of the periodic orbits on the equator of the collision manifold.*
- (2) *If  $0 < \sqrt{1/k} < \sqrt{2b}$  and  $\sqrt{1/k} \neq \sqrt{2b/\mu}$  with  $k = \rho_0/(\bar{v}_0^2 - 2)$ , the above solutions belong to the unstable (stable) manifold of the periodic orbits on the collision manifold with  $v = \sqrt{1/k}$  ( $v = -\sqrt{1/k}$ ).*
- (3) *If  $\sqrt{1/k} = \sqrt{2b/\mu}$  ( $-\sqrt{1/k} = \sqrt{2b/\mu}$ ), then the above solutions belong to the unstable (stable) manifold of one of the fixed points  $A_0^+$ ,  $A_\pi^+$  ( $A_0^-$ ,  $A_\pi^-$ ).*
- (4) *If  $\sqrt{1/k} = \sqrt{2b}$  ( $-\sqrt{1/k} = \sqrt{2b}$ ), then the above solutions belong to the unstable (stable) manifold of one of the fixed points  $A_{-\pi/2}^+$ ,  $A_{\pi/2}^+$  ( $A_{-\pi/2}^-$ ,  $A_{\pi/2}^-$ ).*

*Proof:* To prove (1) observe that  $v = \bar{v}/\rho^{1/2} = \pm\sqrt{2}/\rho^{1/2}$ . Thus  $\lim_{s \rightarrow \infty} v = \lim_{s \rightarrow \infty} \pm\sqrt{2}/\rho^{1/2} = \lim_{\tau \rightarrow \infty} \pm\sqrt{2}/\rho^{1/2} = \lim_{\tau \rightarrow \infty} v = 0$ , since  $\lim_{s \rightarrow \infty} \tau = \infty$ . To prove (2), (3), and (4) we consider the limit  $\lim_{\tau \rightarrow \infty} v$ , which with the help of Eq. (66) becomes

$$\lim_{\tau \rightarrow \infty} v = \lim_{\tau \rightarrow \infty} \frac{\bar{v}}{\rho^{1/2}} = \pm \lim_{\tau \rightarrow \infty} \sqrt{\frac{\rho + 2k}{k}} \rho^{-1/2} = \pm \sqrt{\frac{1}{k}}.$$

Moreover from the energy relation  $\bar{u}^2 + \bar{v}^2 - 2 - 2b\rho/\Delta = 0$  and the fact that  $\bar{v}^2 = (\rho + 2k)/k$  it is easy to see that

$$\bar{u}^2 = -\rho \left( \frac{1}{k} - \frac{2b}{\Delta} \right),$$

and since  $\rho > 0$  and  $\bar{u}^2 \geq 0$ , we have

$$\frac{1}{k} \leq \frac{2b}{\Delta} \leq 2b.$$

Consequently we have shown that

$$0 < \sqrt{\frac{1}{k}} \leq \sqrt{2b}. \quad (67)$$

In particular it is clear that when  $\sqrt{1/k} = \sqrt{2b/\mu}$  ( $-\sqrt{1/k} = \sqrt{2b/\mu}$ ) the solutions lie on the unstable (stable) manifold of one of the fixed points  $A_0^+$ ,  $A_\pi^+$  ( $A_0^-$ ,  $A_\pi^-$ ). Similarly when  $\sqrt{1/k} = \sqrt{2b}$  ( $-\sqrt{1/k} = \sqrt{2b}$ ), the solutions lie on the unstable (stable) manifold of one of the fixed points  $A_{-\pi/2}^+$ ,  $A_{\pi/2}^+$  ( $A_{-\pi/2}^-$ ,  $A_{\pi/2}^-$ ). In the remaining cases the solutions lie on the unstable (stable) manifold of the periodic orbits on the collision manifold with  $v = \sqrt{1/k}$  ( $v = -\sqrt{1/k}$ ).  $\square$

### IX. A PERTURBATIVE APPROACH

In this and the next section we study the appearance of chaos on the zero-energy manifold for the system of Hamiltonian  $H_\beta$  given by Eq. (1). In order to do that we study the problem for small values of  $\epsilon$  and  $b$  and we use an extension of the Melnikov method that is briefly illustrated in the following section. Consider the Hamiltonian

$$\mathcal{H}_\beta = \frac{1}{2}\mathbf{p}^2 - \frac{1}{r} - \frac{b}{r^\beta} + \epsilon b \frac{\beta \cos^2 \theta}{2r^\beta} \equiv \mathcal{H}^0 + b\mathcal{W}_\beta^1(r, \theta) + \epsilon b\mathcal{W}_\beta^2(r, \theta), \quad (68)$$

where  $\beta > 3/2$ ,  $b \ll 1$ ,  $\epsilon \ll 1$ . This is the original Hamiltonian  $H_\beta$  [defined in (1)] truncated to the second order in  $\epsilon$  and  $b$ . Consider, as in Ref. 5, the parabolic solutions of the unperturbed ( $\epsilon = 0, b = 0$ ) problem (68), defined by the Hamiltonian  $\mathcal{H}^0$  of the classical Kepler problem that are on the zero-energy manifold and play the role of homoclinic solutions corresponding to the critical point at infinity, i.e.,  $r = \infty$ ,  $\dot{r} = 0$ . These solutions satisfy the equations

$$\dot{r} = \pm \frac{\sqrt{2r - k^2}}{r}, \quad \dot{\theta} = \frac{k}{r^2}, \quad (69)$$

where  $k \neq 0$  is the (constant) angular momentum and the sign  $-$  (respectively,  $+$ ) holds for  $t < 0$  (respectively,  $t > 0$ ). From (69) we get

$$\begin{aligned} \pm t &= \frac{k^2 + r}{3} \sqrt{2r - k^2} + \text{const}, \\ \theta &= \pm 2 \arctan \frac{\sqrt{2r - k^2}}{\sqrt{k^2}} + \text{const}. \end{aligned} \quad (70)$$

We denote by

$$R = R(t) \quad \text{and} \quad \Theta = \Theta(t) \quad (71)$$

the expressions giving the dependence of  $r$  and of  $\theta$  on the time  $t$ . These are obtained by “inverting” the equations (70) with the conditions  $R(0) = r_{\min} = k^2/2$  and  $\Theta(0) = 0$ . Let us emphasize that, as in Ref. 5, it is not necessary to have the explicit form of these functions. But it is important to remark that  $R(t)$  is even and  $\Theta(t)$  is odd, both in the time variable. The choice  $\Theta(0) = 0$  correspond to selecting the solution describing the parabola with axis coinciding with the  $x$  axis and going to infinity when  $x \rightarrow -\infty$ .

The parabolic orbits can also be described in parametric form.<sup>19</sup> If  $p = k^2 \neq 0$ , we can write

$$r = \frac{p}{2}(1 + \eta^2), \quad t = \frac{p^{3/2}}{2} \eta \left( 1 + \frac{\eta^2}{3} \right), \quad \eta = \tan \frac{\theta}{2}. \quad (72)$$

We also have

$$\cos 2\theta = 2 \frac{(1 - \eta^2)^2}{(1 + \eta^2)^2} - 1. \quad (73)$$

We will further use these remarks to apply the Melnikov method.

### X. THE MELNIKOV METHOD

Consider the problem defined in (68). The homoclinic manifold, i.e., the set of solution of the unperturbed equation which are doubly asymptotic to  $r = \infty$ ,  $\dot{r} = 0$ , is given for each value  $k \neq 0$  of the angular momentum by the two-dimensional manifold described by the family of solutions  $r = R(t - t_0)$ ,  $\vartheta = \Theta(t - t_0) + \theta_0$ , where  $R(t)$  and  $\Theta(t)$  have been defined in (71), with arbitrary  $t_0$ ,  $\theta_0$ .

It is clear from Eq. (68) that the first order in  $b$  of the perturbation (i.e., the term  $b\mathcal{W}_\beta^1$ ) does not contribute to the Melnikov integrals. This is because the perturbed Hamiltonian truncated at the first order, i.e.,  $\mathcal{H}_0 + b\mathcal{W}_\beta^1$ , is integrable and, at this order, the positively and negatively asymptotic sets coincide. Furthermore, for the same reason, the terms in  $b^n$  for  $n \geq 2$  do not contribute to the Melnikov integral. Consequently the first nonvanishing terms of the Melnikov integrals are of order  $\epsilon b$ .

The perturbation resulting from a small anisotropy vanishes as  $r \rightarrow \infty$  since  $\mathcal{W}_\beta^2(r, \theta) \sim 1/r^\beta$  with  $\beta > 3/2$ . This guarantees, among other things, the integrability of the Melnikov integrals and the applicability of the aforementioned method.<sup>5</sup> This allows us to write the first nonvanishing effect on the Melnikov integral in the same form as in Ref. 5, with the difference that here we can drop the dependence on time,

$$M_1(\theta_0) = \int_{-\infty}^{+\infty} \left[ \dot{R}(t) \frac{\partial \mathcal{W}_\beta^2(R(t), \Theta(t) + \theta_0)}{\partial r} + \dot{\Theta}(t) \frac{\partial \mathcal{W}_\beta^2(R(t), \Theta(t) + \theta_0)}{\partial \theta} \right] dt = 0, \quad (74)$$

$$M_2(\theta_0) = \int_{-\infty}^{+\infty} \frac{\partial \mathcal{W}_\beta^2(R(t), \Theta(t) + \theta_0)}{\partial \theta} dt = 0. \quad (75)$$

Since the perturbation  $\mathcal{W}_\beta^2$  vanishes as  $t \rightarrow \pm\infty$ , the first Melnikov condition can be written as

$$M_1(\theta_0) = \int_{-\infty}^{+\infty} \frac{\partial \mathcal{W}_\beta^2(R(t), \Theta(t) + \theta_0)}{\partial t} dt \equiv 0. \quad (76)$$

The above integral is identically zero because the perturbation  $\mathcal{W}_\beta^2$  is not time dependent. This simplifies our discussion since we must only find the solutions of (75). Such solutions correspond, at the order  $\epsilon b$ , to intersections of the positively and negatively asymptotic sets of the critical point at infinity. If one such solution exists then there are infinitely many. Moreover, if the solutions correspond to simple zeroes of  $M_2(\theta_0)$ , the intersection is transversal and, for  $\epsilon$  and  $b$  sufficiently small, higher order terms are not going to destroy the intersection.

It is significant to remark, and easy to verify, that these conditions can be written also in terms of the first integrals of the unperturbed problem as

$$M_1(\theta_0) = \int_{-\infty}^{+\infty} \{H_0, \mathcal{W}_\beta^2\}(\cdots) dt = 0 \quad (77)$$

and

$$M_2(\theta_0) = \int_{-\infty}^{+\infty} \{K, \mathcal{W}_\beta^2\}(\cdots) dt = 0, \quad (78)$$

where  $(\cdots)$  represents the homoclinic orbit.

This resembles some properties obtained for the Gylden problem<sup>5,16</sup> and is related to the symmetries of the problem. In the Gylden problem there is a perturbation that does not depend on the angle  $\theta$ , but depends on time. This means that the perturbation destroys the time homogeneity, so the Hamiltonian is not an integral of motion anymore, but does not destroy the rotational invariance, so the angular momentum is still conserved. Therefore the only one condition is given by (74). However, the anisotropy destroys the rotational symmetry but not the homogeneity of time, so we are left with the condition (75).

Here the Melnikov condition for  $M_2$  becomes

$$M_2(\theta_0) = \frac{\beta}{2} \int_{-\infty}^{+\infty} \frac{\sin[2(\Theta(t) + \theta_0)]}{R(t)^\beta} dt = 0. \quad (79)$$

Using some trigonometry the integral can be written as

$$M_2(\theta_0) = I_1 \cos 2\theta_0 + I_2 \sin 2\theta_0, \quad (80)$$

where  $I_1$  and  $I_2$  are defined by

$$I_1 = \frac{\beta}{2} \int_{-\infty}^{+\infty} \frac{\sin 2\Theta(t)}{R(t)^\beta} dt, \quad (81)$$

$$I_2 = \frac{\beta}{2} \int_{-\infty}^{+\infty} \frac{\cos 2\Theta(t)}{R(t)^\beta} dt.$$

Recall that  $R(t)$  is an even function of time and  $\Theta(t)$  is an odd function. This implies that the integrand of  $I_1$  is an odd function. Therefore  $I_1 \equiv 0$ , and  $M_2$  can be rewritten as

$$M_2(\theta_0) = I_2 \sin 2\theta_0. \quad (82)$$

Thus  $M_2(\theta_0)$  has infinitely many simple zeroes, provided that  $I_2 \neq 0$ . To complete the proof we must verify that  $I_2 \neq 0$ . We compute  $I_2$  using the parametric form of the parabolic orbits defined in Eqs. (72). Since  $dt = (p^{3/2}/2)(1 + \eta^2)d\eta$ , we can write

$$I_2 = 2^{\beta-1} p^{3/2-\beta} \frac{\beta}{2} \int_{-\infty}^{+\infty} \frac{1}{(1 + \eta^2)^{\beta-1}} \left( \frac{2(1 - \eta^2)^2}{(1 + \eta^2)^2} - 1 \right) d\eta. \quad (83)$$

Computing the integral, we find that

$$I_2 = \frac{2^{\beta-1} p^{3/2-\beta}}{2\Gamma(\beta-1)} \sqrt{\pi} \left[ \Gamma\left(\beta - \frac{3}{2}\right) \left( \frac{3}{2(\beta-1)\beta} - 1 \right) + 2 \frac{\Gamma(\beta + \frac{1}{2}) - \Gamma(\beta - \frac{1}{2})}{(\beta-1)\beta} \right], \quad (84)$$

where  $\Gamma(z)$  is Euler's gamma function. Thus  $I_2(\beta)$  is an analytic function in  $\beta$  for  $\beta > 3/2$ , since  $\Gamma(z)$  is analytic for  $z > 0$ . Recall that the gamma function can be expressed as

$$\Gamma(z) = \lim_{n \rightarrow \infty} \frac{n! n^z}{z(z+1) \cdots (z+n)} \quad (85)$$

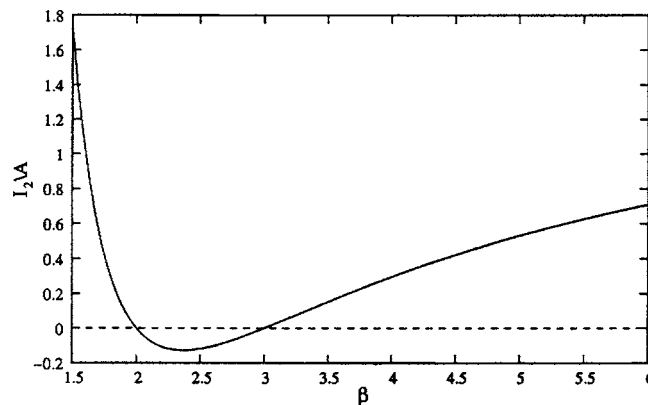
if  $z \neq 0, -1, -2, -3, \dots$ . Using this form of the gamma function, and letting  $A = 2^{\beta-2} p^{3/2-\beta}$ , we find that

$$I_2 = A \lim_{n \rightarrow \infty} n^{3/2} \sqrt{\pi} \frac{\beta(\beta+1)(\beta+2) \cdots (\beta-1+n)}{(\beta - \frac{3}{2})(\beta - \frac{1}{2}) \cdots (\beta - \frac{1}{2} + n)} (\beta^2 - 5\beta + 6)$$

$$= \frac{A \sqrt{\pi} \Gamma(\beta + \frac{1}{2})}{(\beta-1)(\beta - \frac{3}{2})(\beta - \frac{1}{2}) \Gamma(\beta-1)} (\beta^2 - 5\beta + 6), \quad (86)$$

which is zero if and only if  $\beta^2 - 5\beta + 6 = 0$ , since the gamma function  $\Gamma(z)$  is always positive for  $z > 0$  and therefore the first factor never vanishes. Consequently  $I_2$  vanishes if and only if  $\beta^2 - 5\beta + 6 = 0$ , namely for  $\beta = 2$  or  $\beta = 3$ , see Fig. 3.

This proves that for every  $\beta > 3/2$ ,  $\beta \neq 2, 3$ , and  $\epsilon$  small, the system given by the Hamiltonian  $\mathcal{H}_\beta$  of Eq. (68) exhibits chaotic dynamics on the zero energy manifold induced by an infinite sequence of intersections on the Poincaré section of the positively and negatively asymptotic sets of the critical point at infinity. Moreover, if  $\epsilon$  and  $b$  are sufficiently small, one can consider the

FIG. 3.  $I_2/A$  as a function of  $\beta$ .

Hamiltonian  $H_\beta$  of Eq. (1) and the simple zeroes of the Melnikov function cannot be destroyed by the higher-order terms of the perturbation. Therefore chaos persists for the Hamiltonian system of Hamiltonian (1). This proves the following.

**Theorem 10.5:** For every  $\beta > 3/2$ ,  $\beta \neq 2, 3$ , and  $\epsilon$  small, the system given by the Hamiltonian  $H_\beta$  defined by Eq. (1) exhibits chaotic dynamics on the zero energy manifold.

This type of chaotic behavior is induced by a chain of infinitely many intersections of the positively and negatively asymptotic sets to the critical point at infinity. The Smale-Birkhoff theorem does not directly apply to this situation, which is degenerate. But it is well known that the existence of Smale horseshoes and positive topological entropy can arise in the case of nonhyperbolic equilibria.<sup>4</sup>

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