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Self-resonance in cylindrical Josephson junctions*

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The I-V characteristics of self-resonant cylindrical Josephson junctions in the presence of parallel applied magnetic fields have been calculated using a first-order perturbation technique.

I. INTRODUCTION

It has been known for some time[1–3] that the boxlike dielectric cavity which comprises the tunneling barrier in many thin-film Josephson junctions possesses certain resonant modes which, when excited, tend to modify the zero-frequency current-voltage characteristics of these devices. The mechanism has its origin in the ac Josephson effect and arises when the junction is biased at a point where the Josephson frequency is matched to a cavity mode. In such a situation there is strong feedback in the sense that sustained standing waves occur which lead to an extra induced voltage across the junction. It is the perturbing effect of this additional voltage which finally causes the appearance of finite zero-frequency currents; without the feedback only a harmonic Josephson supercurrent would occur. Theory and experiments have clearly demonstrated that the amount of dc current at given bias at or near resonance is sensitive to magnetic fields applied parallel to the plane of the junction.

The only geometry which has been extensively investigated theoretically is that of the above-mentioned rectangular cavity between superconductors. Recently, however, some experimental results were reported by Bermon and Mesak[4] for the case of a cylindrical cavity. Although they were able to verify the expected bias points at which resonance matching would occur, no attempt was made to quantitatively characterize the magnetic field sensitivities. We present in this paper a complete discussion of the effect of applied fields and give some quantitative estimates of appropriate model parameters.

II. THEORY OF THE CYLINDRICAL CAVITY

The partial differential equation governing the self-induced perturbation voltage v was given by Eck *et al.*[5] and may be expressed in polar coordinates (remembering that v does not depend on the axial coordinate) as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}-\frac{1}{\bar{c}^2}\frac{\partial^2 v}{\partial t^2}-\frac{\omega}{\bar{c}^2 Q}\frac{\partial v}{\partial t}=\left(\frac{4\pi l}{\epsilon \bar{c}^2}\right)\frac{\partial j_z}{\partial t}, \tag{1}$$

where $\bar{c}=c(l/d\epsilon)^{1/2}$; $d=2\lambda+l$; and the symbols represent physical quantities as follows: l is the barrier thickness, λ is the penetration depth, j_1 is the tunneling amplitude, ω is the Josephson angular frequency $\omega=2\ \text{eV}/\hbar$, Q is the adjustable damping parameter reflecting losses, ϵ is the dielectric constant of the barrier region, and c, e, $\hbar$ have their usual meanings. Note that all equations and constants are expressed in Gaussian units. The applied magnetic field will be specified in terms of the conventional parameter k as $k=(2ed/\hbar c)H$. Finally, we specify the geometry of the problem by means of Fig. 1.

The perturbation technique of solving Eq. (1) consists of replacing the current term on the right-hand side by its lowest-order approximation $j_z \approx j_1 \sin(\omega t - kr\sin\theta)$, in which case

$$\nabla^2 v-\frac{1}{\bar{c}^2}\frac{\partial^2 v}{\partial t^2}-\frac{\omega}{\bar{c}^2 Q}\frac{\partial v}{\partial t}=\left(\frac{4\pi l j_1 \omega}{\epsilon \bar{c}^2}\right)\cos(\omega t - kr\sin\theta)\ . \tag{2}$$

We now seek solutions to Eq. (2) in the form

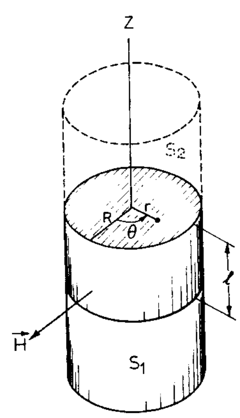

FIG. 1. Cylindrical Josephson junction. The upper superconductor is partially omitted for clarity.

$$v(r,\theta,t)=\sum_{n,m}\{[A^{(1)}_{nm}J_n(\gamma_{nm}r)\cos(n\theta)+A^{(2)}_{nm}J_n(\gamma_{nm}r)\sin(n\theta)]\cos\omega t+[B^{(1)}_{nm}J_n(\gamma_{nm}r)\cos(n\theta)+B^{(2)}_{nm}J_n(\gamma_{nm}r)\sin(n\theta)]\sin\omega t\}\ , \tag{3}$$

the sum over n running from $0\rightarrow\infty$, while m ranges from 1 to infinity. The J_n are nth-order Bessel functions of the first kind. In other words, the self-induced voltage $v(r,\theta,t)$ is expanded in terms of the normal modes of a cylindrical cavity[6] operating in the TM mode and the boundary conditions at the periphery ($r=R$) have been accounted for by requiring

$$J'_n(\gamma_{nm}R)=0\ . \tag{4}$$

Thus if the mth zero of J'_n is denoted X_{nm}, we have

$$\gamma_{nm}=X_{nm}/R\ . \tag{5}$$

If the expression for $v(r,\theta,t)$ given in Eq. (3) is substituted into the differential equation (2), it is possible to show that the expansion coefficients have the following forms:

$$A^{(1)}_{nm}=\left(\frac{4\pi lj_1}{\epsilon\omega}\right)\left(\frac{1/Q}{(1-\gamma^2_{nm}\bar{c}^2/\omega^2)^2+(1/Q)^2}\right)a_{nm}\ , \tag{6a}$$

$$B^{(1)}_{nm}=\left(\frac{4\pi lj_1}{\epsilon\omega}\right)\left(\frac{-(1-\gamma^2_{nm}\bar{c}^2/\omega^2)}{(1-\gamma^2_{nm}\bar{c}^2/\omega^2)^2+(1/Q)^2}\right)a_{nm}\ , \tag{6b}$$

$$A^{(2)}_{nm}=\left(\frac{4\pi lj_1}{\epsilon\omega}\right)\left(\frac{1/Q}{(1-\gamma^2_{nm}\bar{c}^2/\omega^2)^2+(1/Q)^2}\right)b_{nm}\ , \tag{6c}$$

$$B^{(2)}_{nm}=\left(\frac{4\pi lj_1}{\epsilon\omega}\right)\left(\frac{(1-\gamma^2_{nm}\bar{c}^2/\omega^2)}{(1-\gamma^2_{nm}\bar{c}^2/\omega^2)^2+(1/Q)^2}\right)b_{nm}\ , \tag{6d}$$

where

$$\begin{aligned}
a_{nm}&=0 && \text{for } n \text{ odd}\\
&=-\frac{2kR[J_{n-1}(kR)-J_{n+1}(kR)]}{[1-(kR/X_{nm})^2](X^2_{nm}-n^2)J_n(X_{nm})} && \text{for } n \text{ even},\quad X_{nm}\neq kR\\
&=-2 && \text{for } n \text{ even},\quad X_{nm}=kR\\
&=\frac{2kR}{1-(kR/X_{nm})^2}\ \frac{J_1(kR)}{J_0(X_{0m})} && \text{for } n=0,\quad X_{0m}\neq kR\\
&=-1, && \text{for } n=0,\quad X_{0m}=kR;\\
b_{nm}&=0 && \text{for } n \text{ even or zero}\\
&=\frac{2kR[J_{n-1}(kR)-J_{n+1}(kR)]}{[1-(kR/X_{nm})^2](X^2_{nm}-n^2)J_n(X_{nm})} && \text{for } n \text{ odd},\quad X_{nm}\neq kR\\
&=2\ , && \text{for } n \text{ odd},\quad X_{nm}=kR\ .
\end{aligned}$$

With the expansion for v now fully determined, the position- and time-dependent phase may be written

$$\phi=\omega t-kr\sin\theta-\Psi_0+\frac{2e}{\hbar}\int^t v(t')\,dt'\ . \tag{7}$$

The phase constant Ψ_0 has been discussed elsewhere[7] and represents a correction of the approach of Eck *et al.*[5] to this type of problem. It can be thought of as a relative phase shift between the free-junction oscillations $[\approx\sin(\omega t-kr\sin\theta)]$ and the effective self-field generation which also occurs to lowest order at an angular frequency ω. The *net* Josephson current corresponding to Eq. (7) is given by $j=j_1\sin(\phi)$; this is frequency modulated and contains a nonzero dc (zero-frequency) term which we denote $\langle j\rangle$. The time-independent spatially averaged current J_{dc} is just

$$J_{dc}=\frac{1}{\pi R^2}\int_0^R\int_0^{2\pi}\langle j\rangle r\,d\theta\,dr\ . \tag{8}$$

It can be shown that, subject to the definition

$$\psi_{nm}\equiv\tan^{-1}\left(\frac{1/Q}{1-\gamma^2_{nm}\bar{c}^2/\omega^2}\right)\ , \tag{9}$$

$$\langle j\rangle=\left(\frac{4\pi lj_1^2}{\hbar\epsilon\omega^2}\right)\sum_{n,m}\frac{J_n(\gamma_{nm}r)}{[(1-\gamma^2_{nm}\bar{c}^2/\omega^2)^2+(1/Q)^2]^{1/2}}\{\cos(\psi_{nm})\cos(kr\sin\theta+\Psi_0)$$

$$\times[a_{nm}\cos(n\theta)-b_{nm}\sin(n\theta)]+\sin(\psi_{nm})\sin(kr\sin\theta+\Psi_0)[a_{nm}\cos(n\theta)+b_{nm}\sin(n\theta)]\}\ , \tag{10}$$

and thus using Eq. (8),

$$J_{dc}/j_1=(4\pi e l j_1 R^2/\hbar\epsilon\bar{c}^2X_{11}^2)[J^{(1)}\cos(\Psi_0)+J^{(2)}\sin(\Psi_0)]\ , \tag{11}$$

where

$$J^{(1)}=\frac{1}{\eta^2}\sum_{n,m}\left(\frac{b_{nm}[1-(-1)^n](1/Q)+a_{nm}[1+(-1)^n](1-X_{nm}^2/\eta^2X_{11}^2)}{(1-X_{nm}^2/\eta^2X_{11}^2)^2+(1/Q)^2}\,P_{nm}\right) \tag{12}$$

and

$$J^{(2)}=\frac{1}{\eta^2}\sum_{n,m}\left(\frac{b_{nm}[1-(-1)^n](1-X_{nm}^2/\eta^2X_{11}^2)+a_{nm}[1+(-1)^n](1/Q)}{(1-X_{nm}^2/\eta^2X_{11}^2)^2+(1/Q)^2}\,P_{nm}\right)\ . \tag{13}$$

Note that we have introduced the dimensionless voltage $\eta\equiv(\omega R)/(X_{11}\bar{c})$. The term P_{nm} is given by

$$P_{nm}=\frac{(kR)J_n(X_{nm})[J_{n-1}(kR)-J_{n+1}(kR)]}{2[X_{nm}^2-(kR)^2]} \quad \text{for } n\neq 0,\ X_{nm}\neq kR$$

$$=-(kR)J_0(X_{0m})J_1(kR)/[X_{0m}^2-(kR)^2] \quad \text{for } n=0,\ X_{0m}\neq kR$$

$$=[(X_{nm}^2-n^2)/2X_{nm}^2]\,J_n^2(X_{nm}) \quad \text{for } X_{nm}=kR\ .$$

The question now arises as to the appropriate conditions governing the phase-shift parameter Ψ_0; we adopt the view that Ψ_0 is adjusted by the device itself in order to accommodate the conditions imposed by external circuitry. In the case of constant-current sources, a Ψ_0 will be realized such that at each bias point (I, V), $I=I_{max}$ for that V:

$$(\Psi_0)_{opt}=\tan^{-1}(J^{(2)}/J^{(1)}) \tag{14}$$

is the appropriate phase lag for maximum current and this value of Ψ_0 guarantees $J_{dc}\geq 0$; without such optimization (in fact with $\Psi_0\equiv 0$), *negative* currents occur for certain ranges of voltage!

Equations (11)–(14) plus the expressions for a_{nm}, b_{nm}, P_{nm} constitute a complete solution of the problem. We see that resonances may be anticipated at the dimensionless voltages: $\eta=X_{nm}/X_{11}$. The first ten such points have the numerical values 1.000, 1.659, 2.081, 2.282, 2.888, 2.896, 3.485, 3.642, 3.810, 4.074. Clearly, some of these will be grouped quite closely and a high degree of peak overlap is expected unless the effective Q is quite large.

III. COMPARISON WITH EXPERIMENT

For simplicity we shall display $J^*\equiv J^{(1)}\cos(\Psi_0)+J^{(2)}\sin(\Psi_0)$ vs normalized voltage η; results in this form may be scaled to the dimensionless current J_{dc}/j_1 by means of the factor given in Eq. (11). Thus with $I_1\equiv j_1(\pi R^2)$,

$$J_{dc}/j_1=(4edI_1/\hbar c^2X_{11}^2)J^*\ . \tag{15}$$

Using the data of Bermon and Mesak[4] as a guide, we have, with $I_1\sim 5$ mA and $\lambda\sim 500$ Å, the approximate relationship $J_{dc}/j_1\approx(0.09)J^*$.

In Fig. 2 we present the I-V characteristics computed for $Q=10$ and two different magnetic fields determined by the conditions $kR=1.5X_{11}$ and $kR=4X_{11}$. The first case shown in Fig. 2(a) cor-

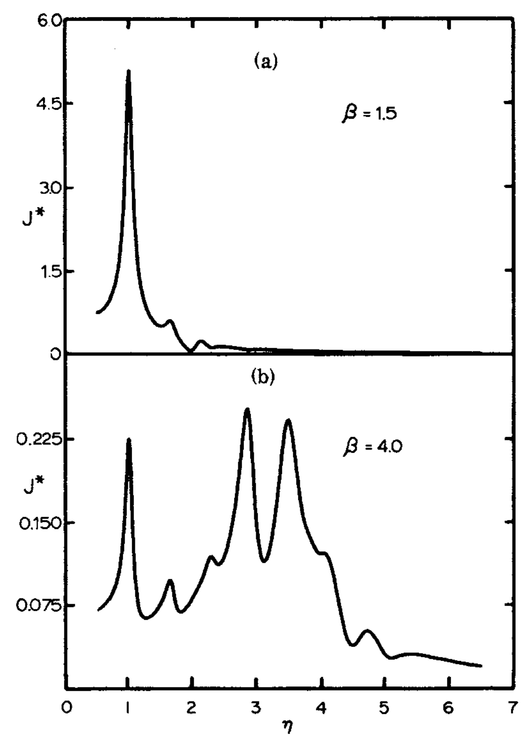

FIG. 2. Computed current–voltage characteristics for a cylindrical junction with $Q=10$ and two different values of applied magnetic field. Dimensionless voltage $[\eta\equiv\omega R/X_{11}\bar{c}]$ and special current $[J^*\equiv J^{(1)}\cos(\Psi_0)+J^{(2)}\sin(\Psi_0)]$ units have been employed. The field parameter is defined as $\beta\equiv kR/X_{11}$.

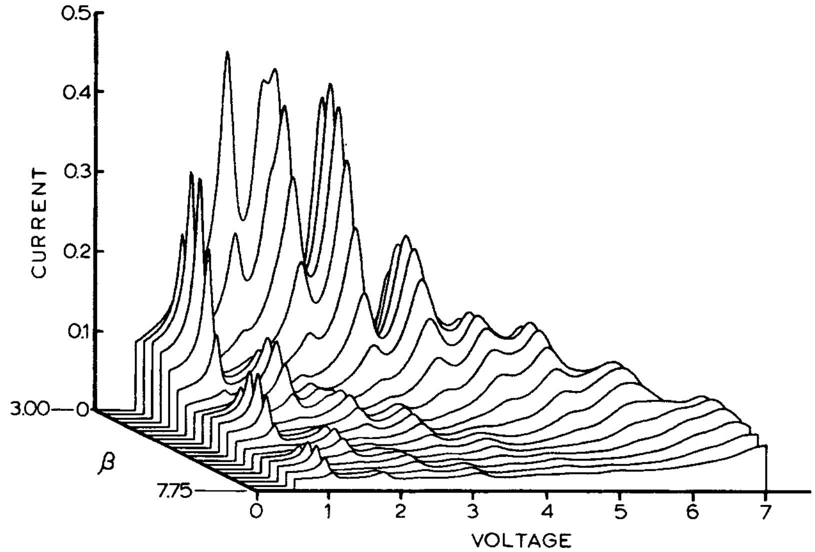

FIG. 3. Three-dimensional perspective view of the $J^* - \eta - \beta$ surface for a cylindrical junction with $Q=10$. The current and voltage units are the same as for Fig. 2.

responds to a field nearly "matched" to the first resonance; the analogous condition in a rectangular junction is $kL=\pi$. Figure 2(b) illustrates the appearance of the I-V curve for $kR=4X_{11}$ and it is seen that higher order resonances now dominate although all current peaks are a good deal smaller than the first peak amplitude at $kR=1.5X_{11}$. Applying the scaling factor of 0.09 to these data, we see that for the first resonance $(J_{dc}/j_1)_{max}\approx 0.45$; in other words the first resonance at optimum bias field should achieve about 45% of the maximum zero-voltage Josephson supercurrent. This estimate is at least reasonable and adjustments to obtain better fits to experimental data only apply to the damping parameter Q (which we assigned the nominal value of 10).

The truly complex structure of the I-V curves and the nature of their field dependence is best illustrated by a three-dimensional plot such as shown in Fig. 3.

IV. CONCLUSIONS

We have found expressions for the zero-frequency current-voltage characteristics of self-resonant cylindrical Josephson junctions by employing a perturbation technique. Current peaks are shown to occur at a discrete, but not evenly spaced, set of voltages and their magnetic field sensitivity has been examined. A sample calculation indicates that these formulas are consistent with the experimental data of Bermon and Mesak.

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