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1998

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STABILITY IMPLICATIONS OF BENDIXSON'S CRITERION*

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Abstract. This note presents a proof that the omega limit set of a solution to a planar system satisfying the Bendixson criterion is either empty or is a single equilibrium. The proof involves elementary techniques which should be accessible to senior undergraduates and graduate students.

Key words. Bendixson's criterion, omega limit set

AMS subject classification. 3401

PII. S0036144597328444

We consider a planar differential equation

$$\dot{x} = P(x, y), \qquad \dot{y} = Q(x, y), \tag{1}$$

where P and Q are real-valued, continuously differentiable functions on $\mathbb{R}^2$. A solution $(x(t), y(t))$ is uniquely determined by its initial value $(x_o, y_o) = (x(0), y(0))$. If the solution exists for all $t \geq 0$, its positive semiorbit is $C^+(x_o, y_o) = \{(x(t), y(t)) : t \in [0, \infty)\}$, and if the solution exists for all real t, its orbit is $C(x_o, y_o) = \{(x(t), y(t)) : t \in (-\infty, \infty)\}$.

In senior undergraduate courses and introductory graduate courses on ordinary differential equations, the questions of existence and stability of periodic solutions of (1) are frequently studied. A solution $(x(t), y(t))$ is periodic with period $\omega > 0$ if $x(t + \omega) = x(t)$ and $y(t + \omega) = y(t)$ for all t. Clearly, the orbit of such a solution is either a simple closed curve or, in the case of constant solutions, a single point called an equilibrium.

The *Poincaré–Bendixson theory* for system (1) shows that if $C^+(x_o, y_o)$ is bounded and the omega limit set $\Omega(x_o, y_o) = \bigcap_{t \geq 0} \overline{C^+(x(t), y(t))}$ contains no equilibria, then $\Omega(x_o, y_o)$ is the orbit of a nonconstant periodic solution of (1), cf [4, p. 46]. On the negative side, if *Bendixson's criterion* $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \neq 0$ on $\mathbb{R}^2$ is satisfied, then no non-constant periodic solutions of (1) exist, cf [4, p. 39]. Thus, every semiorbit of a system satisfying Bendixson's criterion is either unbounded or its omega limit set contains an equilibrium. In fact, a stronger assertion holds. The omega limit set is a single equilibrium or is empty.

Suppose that $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \neq 0$ on $\mathbb{R}^2$ and that $\Omega(x_o, y_o)$ is nonempty. Then

$$\lim_{t \to \infty} \big(x(t), y(t)\big) = \big(\bar{x}, \bar{y}\big),$$

where $P(\bar{x}, \bar{y}) = Q(\bar{x}, \bar{y}) = 0$.

This statement is a special case of a result for higher-dimensional systems satisfying generalized forms of Bendixson's criterion established by Smith [3] and by Li and Muldowney [2]. The proofs are nonelementary in that they rely heavily on the Pugh closing lemma and results such as the centre manifold theorem. We present here a more accessible proof for 2-dimensional systems which relies only on the content of a typical introductory course.

Case 1   **Case 2**   **Case 3**
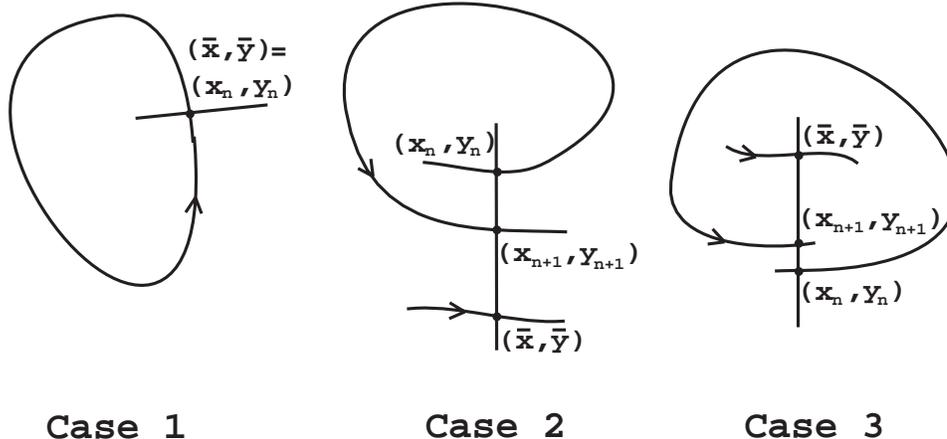
Suppose that $(\bar{x}, \bar{y})$ is an omega limit point which is not an equilibrium, and let T be a transversal through $(\bar{x}, \bar{y})$. That is, let T be a straight line segment through $(\bar{x}, \bar{y})$ such that the vector $(P(x, y), Q(x, y))$ is neither zero nor parallel to T at any $(x, y) \in T$. Thus there is a neighborhood of $(\bar{x}, \bar{y})$ such that the orbit of any point in this neighborhood crosses T and all crossings of T are in the same direction.

Uniqueness of solutions implies that successive intersections of T by an orbit are monotone on T, and so one of the situations displayed in the diagrams as Case 1, Case 2, and Case 3 must occur where $t_n < t_{n+1}$ and $(x_n, y_n) = (x(t_n), y(t_n)) \in T$.

Let T_n be the segment of T joining (x_n, y_n) and (x_{n+1}, y_{n+1}). Let C_n be the segment of $C^+(x_o, y_o)$ joining (x_n, y_n) and (x_{n+1}, y_{n+1}). Let D_n be the region bounded by T_n and C_n. Then, by Green's theorem, we have

$$\int_{D_n} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx\, dy = \int_{T_n} (P\, dy - Q\, dx) + \int_{C_n} (P\, dy - Q\, dx)$$

(2)

$$= \int_{T_n} (P\, dy - Q\, dx)$$

since $\int_{C_n} (P\, dy - Q\, dx) = \pm \int_{t_n}^{t_{n+1}} (P\, \frac{dy}{dt} - Q\, \frac{dx}{dt})\, dt = 0$ from (1) where the sign of the expression is determined by the orientation of the curve.

We can now rule out Case 1. In this case, T_n is merely the point (x_n, y_n) and so the right-hand side of (2) is zero. The left-hand side, however, is nonzero since the integrand is of constant sign and the domain of integration is nontrivial. This contradiction shows that $C^+(x_o, y_o)$ does not self-intersect; there are no nonconstant periodic orbits as asserted by Bendixson's criterion. In fact, this is the classical proof of Bendixson's result.

Consider Case 2. By Case 1, $C^+(x_o, y_o)$ does not self-intersect. Since the (x_n, y_n) are monotone on T we find $D_1 \subset D_2 \subset \ldots$. The integrand has constant sign so that the left-hand side of (2) increases in magnitude as n increases. The sequence $\{(x_n, y_n)\}$ converges to $(\bar{x}, \bar{y})$, so the length of segment T_n approaches zero as n tends to infinity. Combining this with the fact that P and Q are continuous and so must be bounded in a neighborhood of $(\bar{x}, \bar{y})$, we see that $\lim_{n \to \infty} \int_{T_n} (P\, dy - Q\, dx) = 0$. This, from (2), gives us a contradiction, ruling out Case 2.

Finally, consider Case 3. In this case $D_1 \supset D_2 \supset \ldots$, and we will show that there is a nonempty open set U such that $U \subset D_n$ for all n, and therefore, from (2),

$$0 < \left| \int_U \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx\, dy \right| \leq \left| \int_{D_n} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \right) dx\, dy \right| = \left| \int_{T_n} (P\, dy - Q\, dx) \right|.$$

But, $\lim_{n\to\infty} \int_{T_n} (P\, dy - Q\, dx) = 0$ as before, so again we have a contradiction ruling out Case 3. To construct U, observe that the transversal T is divided into two separate line segments by $(\bar{x}, \bar{y})$. One of these line segments T' contains the sequence $\{(x_n, y_n)\}$. The other line segment T'' does not intersect $C^+(x_o, y_o)$, so for all n, we have $T'' \subset D_n$. Each D_n is positively invariant: the positive semiorbit of each point in D_n lies in D_n. Thus semiorbits that begin in T'' lie in each D_n for all $t > 0$. Let I be an open ended subsegment of T''. Then $U = \{(x(t), y(t)) : t \in (0,1),\, (x(0), y(0)) \in I\}$ is a nonempty subset of D_n as asserted. This resolves Case 3.

Thus we see that $\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \neq 0$ implies that the omega limit set of a bounded orbit consists entirely of equilibria. The final step is to show that this omega limit set is a single equilibrium.

Suppose that $\Omega(x_o, y_o)$ contains more than a single point. Either $\Omega(x_o, y_o)$ is connected or each connected component of $\Omega(x_o, y_o)$ is unbounded, as can be seen by slightly modifying the proof of Theorem 1.1 in [1, p. 145]. In either case, there can be no isolated omega limit points. Let $(\bar{x}, \bar{y}) \in \Omega(x_o, y_o)$. Let $M(x, y)$ denote the Jacobian matrix at (x, y) of the map $f : (x, y) \mapsto (P(x, y), Q(x, y))$. This matrix is singular at each $(x, y) \in \Omega(x_o, y_o)$ since the solutions (x, y) of $P(x, y) = 0$, $Q(x, y) = 0$ are not isolated, and therefore f is not locally one-to-one. Thus one of the eigenvalues λ_1, λ_2 of $M(\bar{x}, \bar{y})$ is zero. Since

$$\lambda_1 + \lambda_2 = \operatorname{Tr} M = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} \neq 0,$$

it follows that there is a nonzero eigenvalue of $M(\bar{x}, \bar{y})$. Without loss of generality, it may be assumed that $(\bar{x}, \bar{y}) = (0, 0)$ and

$$P(x, y) = \lambda x + p(x, y), \quad Q(x, y) = q(x, y),$$

where λ is the nonzero eigenvalue of $M(0, 0)$ and $p(0,0) = q(0,0) = 0$, $Dp(0,0) = Dq(0,0) = 0$. This can always be achieved by an affine transformation. The implicit function theorem implies $P(x, y) = 0$ in a neighborhood V of $(0, 0)$ if and only if $x = g(y)$, where g is a continuously differentiable function on a neighborhood of 0 such that $g(0) = 0$ and $g'(y) = -\frac{\partial P}{\partial y} / \frac{\partial P}{\partial x}$. Moreover, $Q(x, y) = 0$ if $x = g(y)$ since $Q(0, 0) = 0$ and $\frac{d}{dy} Q(g(y), y) = \frac{\partial Q}{\partial x} g'(y) + \frac{\partial Q}{\partial y} = -\frac{\partial Q}{\partial x}\left(\frac{\partial P}{\partial y} / \frac{\partial P}{\partial x}\right) + \frac{\partial Q}{\partial y} = \det M(x, y) / \frac{\partial P}{\partial x} = 0$ since $M(x, y)$ is singular when $x = g(y)$.

Each equilibrium $(g(y), y) \in V$ has a one-dimensional stable manifold if $\lambda < 0$ and has a one-dimensional unstable manifold if $\lambda > 0$. Consider a neighborhood B of $(0, 0)$ such that the boundary of B is a simple closed curve formed by arcs from the stable or unstable manifolds of $(g(y_*), y_*)$ and $(g(-y_*), -y_*)$ and from the curves $x = g(y) + c$ and $x = g(y) - c$. If $c > 0$ and $y_* > 0$ are chosen sufficiently small, then $B \subset V$ and there is a point in $\Omega(x_o, y_o)$ which is not in B. Thus the interior of B and that of its complement both intersect $\Omega(x_o, y_o)$, so $C^+(x_o, y_o)$ enters and exits B infinitely many times; it must do so through the arcs $x = g(y) + c$ and $x = g(y) - c$ since stable and unstable manifolds are invariant. At least one of these arcs must therefore contain a point of $\Omega(x_o, y_o)$ and so is an equilibrium. This contradicts the fact that all equilibria in B are in the curve $x = g(y)$.

This establishes that $\Omega(x_o, y_o)$ contains at most one point, and so, as asserted, $\lim_{t\to\infty}(x(t), y(t)) = (\bar{x}, \bar{y})$ since $(\bar{x}, \bar{y})$ is the only omega limit point.

This result can be easily extended to Dulac's criterion. Namely, if there exists a scalar function α defined on $\mathbb{R}^2$ such that $\frac{\partial(\alpha P)}{\partial x} + \frac{\partial(\alpha Q)}{\partial y} \neq 0$ on $\mathbb{R}^2$, then every omega limit set is either empty or a single point.

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