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C. Connell McCluskey

Wilfrid Laurier University, cmccluskey@wlu.ca

James S. Muldowney

University of Alberta

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STABILITY IMPLICATIONS OF BENDIXSON'S CRITERION*

C. CONNELL MCCLUSKEY[†] AND JAMES S. MULDOWNNEY[†]

Abstract. This note presents a proof that the omega limit set of a solution to a planar system satisfying the Bendixson criterion is either empty or is a single equilibrium. The proof involves elementary techniques which should be accessible to senior undergraduates and graduate students.

Key words. Bendixson's criterion, omega limit set

AMS subject classification. 3401

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We consider a planar differential equation

$$(1) \quad \dot{x} = P(x, y), \quad \dot{y} = Q(x, y),$$

where P and Q are real-valued, continuously differentiable functions on \mathbb{R}^2 . A solution $(x(t), y(t))$ is uniquely determined by its initial value $(x_o, y_o) = (x(0), y(0))$. If the solution exists for all $t \geq 0$, its positive semiorbit is $C^+(x_o, y_o) = \{(x(t), y(t)) : t \in [0, \infty)\}$, and if the solution exists for all real t , its orbit is $C(x_o, y_o) = \{(x(t), y(t)) : t \in (-\infty, \infty)\}$.

In senior undergraduate courses and introductory graduate courses on ordinary differential equations, the questions of existence and stability of periodic solutions of (1) are frequently studied. A solution $(x(t), y(t))$ is periodic with period $\omega > 0$ if $x(t + \omega) = x(t)$ and $y(t + \omega) = y(t)$ for all t . Clearly, the orbit of such a solution is either a simple closed curve or, in the case of constant solutions, a single point called an equilibrium.

The *Poincaré–Bendixson theory* for system (1) shows that if $C^+(x_o, y_o)$ is bounded and the omega limit set $\Omega(x_o, y_o) = \bigcap_{t \geq 0} \overline{C^+(x(t), y(t))}$ contains no equilibria, then $\Omega(x_o, y_o)$ is the orbit of a nonconstant periodic solution of (1), cf [4, p. 46]. On the negative side, if *Bendixson's criterion* $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \neq 0$ on \mathbb{R}^2 is satisfied, then no nonconstant periodic solutions of (1) exist, cf [4, p. 39]. Thus, every semiorbit of a system satisfying Bendixson's criterion is either unbounded or its omega limit set contains an equilibrium. In fact, a stronger assertion holds. The omega limit set is a single equilibrium or is empty.

Suppose that $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \neq 0$ on \mathbb{R}^2 and that $\Omega(x_o, y_o)$ is nonempty. Then

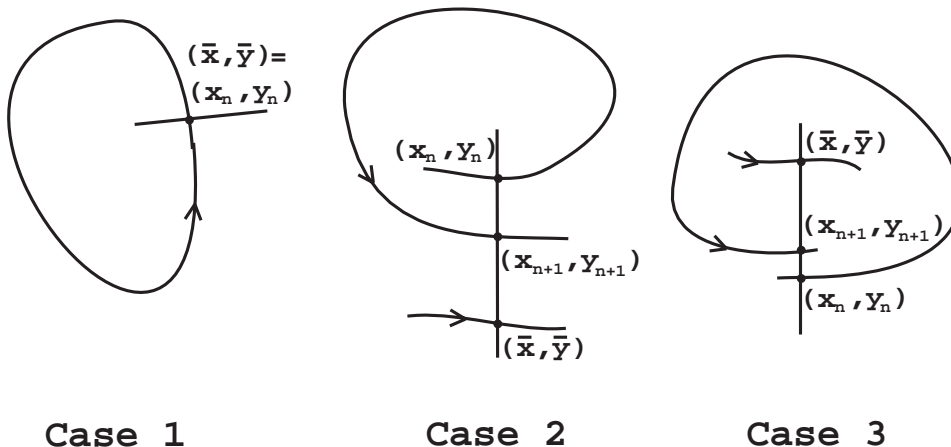
$$\lim_{t \rightarrow \infty} (x(t), y(t)) = (\bar{x}, \bar{y}),$$

where $P(\bar{x}, \bar{y}) = Q(\bar{x}, \bar{y}) = 0$.

This statement is a special case of a result for higher-dimensional systems satisfying generalized forms of Bendixson's criterion established by Smith [3] and by Li and Muldowney [2]. The proofs are nonelementary in that they rely heavily on the Pugh closing lemma and results such as the centre manifold theorem. We present here a more accessible proof for 2-dimensional systems which relies only on the content of a typical introductory course.

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[†]Department of Mathematical Sciences, University of Alberta, Edmonton, AB, T6G 2G1, Canada (jim.muldowney@ualberta.ca).



Suppose that (\bar{x}, \bar{y}) is an omega limit point which is not an equilibrium, and let T be a transversal through (\bar{x}, \bar{y}) . That is, let T be a straight line segment through (\bar{x}, \bar{y}) such that the vector $(P(x, y), Q(x, y))$ is neither zero nor parallel to T at any $(x, y) \in T$. Thus there is a neighborhood of (\bar{x}, \bar{y}) such that the orbit of any point in this neighborhood crosses T and all crossings of T are in the same direction.

Uniqueness of solutions implies that successive intersections of T by an orbit are monotone on T , and so one of the situations displayed in the diagrams as Case 1, Case 2, and Case 3 must occur where $t_n < t_{n+1}$ and $(x_n, y_n) = (x(t_n), y(t_n)) \in T$.

Let T_n be the segment of T joining (x_n, y_n) and (x_{n+1}, y_{n+1}) . Let C_n be the segment of $C^+(x_o, y_o)$ joining (x_n, y_n) and (x_{n+1}, y_{n+1}) . Let D_n be the region bounded by T_n and C_n . Then, by Green's theorem, we have

$$\begin{aligned}
 (2) \quad \int_{D_n} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx dy &= \int_{T_n} (P dy - Q dx) + \int_{C_n} (P dy - Q dx) \\
 &= \int_{T_n} (P dy - Q dx)
 \end{aligned}$$

since $\int_{C_n} (P dy - Q dx) = \pm \int_{t_n}^{t_{n+1}} (P \frac{dy}{dt} - Q \frac{dx}{dt}) dt = 0$ from (1) where the sign of the expression is determined by the orientation of the curve.

We can now rule out Case 1. In this case, T_n is merely the point (x_n, y_n) and so the right-hand side of (2) is zero. The left-hand side, however, is nonzero since the integrand is of constant sign and the domain of integration is nontrivial. This contradiction shows that $C^+(x_o, y_o)$ does not self-intersect; there are no nonconstant periodic orbits as asserted by Bendixson's criterion. In fact, this is the classical proof of Bendixson's result.

Consider Case 2. By Case 1, $C^+(x_o, y_o)$ does not self-intersect. Since the (x_n, y_n) are monotone on T we find $D_1 \subset D_2 \subset \dots$. The integrand has constant sign so that the left-hand side of (2) increases in magnitude as n increases. The sequence $\{(x_n, y_n)\}$ converges to (\bar{x}, \bar{y}) , so the length of segment T_n approaches zero as n tends to infinity. Combining this with the fact that P and Q are continuous and so must be bounded in a neighborhood of (\bar{x}, \bar{y}) , we see that $\lim_{n \rightarrow \infty} \int_{T_n} (P dy - Q dx) = 0$. This, from (2), gives us a contradiction, ruling out Case 2.

Finally, consider Case 3. In this case $D_1 \supset D_2 \supset \dots$, and we will show that there is a nonempty open set U such that $U \subset D_n$ for all n , and therefore, from (2),

$$0 < \left| \int_U \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx dy \right| \leq \left| \int_{D_n} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx dy \right| = \left| \int_{T_n} (P dy - Q dx) \right|.$$

But, $\lim_{n \rightarrow \infty} \int_{T_n} (P dy - Q dx) = 0$ as before, so again we have a contradiction ruling out Case 3. To construct U , observe that the transversal T is divided into two separate line segments by (\bar{x}, \bar{y}) . One of these line segments T' contains the sequence $\{(x_n, y_n)\}$. The other line segment T'' does not intersect $C^+(x_o, y_o)$, so for all n , we have $T'' \subset D_n$. Each D_n is positively invariant: the positive semiorbit of each point in D_n lies in D_n . Thus semiorbits that begin in T'' lie in each D_n for all $t > 0$. Let I be an open ended subsegment of T'' . Then $U = \{(x(t), y(t)) : t \in (0, 1), (x(0), y(0)) \in I\}$ is a nonempty subset of D_n as asserted. This resolves Case 3.

Thus we see that $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \neq 0$ implies that the omega limit set of a bounded orbit consists entirely of equilibria. The final step is to show that this omega limit set is a single equilibrium.

Suppose that $\Omega(x_o, y_o)$ contains more than a single point. Either $\Omega(x_o, y_o)$ is connected or each connected component of $\Omega(x_o, y_o)$ is unbounded, as can be seen by slightly modifying the proof of Theorem 1.1 in [1, p. 145]. In either case, there can be no isolated omega limit points. Let $(\bar{x}, \bar{y}) \in \Omega(x_o, y_o)$. Let $M(x, y)$ denote the Jacobian matrix at (x, y) of the map $f : (x, y) \mapsto (P(x, y), Q(x, y))$. This matrix is singular at each $(x, y) \in \Omega(x_o, y_o)$ since the solutions (x, y) of $P(x, y) = 0$, $Q(x, y) = 0$ are not isolated, and therefore f is not locally one-to-one. Thus one of the eigenvalues λ_1, λ_2 of $M(\bar{x}, \bar{y})$ is zero. Since

$$\lambda_1 + \lambda_2 = \text{Tr } M = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \neq 0,$$

it follows that there is a nonzero eigenvalue of $M(\bar{x}, \bar{y})$. Without loss of generality, it may be assumed that $(\bar{x}, \bar{y}) = (0, 0)$ and

$$P(x, y) = \lambda x + p(x, y), \quad Q(x, y) = q(x, y),$$

where λ is the nonzero eigenvalue of $M(0, 0)$ and $p(0, 0) = q(0, 0) = 0$, $Dp(0, 0) = Dq(0, 0) = 0$. This can always be achieved by an affine transformation. The implicit function theorem implies $P(x, y) = 0$ in a neighborhood V of $(0, 0)$ if and only if $x = g(y)$, where g is a continuously differentiable function on a neighborhood of 0 such that $g(0) = 0$ and $g'(y) = -\frac{\partial P}{\partial y} / \frac{\partial P}{\partial x}$. Moreover, $Q(x, y) = 0$ if $x = g(y)$ since $Q(0, 0) = 0$ and $\frac{d}{dy} Q(g(y), y) = \frac{\partial Q}{\partial x} g'(y) + \frac{\partial Q}{\partial y} = -\frac{\partial Q}{\partial x} \left(\frac{\partial P}{\partial y} / \frac{\partial P}{\partial x} \right) + \frac{\partial Q}{\partial y} = \det M(x, y) / \frac{\partial P}{\partial x} = 0$ since $M(x, y)$ is singular when $x = g(y)$.

Each equilibrium $(g(y), y) \in V$ has a one-dimensional stable manifold if $\lambda < 0$ and has a one-dimensional unstable manifold if $\lambda > 0$. Consider a neighborhood B of $(0, 0)$ such that the boundary of B is a simple closed curve formed by arcs from the stable or unstable manifolds of $(g(y_*), y_*)$ and $(g(-y_*), -y_*)$ and from the curves $x = g(y) + c$ and $x = g(y) - c$. If $c > 0$ and $y_* > 0$ are chosen sufficiently small, then $B \subset V$ and there is a point in $\Omega(x_o, y_o)$ which is not in B . Thus the interior of B and that of its complement both intersect $\Omega(x_o, y_o)$, so $C^+(x_o, y_o)$ enters and exits B infinitely many times; it must do so through the arcs $x = g(y) + c$ and $x = g(y) - c$ since stable and unstable manifolds are invariant. At least one of these arcs must therefore contain a point of $\Omega(x_o, y_o)$ and so is an equilibrium. This contradicts the fact that all equilibria in B are in the curve $x = g(y)$.

This establishes that $\Omega(x_o, y_o)$ contains at most one point, and so, as asserted, $\lim_{t \rightarrow \infty} (x(t), y(t)) = (\bar{x}, \bar{y})$ since (\bar{x}, \bar{y}) is the only omega limit point.

This result can be easily extended to Dulac's criterion. Namely, if there exists a scalar function α defined on \mathbb{R}^2 such that $\frac{\partial(\alpha P)}{\partial x} + \frac{\partial(\alpha Q)}{\partial y} \neq 0$ on \mathbb{R}^2 , then every omega limit set is either empty or a single point.

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