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On Congruence Compact Monoids

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Abstract. A universal algebra is called *congruence compact* if every family of congruence classes with the finite intersection property has a non-empty intersection. This paper determines the structure of all right congruence compact monoids S for which Green's relations $\mathscr{J}$ and $\mathscr{H}$ coincide. The results are thus sufficiently general to describe, in particular, all congruence compact commutative monoids and all right congruence compact Clifford inverse monoids.

§1. *Introduction.* An algebra A is called *congruence compact* if every filter base consisting of congruence classes of A has a non-empty intersection. An equivalent definition results if one uses chains of congruence classes in place of filter bases. This is proved in the same way as the corresponding result for topological compactness (see for example **[15]**, p. 163). It follows that congruence compactness can be regarded as a finiteness condition that generalizes the descending chain condition for congruences, because the latter implies the descending chain condition for congruence classes. It is also straightforward to show that homomorphic images of congruence compact algebras are again congruence compact. These observations will be used freely in the sequel.

If one deals only with the discrete topology, the notion of *liner compactness* for rings and modules, introduced in **[19]**, coincides with congruence compactness in those categories. A ring is called right congruence compact if it is congruence compact as a right module over itself. In **[17]**, congruence compact acts over a monoid are discussed. (If S is a monoid with identity element 1, a *right S-act* A_S, the non-additive equivalent of a right R-module, is simply a non-empty set A equipped with a mapping $(a, s) \rightsquigarrow as$ from $A \times S$ into A satisfying the conditions $(as)t = a(st)$ and $a1 = a$ for every $a \in A$ and $s, t \in S$. Left S-acts are defined similarly.) A monoid will be called *right* (*left*) *congruence compact* if it is congruence compact as a right (left) act over itself. If a commutative monoid S is considered as a left or right act over itself, then the act congruences and semigroup congruences on S coincide, so that, when considering congruence compactness in this case, it is immaterial whether one views S as a monoid or as an S-act.

The main result of this paper is a description of the right congruence compact monoids in which Green's relations $\mathscr{J}$ and $\mathscr{H}$ coincide. With some effort, the description could be extended to a slightly wider class, but the problem of describing right congruence compact monoids in general appears to be out of reach. Also, in view of the fact that every congruence-free (*i.e.*, simple, in

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universal algebra parlance) monoid is congruence compact, it seems unreasonable at this time to hope for a general description of all (two-sided) congruence compact monoids.

Throughout this paper, we adopt the notation that, if θ is an equivalence relation on a set X and x is an element of X, then $x\theta$ stands for the θ-class of x. The reader is referred to **[9, 10]** for definitions and standard results from the algebraic theory of semigroups that are not fully explained here.

§2. *Some necessary conditions.* We first note that, for right congruence compactness or congruence compactness, it makes no difference whether one considers semigroups or monoids. (In the sequel, we will freely switch between the two.)

PROPOSITION 2.1. *Let S be a semigroup without identity (resp. zero) element. Then S is right congruence compact if and only if S^1 (resp. S^0) is right congruence compact.*

Proof. This is straightforward.

We now present conditions that are necessary for a monoid to be right congruence compact.

PROPOSITION 2.2. *Every right congruence compact monoid is periodic.*

Proof. Let S be a right congruence compact monoid and let s be any element of S. Suppose that

$$p_1, p_2, p_3, \ldots$$

is a list of the prime numbers in ascending order. For each $k \in \mathbb{N}$, we let ρ_k denote the principal right congruence on S generated by the pair (s^k, s^{k+p_k}). The set $\{s^k\rho_k : k \in \mathbb{N}\}$ is a filter sub-base consisting of right congruence classes because, for any finite set $K \subset \mathbb{N}$, the Chinese remainder theorem asserts the existence of an integer u such that $u \equiv k \pmod{p_k}$ for each $k \in K$. Hence, for each k, $u = k + z_k p_k$ for some integer z_k. Addition of a suitable multiple of $\prod_{k \in K} p_k$ yields an element $t \in \mathbb{N}$ such that $t = k + r_k p_k$ with each $r_k \in \mathbb{N}$. It follows easily that s^t belongs to $\bigcap_{k \in K} s^k\rho_k$. It now follows by assumption that there exists $x \in \bigcap_{k \in \mathbb{N}} s^k\rho_k$.

In any monoid, the right congruence $\rho(wt, t)$ generated by a pair of the form (wt, t) is easily seen to be contained in the right congruence $\rho(w, 1)$, which consists of all elements (a, b) such that $w^m a = w^n b$ for some non-negative integers m and n. (A full characterization of the relation $\rho(wt, t)$ is given in Lemma 2.2 of [2].) Thus, in the present situation, there exist non-negative integers m_k, n_k for each $k \in \mathbb{N}$ such that

$$s^{m_k p_k} x = s^{k + n_k p_k}.$$

Suppose that s has infinite order. We claim that

$$m_k p_k - k - n_k p_k$$

assumes the same value c for all natural numbers k. If not, there exist $k \neq l$ such that

$$m_k p_k - k - n_k p_k < m_l p_l - l - n_l p_l,$$

which may be written

$$m_k p_k + l + n_l p_l < m_l p_l + k + n_k p_k.$$

We may now calculate

$$\begin{aligned} s^{m_l p_l + k + n_k p_k} &= s^{m_l p_l} s^{k + n_k p_k} \\ &= s^{m_l p_l} s^{m_k p_k} x \\ &= s^{m_k p_k} s^{m_l p_l} x \\ &= s^{m_k p_k} s^{l + n_l p_l} \\ &= s^{m_k p_k + l + n_l p_l}, \end{aligned}$$

contradicting the assumption that s has infinite order. But now, if p_k is one of the prime factors of c, then from

$$p_k \mid m_k p_k - k - n_k p_k$$

we conclude that p_k divides k, which is surely impossible. If the constant c were equal to ± 1, then $(m_k - n_k) p_k = k \pm 1$ for all k, which is again impossible.

The following result is now clear (*cf.* [4], Exercise 4, p. 21).

COROLLARY 2.3. *Every congruence compact commutative monoid S is a semilattice of its maximal one-idempotent subsemigroups S_e, where*

$$S_e = \{s \in S : s^n = e \text{ for some } n \in \mathbb{N}\}$$

for each $e \in E = E(S)$.

(Note that the subsemigroups S_e are both the archimedean components of S and the η-classes of S, where η is the least semilattice congruence on S.)

Example 5.1 will show that, in general, the subsemigroups S_e of a congruence compact commutative semigroup S need not be congruence compact. Right congruence compactness is however inherited by *subgroups*, as will now be demonstrated.

PROPOSITION 2.4. *Every subgroup of a right congruence compact monoid is right congruence compact.*

Proof. Let G be a subgroup of a right congruence compact monoid S. Any right congruence class of G is of the form Hg, where $g \in G$ and H is a subgroup of G. Any such right congruence class is actually equal to the class $g\rho_{(H \times H)}$ of S, where $\rho_{(H \times H)}$ denotes the smallest right congruence on S identifying all elements of H.

Thus any filter base of right congruence classes of G is a filter base of right congruence classes of S, and so has non-empty intersection.

It is known that the right (or left) congruence compact groups are precisely the artinian groups, *i.e.*, the groups satisfying the descending chain condition on subgroups [13]. Thus right congruence compactness and left congruence compactness coincide for groups. If, for each $n \in \mathbb{N}$, we let H_n be the alternating group on the set $\{n, n+1, \ldots\}$, then, being simple, $G = H_1$ is a congruence compact group. However, G is neither left nor right congruence compact, because

$$G \supset H_2 \supset H_3 \supset \cdots$$

is an infinite descending chain of subgroups of G.

§3. *Right congruence compact monoids in which* $\mathcal{J} = \mathcal{H}$. In the next part of the paper, we restrict our attention to monoids in which all of Green's relations coincide. We first consider chain conditions for idempotent-generated one- and two-sided ideals of such a monoid S. In case S is periodic (*e.g.*, if S is left or right congruence compact), we will show that S satisfies the a.c.c. or d.c.c. for idempotent-generated principal left or right ideals if and only if it does so for idempotent-generated principal two-sided ideals. A preliminary lemma is needed.

LEMMA 3.1. *Let S be a monoid on which Green's relation $\mathcal{J}$ is a right congruence. Then, for any $p, q \in S$,*

$$SpSSqS = SpS \quad \text{if and only if} \quad pq\mathcal{J}p.$$

Proof. If $pq\mathcal{J}p$, then

$$SpS = SpqS \subseteq SpSqS \subseteq SpS,$$

whether or not $\mathcal{J}$ is a right congruence.

On the other hand, suppose that $SpSqS = SpS$, so that $p = upxqy$ for some $u, x, y \in S$. It follows that $p\mathcal{J}pxq$ and $px\mathcal{J}p$. Hence, $pxq\mathcal{J}pq$, and so $p\mathcal{J}pq$, as required.

PROPOSITION 3.2. *Let S be a periodic monoid in which $\mathcal{J} = \mathcal{H}$. Then the following conditions are equivalent:*

1. *every ascending (resp. descending) chain of idempotent-generated principal right ideals of S is finite;*
2. *every ascending (resp. descending) chain of idempotent-generated principal left ideals of S is finite;*
3. *every ascending (resp. descending) chain of idempotent-generated principal ideals of S is finite.*

Proof. The proof consists of verifying each of the following claims. We omit those verifications that we judge to be straightforward.

(a) Suppose that S is a monoid on which $\mathcal{J}$ (resp. $\mathcal{L}$, $\mathcal{R}$) is a congruence. Then $\bar{S} = S/\mathcal{J}$ (resp. $S/\mathcal{L}$, $S/\mathcal{R}$) is $\mathcal{J}$- (resp. $\mathcal{L}$-, $\mathcal{R}$-) trivial. There is an order-isomorphism between the poset of principal (left, right) ideals of S and the poset of principal (left, right) ideals of $S/\mathcal{J}$ (resp. $S/\mathcal{L}$, $S/\mathcal{R}$), under which a (left, right) principal ideal I of S is mapped to the (left, right) principal ideal $\bar{I}=\{\bar{x}: x\in I\}$ of $\bar{S}$. (If $x\in S$, then $\bar{x}$ denotes the congruence class of x, modulo the relevant Green's relation.) Under this correspondence, globally idempotent principal (left, right) ideals of S correspond to globally idempotent (left, right) ideals of $\bar{S}$. (A one- or two-sided ideal I is called *globally idempotent* if $I^2=I$.)

(b) In a periodic monoid S, a principal (left, right) ideal is globally idempotent if and only if it is idempotent-generated. The non-obvious part of this is proved as follows. Suppose that $aSaS=aS$ for some $a\in S$. If u, $v\in S$ are such that $a=auav$, then

$$a=auav=(au)auav(v)=(au)^2a(v^2)=\cdots=(au)^na(v^n)$$

for all natural numbers n. If $e=(au)^n$ is idempotent, then $aS=eS$, as desired. (A similar argument prevails for left or two-sided principal ideals.)

(c) Suppose that S is a monoid in which $\mathcal{J}=\Delta$. Then S satisfies the ascending (resp. descending) chain condition for idempotent-generated right ideals if and only if S satisfies the corresponding chain condition for idempotent-generated two-sided ideals. We present a proof of (c).

Suppose first that

$$e_1S\subseteq e_2S\subseteq\cdots$$

is an ascending chain of idempotent-generated right ideals of S, and that S satisfies the ascending chain condition for idempotent-generated principal two-sided ideals. Suppose that n is a natural number for which $Se_nS=Se_{n+1}S$. Then, because S is $\mathcal{J}$-trivial, $e_n=e_{n+1}$, and so $e_nS=e_{n+1}S$, as desired.

Now assume that

$$Se_1S\subseteq Se_2S\subseteq\cdots,$$

and that S satisfies the ascending chain condition for idempotent-generated principal right ideals. For any n, from $Se_nS\subseteq Se_{n+1}S$, we deduce that

$$Se_{n+1}SSe_nS=Se_nS.$$

From Lemma 3.1 (in its left–right symmetric form), together with $\mathcal{J}$-triviality, we obtain $e_{n+1}e_n=e_n$, and so in fact we have a chain

$$e_1S\subseteq e_2S\subseteq\cdots$$

of idempotent-generated principal right ideals of S. If $e_mS=e_{m+1}S$, then also $Se_mS=Se_{m+1}S$, completing the proof of (c).

The equivalence of, say, the "ascending" parts of 1 and 3 of the proposition is now proved as follows. Note first that because $\mathcal{J}=\mathcal{R}=\mathcal{L}=\mathcal{H}$, it follows that $\mathcal{J}$ is a congruence. Using (a) and (b), we see that S satisfies the ascending chain condition for idempotent-generated principal right ideals if and only if the monoid $S/\mathcal{R}$ does. But $\mathcal{R}=\mathcal{J}$, and $S/\mathcal{J}$ is $\mathcal{J}$-trivial, so this is true (using (c)) if and only if $S/\mathcal{J}$ satisfies the a.c.c. for idempotent-generated principal two-sided ideals, which occurs (using (a) and (b) again) if and only if S itself satisfies the a.c.c. for idempotent-generated principal two-sided ideals.

The remaining parts of the proof follow similarly.

Our next goal is to show that, in a right congruence compact monoid in which $\mathcal{J}=\mathcal{H}$, all chains of idempotent-generated principal left (or right) ideals are finite. This is equivalent to showing that such a right congruence compact monoid satisfies both the ascending and descending chain conditions for idempotent-generated principal left (or right) ideals.

LEMMA 3.3. *Let C be a non-empty subset of a monoid S. Then C is a right congruence class of S if and only if, for all $s, t, u \in S$,*

$$s, t, su \in C \Rightarrow tu \in C.$$

Further, C is a congruence class of S if and only if, for all $s, t, u, v \in S$,

$$s, t, vsu \in C \Rightarrow vtu \in C.$$

For $x \in S$ and $Y \subseteq S$ with $\{Sy : y \in Y\}$ a chain, define

$$T_{x,Y} = \{s \in S : xs \notin x\mathcal{R}, ys = y \text{ for some } y \in Y\},$$

and

$$T_{Y,x} = \{s \in S : ys \notin y\mathcal{R} \text{ for some } y \in Y, xs = x\}.$$

Note that $xs \notin x\mathcal{R}$ is equivalent to $xsS \subset xS$.

LEMMA 3.4.

(a) *$T_{x,Y}$ is a subsemigroup of S, which is either empty or a right congruence class of S.*
(b) *If $\mathcal{J}=\Delta$, then $T_{x,Y}$ is either empty or a congruence class of S.*
(c) *If $Y \subseteq E(S) \cap Sx$, and $(x, y) \notin \mathcal{J}$ for all $y \in Y$, then $Y \subseteq T_{x,Y}$.*

Proof. (a) If $s \in T_{x,Y}$ and $t \in S$, then $xtsS \subseteq xtS \subset xS$. If t also belongs to $T_{x,Y}$, let $y, y' \in Y$ be such that $ys = y$ and $y't = y'$. Let y'' be such that

$$Sy'' \subseteq Sy \cap Sy'.$$

Then $y''st = y''t = y''$. This shows that $T_{x,Y}$, if non-empty, is a subsemigroup of S. To show that it is a right congruence class, it suffices, by Lemma 3.3, to show that

$$s, t, su \in T_{x,Y} \Rightarrow tu \in T_{x,Y}.$$

Given $s, t, su \in T_{x,Y}$, we see first that $xtuS \subseteq xtS \subset xS$. Supposing that y, y', y'' and $\tilde{y}$ are elements of Y such that

$$ys = y,\ y't = y',\ y''su = y''$$

and

$$S\tilde{y} \subseteq Sy \cap Sy' \cap Sy'',$$

we see that

$$\tilde{y}tu = \tilde{y}u = \tilde{y}su = \tilde{y},$$

and so the proof of part (a) is complete.

(b) Assume that $\mathscr{J}=\Delta$. Note that this implies that $\mathscr{R}=\Delta$, which in turn implies that, for all $x, y, z \in S$,

$$xyz=x \Rightarrow xy=x=xz.$$

Now, using the second part of Lemma 3.3, suppose that $s, t, vsu \in T_{x,Y}$. We seek to show that $vtu \in T_{x,Y}$ also. To show that $xvtuS \subset xS$, we consider two cases. If $xv=x$, then

$$xvtuS=xtuS \subseteq xtS \subset xS,$$

because $t \in T_{x,Y}$. If $xv \neq x$, then $xvtuS \subseteq xvS \subset xS$, the last relation following from $\mathscr{R}$-triviality. Finally, if $y, y' \in Y$ are such that $yvsu=y$ and $y't=y'$, then we also have $yu=yv=ys=y$, as noted above. Now, for any $y'' \in Y$ with $Sy'' \subseteq Sy \cap Sy'$, we easily see that $y''vtu=y''$, as needed.

(c) Suppose that $Y \subseteq E(S) \cap Sx$, and that $(x, y) \notin \mathscr{J}$ for all $y \in Y$. To show that $Y \subseteq T_{x,Y}$, suppose that $xyS=xS$ for some $y \in Y$. Then we would have $x=xyz$ for some $z \in S$, and also $y=tx$ for some $t \in S$. Together these imply that $(x, y) \in \mathscr{J}$, a contradiction. The fact that each $y \in Y$ is idempotent finishes the proof.

LEMMA 3.5.

(a) *$T_{Y,x}$ is a subsemigroup of S, which is either empty or a right congruence class of S.*

(b) *If $\mathscr{J}=\Delta$, then $T_{Y,x}$ is either empty or a congruence class of S.*

(c) *If $x \in E(S)$ and $(x, y) \notin \mathscr{J}$ for some $y \in Y$ such that $Sx \subset Sy$, then $x \in T_{Y,x}$ (so that $T_{Y,x}$ is a right congruence class).*

Proof. (a) Let s, t belong to $T_{Y,x}$, so that $ysS \subset yS$ for some $y \in Y$, and $xs=xt=x$. Then $ystS \subseteq ysS \subset yS$, and $xst=xt=x$, showing that st belongs to $T_{Y,x}$. To show that $T_{Y,x}$ is a right congruence class, it suffices to show that

$$s, t, su \in T_{Y,x} \Rightarrow tu \in T_{Y,x}.$$

Because $ytS \subset yS$ for some $y \in Y$, it follows that $ytuS \subseteq ytS \subset yS$. Moreover, $xtu=xu=xsu=x$. This completes the proof of (a).

(b) To show that $T_{Y,x}$ is a congruence class, under the assumption $\mathscr{J}=\Delta$, it suffices to show that

$$s, t, vsu \in T_{Y,x} \Rightarrow vtu \in T_{Y,x}.$$

If $yv=y$, then $yvtuS=ytuS \subseteq ytS \subset yS$, whereas if $yv \neq y$, then $yvtuS \subseteq yvS \subset yS$ (because $\mathscr{J}=\mathscr{R}=\Delta$). Also, from $x=xvsu$ we see that $xu=xs=xv=x$, and so $xvtu=xtu=xu=x$ as required. Thus $T_{Y,x}$ is a congruence class.

(c) We must show that $T_{Y,x}$ contains x. If $yxS=yS$ were true, then because $Sx \subset Sy$ we would have $x \mathscr{J} y$, contradicting our assumption. The fact that $x^2=x$ finishes the proof.

LEMMA 3.6. *Let S be a congruence compact monoid which is $\mathscr{J}$-trivial. Then every descending chain of idempotent-generated principal left, right, or two-sided ideals of S is finite.*

Proof. Assume that there exists a strictly descending chain of idempotent-generated principal left ideals. That is, there exists a sequence

$$e_1, e_2, \ldots$$

of idempotents such that

$$Se_1 \supset Se_2 \supset \cdots.$$

Define

$$E_i = \{e_j : j > i\}.$$

Because $E_i \subseteq E(S) \cap Se_i$ and $(e_i, e_j) \notin \mathcal{J}$ whenever $j > i$, using Lemma 3.4(c) we conclude that $E_i \subseteq T_{e_i,E_i}$ for each i. Now from part (b) of the same lemma we know that T_{e_i,E_i}, being non-empty, is a congruence class of S. If $i<j$ then $T_{e_i,E_i} \supseteq T_{e_j,E_j}$. (If $e_i s = e_i$ were true, then

$$e_j s = e_j e_i s = e_j e_i = e_j$$

would also hold; in other words, $s \in T_{e_j,E_j}$ implies that $e_i s S \subset e_i S$. Moreover, if $ys = s$ for $y \in E_j$, then the inclusion $E_j \subseteq E_i$ is used to complete the argument.) Finally, note that

$$\bigcap_{i \in \mathbb{N}} T_{e_i,E_i} = \varnothing.$$

This is because, if $x \in \bigcap_{i \in \mathbb{N}} T_{e_i,E_i}$, then $e_j x = e_j$ for at least one index j. However, if x were an element of T_{e_j,E_j}, then we would have $e_j x \neq e_j$.

COROLLARY 3.7. *Let S be a right congruence compact monoid for which $\mathcal{J} = \mathcal{H}$. Then every descending chain of idempotent-generated left, right, or two-sided principal ideals of S is finite.*

Proof. Note that $\mathcal{J}$ is a congruence on S. The monoid $S/\mathcal{J}$ is $\mathcal{J}$-trivial and right congruence compact (and therefore congruence compact). We have shown in the preceding lemma that every descending chain of idempotent-generated principal left ideals of $S/\mathcal{J}$ is finite. The correspondence principle stated in part (a) of the proof of Proposition 3.2 can now be used to conclude that every descending chain of idempotent-generated principal left ideals of S is finite. Proposition 3.2 itself then gives the result for right or two-sided ideals of S.

LEMMA 3.8. *Let S be a congruence compact monoid which is $\mathcal{J}$-trivial. Then every ascending chain of idempotent-generated principal left, right, or two-sided ideals of S is finite.*

Proof. Assume that there exists a strictly ascending chain of idempotent-generated principal left ideals. That is, there exists a sequence

$$e_1, e_2, \ldots$$

of idempotents such that

$$Se_1 \subset Se_2 \subset \cdots.$$

Define

$$E_i = \{e_j : j > i\}.$$

Then, for each i, T_{E_i,e_i} is a congruence class (because $e_i \in T_{E_i,e_i}$, by Lemma 3.5 (c)). Next, observe that

$$i \leqslant j \Rightarrow T_{E_i,e_i} \supseteq T_{E_j,e_j};$$

if $s \in T_{E_j,e_j}$ then $e_k sS \subset e_k S$ for some $k > j \geqslant i$, and also

$$e_i s = ue_j s = ue_j = e_i$$

for some $u \in S$.

Finally, note that

$$\bigcap_{i \in \mathbb{N}} T_{E_i,e_i} = \varnothing.$$

This is because, if $x \in \bigcap_{i \in \mathbb{N}} T_{E_i,e_i}$, then $e_k x \neq e_k$ for some k. But then x does not belong to T_{E_k,e_k}.

Corollary 3.9. *Let S be a right congruence compact monoid for which $\mathscr{J} = \mathscr{H}$. Then every chain of idempotent-generated left, right, or two-sided principal ideals of S is finite.*

Proof. Corollary 3.7 shows that every descending chain of idempotent-generated left (or right) ideals of S is finite. A similar proof, in conjuction with Lemma 3.8, establishes the same thing for ascending chains, and so the result follows.

Lemma 3.10. *Let S be a monoid for which $\mathscr{J} = \mathscr{H}$. If $x, z \in S$, then $xSzS = xS$ if and only if $xzS = xS$.*

Proof. Suppose that $x = xuzv$ for elements $u, v \in S$. Then $x \mathscr{R} xu$ and so, because $\mathscr{R} = \mathscr{L}$ is a right congruence, $x = xuzv \mathscr{R} xzv$. From this, $xS \subseteq xzS$. The remainder of the proof is clear.

Proposition 3.11. *Let S be a right congruence compact monoid for which $\mathscr{J} = \mathscr{H}$. Then, for every infinite set X of incomparable principal right ideals of S, there exist $z \in S$ and $P, Q \in X$ such that $PzS = P$ and $QzS \neq Q$.*

Proof. Assuming that the condition fails, let $X = \{x_i S : i \in I\}$ be an infinite set of incomparable principal right ideals of S such that, for all $z \in S$ and all $i, j \in I$, we have $x_i zS = x_i S$ if and only if $x_j zS = x_j S$. (Note that Lemma 3.10 was used here.) The assumption that $\mathscr{J} = \mathscr{H}$ implies that $\mathscr{R}$ is a congruence on S. For each $s \in S$, let $\bar{s}$ denote the congruence class $s\mathscr{R}$, and let $Y = \{\bar{x}_i : i \in I\}$. Note that, if J is a non-empty subset of I, then $C_J = \{\bar{x}_j : j \in J\}$ is a right congruence class of $S/\mathscr{R}$. For, using Lemma 3.3, suppose that $\bar{x}_j$, $\bar{x}_k$ and $\bar{x}_j \bar{z}$ belong to C_J for some $x \in S$. If $\bar{x}_j \bar{z} = \bar{x}_l$, then, from the ensuing relation $x_j zS = x_l S$, we obtain $j = l$, and so necessarily $x_j zS = x_j S$. By our assumption,

$x_k zS = x_k S$, and so $\bar{x}_k \bar{z} = \bar{x}_k \in C_J$. We now see that

$$\{C_J : J \text{ is a cofinite subset of } I\}$$

is a filter-base of congruence classes of $S/\mathcal{R}$ having empty intersection, a contradiction to the fact that the monoid $S/\mathcal{R}$ is congruence compact.

§4. *The main theorem.* In this section, we will show that the necessary conditions obtained heretofore for a monoid S with $\mathcal{H} = \mathcal{J}$ to be right congruence compact are also sufficient. Some preliminary lemmas are required.

Let P and Q be principal ideals of a monoid S. Then either $PQ = P$ or $PQ \subset P$. Consider the corresponding $\mathcal{J}$-classes $P \backslash \hat{P}$ and $Q \backslash \hat{Q}$, where $\hat{P}$ and $\hat{Q}$ denote the largest (possibly empty) ideals properly contained in P and Q, respectively.

LEMMA 4.1. *$PQ = P$ if and only if $(P \backslash \hat{P})(Q \backslash \hat{Q}) \nsubseteq \hat{P}$.*

Proof. If $PQ = P$, and $p, q \in S$ are such that $P = SpS$, $Q = SqS$, then $p = xpyqz$ for certain $x, y, z \in S$. Because $py \notin \hat{P}$ and $q \notin \hat{Q}$, it follows that pyq belongs to $(P \backslash \hat{P})(Q \backslash \hat{Q})$ but not to $\hat{P}$.

On the other hand, if $PQ \subset P$, we would have

$$(P \backslash \hat{P})(Q \backslash \hat{Q}) \subseteq PQ \subseteq \hat{P}.$$

LEMMA 4.2. *Let P be a principal ideal of a monoid S. If a principal ideal Q is minimal among the principal ideals Q' such that $PQ' = P$, then Q is the* smallest *among them. Moreover,*

$$P \backslash \hat{P} \subseteq (P \backslash \hat{P})(Q \backslash \hat{Q}).$$

If $\mathcal{J}$ is a congruence on S, then

$$P \backslash \hat{P} = (P \backslash \hat{P})(Q \backslash \hat{Q}).$$

Proof. Let Q' be a principal ideal satisfying $PQ' = P$. If $Q \nsubseteq Q'$, then $QQ' \subset Q$ (otherwise $Q = QQ' \subseteq Q'$). Therefore we know that

$$QQ' \subseteq \hat{Q} = \bigcup_{a \in \hat{Q}} SaS,$$

and so

$$P = PQQ' \subseteq P\hat{Q} = \bigcup_{a \in \hat{Q}} PSaS.$$

If $P = SpS$ for $p \in S$, then there exists $a \in \hat{Q}$ such that $p \in PSaS$, and so $PSaS = P$ with $SaS \subset Q$. However, this contradicts the minimality of Q.

Any element p of $P \backslash \hat{P}$ (*i.e.*, any generator of P) can be written as $p = p_1 q_1$ where $p_1 \in P \backslash \hat{P}$ and $q_1 \in Q \backslash \hat{Q}$ (for $PQ = P$, and if $q_1 \in \hat{Q}$ we would have $PSq_1S = P$ with $Sq_1S \subseteq \hat{Q} \subset Q$, contradicting the minimality of Q).

Finally, assume that $\mathcal{J}$ is a congruence on S. If $P \backslash \hat{P} = J_p$ and $Q \backslash \hat{Q} = J_q$, then, from above, $J_p \subseteq J_p J_q \subseteq J_{pq}$. But comparable congruence classes of the same congruence are equal, and so $J_p = J_p J_q$, as required.

LEMMA 4.3. *Let C be a right congruence class of a monoid S for which Green's relations $\mathcal{J}$ and $\mathcal{H}$ coincide. Suppose in addition that S is periodic and that all descending chains of idempotent-generated principal ideals are finite. Then there exists an idempotent element e_C such that Se_CS is the smallest element in the set $\{SpS: p\in S$ and $ScSSpS=ScS$ for some $c\in C\}$.*

Proof. Consider

$$F_C=\{SfS: f\in E(S),\ ScSSfS=ScS \text{ for some } c\in C\},$$

a subset of the aforementioned set of principal ideals. Clearly $S=S1S\in F_C$. Our supposition about S implies that F_C has a minimal element Se_CS, where $e_C\in E(S)$ and $SySSe_CS=SyS$ for some $y\in C$.

We claim that Se_CS is in fact the smallest element of F_C. For, suppose that

$$Se_CS \nsubseteq Se_xS$$

with $x\in C, e_x\in E(S)$, $SxSSe_xS=SxS$, and Se_xS minimal in F_C. From $SxSSe_xS=SxS$ and Lemma 3.1, we conclude that $xe_x\mathcal{J}x$. But $\mathcal{J}=\mathcal{L}$, and so $Sxe_x=Sx$. Because e_x is idempotent, this implies that $x=xe_x$. By the same token, $y=ye_C$. Suppose that C is a class of the right congruence ρ. Then

$$ye_Ce_x=ye_x\rho xe_x=x,$$

and so $ye_Ce_x\in C$. Moreover, by periodicity, there exists $m\in\mathbb{N}$ such that

$$y\rho ye_Ce_x\rho\cdots\rho y(e_Ce_x)^m=ye$$

where $e=(e_Ce_x)^m\in E(S)$, and so $ye\in C$ also. Now observe that

$$SyeSSeS=SyeS,$$

and so SeS belongs to F_C. But $Ses\subseteq Se_CS$, whence $SeS=Se_CS$. But because e and e_C are idempotents in the same $\mathcal{J}=\mathcal{H}$ class, $e_C=e\in Se_xS$, a contradiction.

Now let p be such that $ScSSpS=ScS$ for some $c\in C$. If $c=ucvpw$ for $u, v, w\in S$, then

$$c=ucvpw=\cdots=u^ncf,$$

where n is a positive integer and $f=(vpw)^n$ is idempotent. Because

$$ScSSfS=ScS,$$

from the previous paragraph it follows that

$$Se_CS\subseteq SfS\subseteq SpS,$$

as claimed

The next theorem is the main result of this paper. In connection with clause 2, recall that the right or left congruence compact groups are described exactly in [13] as the groups in which every descending chain of subgroups is finite.

THEOREM 4.4. *Let S be a monoid for which Green's relations $\mathscr{J}$ and $\mathscr{H}$ coincide. Then S is right congruence compact if and only if*

1. *S is periodic,*
2. *all subgroups of S are right congruence compact,*
3. *all chains of idempotent-generated principal left (or right, or two-sided) ideals of S are finite, and*
4. *for every infinite set X of incomparable principal right ideals there exist $z \in S$ and $P, Q \in X$ such that $PzS = P$ and $QzS \neq Q$.*

Proof. The necessity follows from Propositions 2.2 and 2.4, Corollary 3.9, and Proposition 3.11.

For the sufficiency, suppose that S satisfies conditions 1–4, and that $\mathscr{C}$ is a chain of right congruence classes of S.

For any $C \in \mathscr{C}$, consider the idempotent e_C described in Lemma 4.3. (This element is uniquely determined because $\mathscr{J} = \mathscr{H}$.) If C and C' are elements of $\mathscr{C}$ with $C \subseteq C'$, then, from $ScSSe_CS = ScS$ with $c \in C \subseteq C'$, we deduce that Se_CS belongs to $\{SpS : p \in S \text{ and } ScSSpS = ScS \text{ for some } c \in C'\}$, and so $Se_{C'}S \subseteq Se_CS$. Using condition 3, we may assume that $C_0 \in \mathscr{C}$ is such that $Se_{C_0}S$ is the largest element of the chain $\{Se_CS : C \in \mathscr{C}\}$, so that $e_C = e_{C_0} = f$ (say) for all $C \in \mathscr{C}$ with $C \subseteq C_0$.

Consider any $C \in \mathscr{C}$ with $C \subseteq C_0$. Define

$$U_C = \{x\mathscr{J} : x \in C, xf = x\}.$$

We aim to show that U_C is finite. From the definition of e_C and Lemma 3.1, U_C is non-empty.

Suppose that $x\mathscr{J}$ and $x'\mathscr{J}$ $(x, x' \in C)$ are distinct elements of U_C. Let us show that xS and $x'S$ are incomparable. If $xS \subseteq x'S$ were true, then, because C is a right congruence class, $x = x's$ for some $s \in S$ would imply that $xs^m \in C$ for some positive integer m such that $e = s^m$ is idempotent. Because $SxeSSeS = SxeS$, it would follow that $SfS \subseteq SeS$, and so $SfSSeS = SfS$, leading to $fe\mathscr{R}f$ (using Lemma 3.1). Thus $f = feu$ for some $u \in S$, and therefore

$$x' = x'f = x'feu = x'eu = x'ss^{m-1}u \in xS,$$

implying that $xS = x'S$. Hence we would reach the contradiction $x\mathscr{J} = x'\mathscr{J}$. This shows that distinct elements $x\mathscr{J}$ and $x'\mathscr{J}$ of U_C correspond to incomparable principal right ideals xS and $x'S$.

Thus, if U_C were infinite, then $\{xS : x \in C, xf = x\}$ would be an infinite set of incomparable principal right ideals of S. But now, using condition 4, we would obtain $z \in S$ and $x, x' \in C$ such that $xf = x$, $x'f = x'$, $xSzS = xS$, and $x'SzS \subset x'S$. From $SxSSzS = SxS$ and from $SfS \subseteq SzS$ (Lemma 4.3), we have $f = szt$ for some $s, t \in S$, and so

$$x' = x'f = x'szt \in x'SSzS,$$

a contradiction. Hence U_C is indeed finite.

We may now choose $C_1 \in \mathscr{C}$, $C_1 \subseteq C_0$ such that U_{C_1} is minimal. For ease of notation, let $\mathscr{Y} = U_{C_1}$. Then each $C \in \mathscr{C}$ has non-empty intersection with each $Y \in \mathscr{Y}$. If we can show that

$$\bigcap \{Y \cap C: C \in \mathscr{C}, C \subseteq C_0\}$$

is non-empty for some $Y \in \mathscr{Y}$, then this will certainly prove our result. In fact, we show that this is true for *every* $Y \in \mathscr{Y}$.

Fix $Y = y\mathscr{J} \in \mathscr{Y}$. Because $\mathscr{J}$ is a congruence, Y is a right congruence class of S, as is, therefore, each $Y \cap C$, with $C \in \mathscr{C}$ and $C \subseteq C_0$. Let $F = f\mathscr{J}$ (a group, because $\mathscr{J} = \mathscr{H}$). It follows from Lemma 4.2 and the minimality of SfS (Lemma 4.3) that $YF = Y$. In fact, $Y = yF$. (For the proof, consider $y' \in Y$. From $YF = Y$ we have $yf \in Y = y\mathscr{R}$, and so $y' = yfs$ for some $s \in S$. From this, $SySSfsS = SyS$ and so, as before, $SfS \subseteq SfsS$. Because $SfsS \subseteq SfS$, we conclude that $fs \in f\mathscr{J} = F$. Hence $y' \in yF$.)

Because $Y = yF$ is a homomorphic image of the congruence compact group F (in the category of right F-acts), Y is congruence compact. It now follows that

$$\bigcap \{Y \cap C: C \in \mathscr{C}, C \subseteq C_0\}$$

is non-empty, and the proof is complete.

COROLLARY 4.5. *A Clifford inverse semigroup S is right congruence compact if and only if $E(S)$ has no infinite chains, and all of its maximal subgroups are right congruence compact groups.*

COROLLARY 4.6. *S semilattice is congruence compact if and only if all its chains are finite.*

In connection with the following result, the reader may wish to keep in mind that the congruence compact abelian groups can be described as those isomorphic to a direct sum of a finite number of groups, each isomorphic to some $\mathbb{Z}_{p^k}$ or to some $\mathbb{Z}_{p^\infty}$ [**16**].

COROLLARY 4.7. *A commutative monoid S is congruence compact if and only if*

1. *S is periodic,*
2. *the maximal subgroups of S are congruence compact abelian groups,*
3. *$E(S)$ is congruence compact* (i.e., *contains no infinite chains*), *and*
4. *for every infinite set X of incomparable principal ideals there exist $z \in S$ and P, $Q \in X$ such that $Pz = P$ and $Qz \neq Q$.*

§5. *Examples.* The following example shows that the maximal one-idempotent subsemigroups S_e of a congruence compact commutative semigroup need not be congruence compact as semigroups.

EXAMPLE 5.1. *Let $S=\{a_i : i\in\mathbb{N}\}\cup\{e_i : i\in\mathbb{N}\}\cup\{0\}$, where 0 is the zero element of S, and set*

$$a_i a_j = 0, \quad \text{for all } i, j\in\mathbb{N},$$

$$e_i e_j = \begin{cases} e_i, & \text{if } i=j, \\ 0, & \text{if } i\neq j, \end{cases}$$

$$e_i a_j = a_j e_i = \begin{cases} a_i, & \text{if } i=j, \\ 0, & \text{if } i\neq j, \end{cases}$$

Then S is congruence compact, but its ideal $S_0=\{a_i : i\in\mathbb{N}\}\cup\{0\}$ is not.

One way to show that S is a semigroup is to represent it by transformations, as follows. For each $n\in\mathbb{N}$ define elements α_n and ε_n of $\mathcal{T}(\mathbb{Z})$ as follows:

$$i\alpha_n = \begin{cases} -i, & \text{if } i=n, \\ 0, & \text{otherwise;} \end{cases}$$

$$i\varepsilon_n = \begin{cases} i, & \text{if } i\in\{n, -n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Let ω denote the element of $\mathcal{T}(\mathbb{Z})$ having constant value 0. One can now check that

$$\varepsilon_i \varepsilon_j = \begin{cases} \varepsilon_i, & \text{if } i=j, \\ \omega, & \text{otherwise,} \end{cases}$$

$$\alpha_i \alpha_j = \omega, \quad \text{for all } i, j\in\mathbb{N},$$

$$\varepsilon_i \alpha_j = \alpha_j \varepsilon_i = \begin{cases} \alpha_i, & \text{if } i=j, \\ \omega, & \text{otherwise.} \end{cases}$$

Because S_0 is an infinite null semigroup, it is not congruence compact. (The family of cofinite subsets of S_0 that do not contain 0 forms a filter-base of congruence classes having empty intersection.) However, S itself is congruence compact. To see this, observe that any congruence class that does not contain 0 must have at most 2 elements. (Alternatively, apply Corollary 4.7.)

The next example shows that, in Corollary 4.7, condition 4 is independent of conditions 1–3.

EXAMPLE 5.2. *Let $S=\bigcup_{i\in\mathbb{N}}\{a_i, e_i\}\cup\{0\}$ be a 0-disjoint union of (commutative) null semigroups $\{a_i, e_i\}$, in which e_i serves as the zero element for each i. Then S is periodic, the subgroups of S are congruence compact, and $E(S)$ has no infinite chains. However, S is not congruence compact.*

The monoid S^1 clearly satisfies conditions 1, 2 and 3 of Corollary 4.7. (Also note that each subsemigroup S^1_e which has either one or two elements is congruence compact.) However, the set $X=\{a_i S^1 : i\in\mathbb{N}\}$ has the property that, for all $z \in S^1$ and all $i, j \in \mathbb{N}$,

$$a_i S^1 z = a_i S^1 \Leftrightarrow a_j S^1 z = a_j S^1 (\Leftrightarrow z = 1),$$

and so condition 4 of Corollary 4.7 is not satisfied.

Finally, we present an example showing that, in condition 4 of Theorem 4.4, one cannot replace "right ideals" by "two sided ideals".

EXAMPLE 5.3. *Let U be an infinite nil semigroup with the property that every set of incomparable principal left ideals is finite. (For example, take U to be the multiplicative semigroup H/Hr, where H is the right open real unit interval under multiplication, and $0 < r < 1$. The ideals of this commutative semigroup form a chain (see* **[11]**). *Let U' be a set in one-to-one correspondence with U under the bijection $u \leftrightarrow u'$ for $u \in U$, and such that $U \cap U' = \{0\}$. Consider the semigroup on $U \cup U'$ that has U as a subsemigroup, and satisfies $U' \cdot (U \cup U') = \{0\}$ and $uv' = (uv)'$ for all $u, v \in U$. Finally, let S be the monoid $(U \cup U')^1$. Because S is itself a nil monoid, it is periodic and satisfies the condition $\mathcal{J} = \mathcal{H}$. The principal right ideals $u'S = \{u', 0\}$ with $u \in U \backslash \{0\}$, are incomparable, yet $u'SzS = u'S$ holds only when $z = 1$. Thus condition* 4 *of Theorem* 4.4 *is violated. However, because there is no infinite set of incomparable principal left ideals, there is no infinite set of incomparable principal two-sided ideals.*

By a similar construction (employing a set U'' such that $(S \backslash \{1\}) \cdot U'' = \{0\}$), it can be seen that a nil semigroup may have infinite sets of incomparable principal left ideals and of incomparable principal right ideals without having any infinite set of incomparable principal two-sided ideals. It can be deduced from this that a nil monoid may be congruence compact without being left or right congruence compact. (Recall that a group having this property was mentioned after Proposition 2.4.) Finally, for the semigroup $U = H/Hr$ in the example above, U^1 is a congruence compact monoid in which infinite chains of principal ideals exist. Thus the term "idempotent-generated" may not be omitted from condition 3 of Theorem 4.4.

§6. *Further results.* We now point out a connection between congruence compactness and equational compactness for semilattices. Recall that an algebra A is called *equationally compact* if every system of polynomial equations (with constants from A) has a solution in A provided that every finite subsystem has a solution in A. (A convenient reference on equational compactness is the survey article by G. H. Wenzel which forms Appendix 6 of **[7]**.) It was proved in **[8]** that a meet semilattice S is equationally compact if and only if it is meet-complete, every subchain of S has a join, and

$$s \wedge \bigvee_{c \in C} c = \bigvee_{c \in C} (s \wedge c)$$

for every $s \in S$ and every subchain C of S. Since every well-ordered chain has these properties, we conclude from Corollary 4.6 that the congruence compact semilattices form a proper subclass of the equationally compact ones.

An alternative view may be given as follows. It is easy to see that any right congruence compact monoid S is 1-equationally compact as a left act over itself. This means that every system of equations

$$\{a_i x = b_i : i \in I\} \cup \{c_k x = d_k x : k \in K\},$$

with $a_i, b_i \in S$ for $i \in I$ and $c_k, d_k \in S$ for $k \in K$, has a solution in S provided that every finite subsystem has a solution in S. In fact, a slightly sharper result, involving systems of equations over a finite set of variables, can be shown to hold (Proposition 2.12 of [**17**]). Within the class of semilattices, 1-equational compactness of S as a monoid is the same as 1-equational compactness as an S-act. Therefore every congruence compact semilattice is 1-equationally compact. From [**8**] we see that 1-equationally compact semilattices are equationally compact, so that the congruence compact semilattices form a subclass of the equationally compact ones. Again, a complete, infinite chain is equationally compact but not congruence compact, so the subclass is proper.

In fact, we can do slightly better. The dual of Theorem 2.1 of [**1**] states that, if every subchain of a semilattice S is dually well ordered, then S is a compact topological semilattice. It follows that congruence compact semilattices are compact topological semilattices (which in turn form a proper subclass of the equationally compact semilattices: see §2 of [**1**]). The closed unit interval in its usual order provides an example of a compact topological semilattice that is not congruence compact. Thus the classes of congruence compact semilattices, of compact semilattices, and of equationally compact semilattices form a proper chain.

Not quite so much is true for arbitrary right congruence compact monoids, as may be seen from abelian groups. Every congruence compact abelian group satisfies the descending chain condition on subgroups, hence is equationally compact (=algebraically compact), and the converse does not hold. (For information on algebraically compact abelian groups, consult [**5**].) On the other hand, the compact congruence compact abelian groups are precisely the finite ones. Here, we use the fact that the Prüfer groups $\mathbb{Z}_{p^\infty}$ are divisible, hence equationally compact, but do not admit a compact Hausdorff topology (see [**12**]).

Proposition 2.11 of [**17**] shows that congruence compactness does not imply equational compactness. It remains to be seen, however, whether every right congruence compact semigroup is equationally compact, at least if $\mathscr{J} = \mathscr{H}$. We postpone the investigation of this question, even for nil semigroups and commutative regular semigroups.

Giving a characterization of the equationally compact distributive lattices is a notoriously difficult unsolved problem, whereas it has long been known that the equationally compact boolean algebras are exactly the complete ones. The congruence compact distributive lattices and boolean algebras turn out to be precisely the finite ones, as we now show.

If A is any algebra and $a, b \in A$, let $\theta(a, b)$ denote the smallest congruence relation on A containing the pair (a, b). In the case of semilattices and distributive lattices we have the following, well-known, descriptions of $\theta(a, b)$ when $a < b$ (*cf.* [**6**], Theorem II.3.3).

LEMMA 6.1.
(1) *If a and b are elements of a semilattice S, with* $a<b$, *then* $(x, y)\in\theta(a, b)$ *if and only if either* $x=y$, *or else* $xa=ya$ *and* $x, y\leqslant b$.
(2) *If a and b are elements of a distributive lattice L with* $a<b$, *then* $(x, y)\in\theta(a, b)$ *if and only if* $a\wedge x=a\wedge y$ *and* $b\vee x=b\vee y$.

It follows that every interval $[a, b]$ (where $a<b$) in a semilattice or distributive lattice is a congruence class, modulo $\theta(a, b)$. (The reader is reminded that the congruences on a boolean algebra are the same as its congruences *qua* distributive lattice. However, the congruences on a lattice may differ from its congruences when it is considered as a meet- or join-semilattice.)

PROPOSITION 6.2. *Let S be a congruence compact algebra whose universe has a partial order* $\leqslant$ *with respect to which, for every* $a, b\in S$ *with* $a<b$, *the closed interval* $[a, b]$ *is a congruence class. Then every subchain C of S in this partial order is finite.*

Proof. Let C be any subchain of S. If C has no largest element, then an ascending chain

$$c_1<c_2<\cdots$$

of elements of C exists. For each $m\in\mathbb{N}$, define $\theta_m=\bigcup_{n>m}\theta_{m,n}$, where $\theta_{m,n}=\theta_{[c_m,c_n]\times[c_m,c_n]}$. Being the union of a chain of congruences, each θ_m is a congruence on S. One easily shows that $c_m\,\theta_{m,n}=[c_m, c_n]$ whenever $m<n$, from which it follows that

$$c_m\,\theta_m=\bigcup_{n>m}[c_m, c_n].$$

Because $k\leqslant m \Rightarrow \theta_k\supseteq\theta_m$, we see that

$$\mathscr{F}=\{c_m\,\theta_m : m\in\mathbb{N}\}$$

is a chain of congruence classes of S. Because S is congruence compact, there exists $t\in\bigcap_{m\in\mathbb{N}} c_m\theta_m$. This implies that $t\geqslant c_m$ for all $m\in\mathbb{N}$, and also $t\leqslant c_n$ for at least one $n\in\mathbb{N}$. Therefore $c_n=c_{n+1}=\cdots$, contrary to our initial assumption.

It can be shown in a completely analogous manner that C has a smallest element, and so the proof is complete.

COROLLARY 6.3. *Every congruence compact semilattice has only finite subchains. A distributive lattice or a boolean algebra is congruence compact if and only if it is finite.*

Proof. Use the remarks preceding the proposition, together with the well-known fact (see for example [**3**], pp. 26–27) that distributive lattices containing only finite subchains are finite.

We remark that, for lattices in general, congruence compactness does not force all subchains to be finite: just recall that any lattice can be embedded into a simple, and therefore, congruence compact, lattice (*cf.* [**6**], Theorems IV.4.4 and IV.4.2). Figure 1 on p. 150 of [**6**] depicts a simple, hence congruence compact, *modular* lattice that contains infinite subchains.

Although a general description of the right congruence compact monoids, even the idempotent ones, remains unknown, the remaining results of this paper illustrate that some progress is nonetheless possible beyond the restricted class of monoids considered so far.

PROPOSITION 6.4. *Let $S=\bigcup_{\gamma\in\Gamma} S_\gamma$ be a semilattice Γ of right congruence compact semigroups S_γ such that every chain in Γ is finite and such that $x\in xS_\gamma$ for each $x\in S_\gamma$, for every $\gamma\in\Gamma$. Then S is right congruence compact.*

Proof. Observe first that every upward-directed (resp. downward-directed) subset of Γ has a greatest (resp. least) element.

Suppose that $\mathscr{C}=\{a_i\rho_i : i\in I\}$ is a chain of right congruence classes of S. Then, for each $i\in I$, the set $E_i=\{\gamma\in\Gamma : S_\gamma\cap a_i\rho_i\neq\varnothing\}$ is a subsemilattice of Y, and therefore has a smallest element γ_i. (For the proof, let $\gamma,\delta\in E_i$, and suppose that $x\in S_\gamma\cap a_i\rho_i$, $y\in S_\delta\cap a_i\rho_i$. Then $xz=x$ for some $z\in S_\gamma$, and so $x=xz\rho_i yz\in S_{\gamma\delta}\cap a_i\rho_i$, showing that $\gamma\delta\in E_i$.)

Now, for each $i\in I$, pick an element $x_i\in S_{\gamma_i}\cap a_i\rho_i$. Let

$$U=\{\gamma_i : i\in I\}.$$

To show that U is a chain in Γ, let $i, j\in I$. If $a_i\rho_i\subseteq a_j\rho_j$, then we have $x_i\in S_{\gamma_i}\cap a_i\rho_i\subseteq S_{\gamma_i}\cap a_j\rho_j$. This means that $S_{\gamma_i}\cap a_j\rho_j$ is non-empty, that γ_i belongs to E_j, and so $\gamma_j\leqslant\gamma_i$. Similarly, $a_i\rho_i\supseteq a_j\rho_j$ implies that $\gamma_j\geqslant\gamma_i$.

Let γ_{i_0} denote the largest element of U. We claim that $a_i\rho_i\cap S_{\gamma_{i_0}}\neq\varnothing$ for each $i\in I$. Take any $i\in I$. If $a_i\rho_i\subseteq a_{i_0}\rho_{i_0}$, then $\gamma_i=\gamma_{i_0}$, and then, because x_i belongs to $S_{\gamma_i}\cap a_i\rho_i$, it also belongs to $a_i\rho_i\cap S_{\gamma_{i_0}}$. If, on the other hand, $a_i\rho_i\supseteq a_{i_0}\rho_{i_0}$, then $x_{i_0}\in S_{\gamma_{i_0}}\cap a_{i_0}\rho_{i_0}\subseteq S_{\gamma_{i_0}}\cap a_i\rho_i$.

Now, because $\mathscr{C}_0=\{a_i\rho_i\cap S_{\gamma_{i_0}} : i\in I\}$ is a chain of right congruence classes of the right congruence compact semigroup $S_{\gamma_{i_0}}$, it follows that $\bigcap\mathscr{C}_0$ is non-empty, and so $\bigcap\mathscr{C}$ is also non-empty, as required.

A corresponding proposition concerning two-sided congruence compactness is obtained by replacing the condition $x\in xS_\gamma$ by $x\in SxS_\gamma\cup S_\gamma xS$.

Consider now a normal band S (a strong semilattice of rectangular bands) which is congruence compact. Corollary 4.6 assures us that the "structure semi-lattice" Γ (the greatest semilattice image of S) has no infinite chains. Assume that

$$S=\bigcup_{\gamma\in\Gamma} S_\gamma$$

is the greatest semilattice decomposition of S. Then every congruence of S_γ can be extended to a congruence of S. It follows that each S_γ is congruence compact, and hence finite (because any union of $\mathscr{R}$ or $\mathscr{L}$ classes of a rectangular band is a congruence class). We therefore have the following result.

Corollary 6.5. *For a normal band S, the following statements are equivalent.*

1. *S is congruence compact.*
2. *S is right (or left) congruence compact.*
3. *The structure semilattice Γ of S has no infinite chains, and all of the rectangular components of S are finite.*

It is obvious that we may replace normal bands in the previous corollary by bands having the congruence extension property (*cf.* **[14]**, **[18]**).

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References

1. S. Bulman-Fleming. On equationally compact semilattices. *Algebra Universalis*, 2 (1972), 146–151.
2. S. Bulman-Fleming and P. Normak. Flatness properties of monocyclic acts. *Mh. Math.*, 122 (1996), 307–323.
3. P. Crawley and R. P. Dilworth. *The Algebraic Theory of Lattices*. Prentice-Hall, Englewood Cliffs, N.J., 1973.
4. A. H. Clifford and G. B. Preston. *The Algebraic Theory of Semigroups, Volume 1*. Mathematical Surveys of the American Mathematical Society, Number 7. Providence, R.I., 1961.
5. L. Fuchs. *Infinite Abelian Groups*. Academic Press, New York, 1970.
6. G. Grätzer. *General Lattice Theory*. Academic Press, New York, 1978.
7. G. Grätzer. *Universal Algebra, 2nd Edition*. Springer, New York, 1979.
8. G. Grätzer and H. Lakser. Equationally compact semilattices. *Colloq. Math.*, 20 (1969), 27–30.
9. P. A. Grillet. *Semigroups: An Introduction to the Structure Theory*. Marcel Dekker Inc., New York, 1995.
10. J. M. Howie. *Fundamentals of Semigroup Theory*. Oxford University Press, Oxford, 1995.
11. E. Hotzel. Halbgruppen mit ausschliesslich reesschen Linkskongruenzen. *Math. Z.*, 112 (1969), 300–320.
12. A. Hulanicki. Algebraic characterization of abelian divisible groups which admit compact topologies. *Fund. Math.*, 44 (1957), 192–197.
13. M. Huber and W. Meier. Linearly compact groups. *J. Pure Appl. Algebra*, 16 (1980), 167–182.
14. P. R. Jones. On the congruence extension property for semigroups. In *Semigroups, Algebraic Theory and Applications to Formal Languages and Codes, Proceedings, Luino, 1992*. World Scientific, Singapore, 1993, 133–143.
15. J. L. Kelley. *General Topology*. Van Nostrand, Princeton, 1955.
16. H. Leptin. Über eine Klasse linear kompakter abelscher Gruppen I. *Abh. Math. Sem. Hamb.*, 19 (1954), 23–40.
17. P. Normak. Congruence compact acts. *Semigroup Forum*, 55 (1997), 299–308.
18. X. Tang. Semigroups with the congruence extension property. *Semigroup Forum*, 56 (1998), 226–264.

19. D. Zelinsky. Linearly compact modules and rings. *Amer. J. Math.*, 75 (1953), 79–90.

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