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MONOIDS OVER WHICH ALL WEAKLY FLAT ACTS ARE FLAT

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If R is a ring with identity and M is a left R-module then it is well known that the following statements are equivalent:
(1) M is flat.
(2) The functor $-\otimes M$ preserves embeddings of right ideals into R.
This paper investigates situations in which the analogous statements are equivalent in the context of S-sets over a monoid S.

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1. Introduction

If R is a ring with identity and M is a left R-module then it is well known that the following statements are equivalent [**1**]:

(1) M is flat.

(2) The functor $-\otimes M$ preserves embeddings of (finitely generated) right ideals into R.

In this paper we investigate the equivalence of the analogous statements in the context of S-sets over a monoid S.

Let S be monoid. If A is a right S-set and B is a left S-set the tensor product $A\otimes B$ has been studied extensively (see, for example, [**7**] or [**2**]). We will use the following terminology, which is consistent with that appearing in [**14**], [**10**] and [**8**].

Definition 1.1. Let S be a monoid and B be a left S-set. B is called flat if the functor $-\otimes B$ preserves embeddings of right S-sets. If this functor preserves embeddings of [principal] right ideals of S into S (considered as a right S-set) then B is called [principally] weakly flat.

Let A be a right S-set, B be a left S-set, $a, a'\in A$ and $b, b'\in B$. It is shown in [**2**] that

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$a \otimes b = a' \otimes b'$ in $A \otimes B$ if and only if there exist $a_1, \ldots, a_n \in A$, $b_2, \ldots, b_n \in B$ and $s_1, t_1, \ldots, s_n, t_n \in S$ such that

$$\begin{array}{ll} a = a_1 s_1 & \\ a_1 t_1 = a_2 s_2 & s_1 b = t_1 b_2 \\ a_2 t_2 = a_3 s_3 & s_2 b_2 = t_2 b_3 \\ \vdots & \vdots \\ a_n t_n = a' & s_n b_n = t_n b' \end{array}$$

This system of equalities is called a scheme over A and B of length n joining (a, b) to (a', b'). Thus B is flat if and only if, for every right S-set A, and every $a, a' \in A$, $b, b' \in B$ such that there exists a scheme over A and B joining (a, b) to (a', b'), there exists a scheme (of possibly different length) over $aS \cup a'S$ and B joining (a, b) to (a', b'). Furthermore, using the fact that $S \otimes B \cong B$, it is easy to see that B is weakly flat if and only if for every $x, y \in S$ and $b, b' \in B$, $xb = yb'$ implies there exist $x_1, \ldots, x_n \in \{x, y\}$, $b_2, \ldots, b_n \in B$ and $u_1, v_1, \ldots, u_n$, v_n in S such that

$$\begin{array}{ll} x = x_1 u_1 & \\ x_1 v_1 = x_2 u_2 & u_1 b = v_1 b_2 \\ \vdots & \vdots \\ x_n v_n = y & u_n b_n = v_n b'. \end{array}$$

A similar description in which $x = y$ characterizes when B is principally weakly flat.

A monoid S is called [weakly] [left, right] absolutely flat if all its [left, right] S-sets are [weakly] flat.

Let $E(S)$ denote the set of idempotents of a monoid S. If $x \in S$ the principal right ideal xS is projective if and only if there exists $u \in E(S)$ such that $xu = x$, and whenever $xs = xt$ $(s, t \in S)$ it follows that $us = ut$ (see [12]). S is called right PP [9] if all of its principal right ideals are projective. The class of these monoids, which we shall consider in Section 2, is extensive in that it contains all regular and all left cancellative monoids.

To conclude this introduction we provide a description of weak flatness which will be used throughout the paper. This generalizes Lemma 2 of [10].

Proposition 1.2. *A left S-set B is weakly flat if and only if*

(*i*) *B is principally weakly flat, and*
(*ii*) *for all right ideals I and J of S,* $IB \cap JB = (I \cap J)B$.

Remark. In the sequel we use the following, equivalent, formulation of (ii): for all $x, y \in S$ and $b, b' \in B$, if $xb = yb'$ then there exist $b'' \in B$ and $z \in xS \cap yS$ such that $xb = yb' = zb''$.

Proof of 1.2. Assume first that B is weakly flat. Then condition (i) follows immediately. To obtain (ii) suppose $x, y \in S$ and $b, b' \in B$ are such that $xb = yb'$. Because B is weakly flat, there exists a scheme

$$\begin{array}{ll} x = x_1 u_1 & \\ x_1 v_1 = x_2 u_2 & u_1 b = v_1 b_2 \\ \vdots & \vdots \\ x_n v_n = y & u_n b_n = v_n b' \end{array}$$

where $x_i \in \{x, y\}$ for $1 \leqq i \leqq n$, $u_1, v_1, \ldots, u_n, v_n \in S$, and $b_2, \ldots, b_n \in B$. Let $z_1 = x$ and for $2 \leqq i \leqq n+1$ let $z_i = x_{i-1} v_{i-1}$. There exists some $k \in \{1, \ldots, n+1\}$ such that $z_k \in xS \cap yS$. Since it is easy to check that $xb = yb' = z_1 b = z_{n+1} b' = z_k b_k$ for $2 \leqq k \leqq n$, (ii) holds.

Now assume conditions (i) and (ii) and suppose $x \otimes b = y \otimes b'$ in $S \otimes B$ (i.e. $xb = yb'$ in B) for some $b, b' \in B$ and $x, y \in S$. We must show $x \otimes b = y \otimes b'$ in $(xS \cup yS) \otimes B$. By (ii) there exists $z \in xS \cap yS$ and $b'' \in B$ such that $xb = yb' = zb''$. Because $x \otimes b = z \otimes b''$ in $S \otimes B$ this equality also holds in $xS \otimes B$ by (i). Similarly, $y \otimes b' = z \otimes b''$ in $yS \otimes B$. Hence, $x \otimes b = y \otimes b'$ in $(xS \cup yS) \otimes B$. □

2. Right *PP* monoids

Lemma 2.1. *Let S be a right PP monoid. A left S-set B is principally weakly flat if and only if, for every $b, b' \in B$ and $x \in S$, $xb = xb'$ implies there exists $u \in E(S)$ such that $xu = x$ and $ub = ub'$.*

Proof. Suppose B is principally weakly flat, $b, b' \in B$, $x \in S$, and $xb = xb'$. Then there exist $u_1, \ldots, u_n$, $v_1, \ldots, v_n \in S$ and $b_2, \ldots, b_n \in B$ such that

$$\begin{array}{ll} x = x u_1 & \\ x v_1 = x u_2 & u_1 b = v_1 b_2 \\ \vdots & \vdots \\ x v_n = x & u_n b_n = v_n b'. \end{array}$$

Since S is right *PP* there exists $u \in E(S)$ such that $xu = x$ and, whenever $xs = xt$ $(s, t \in S)$, it follows that $us=ut$. Therefore, $u=uu_1$, $uv_i=uu_{i+1}$ $(1\leqq i\leqq n-1)$, $uv_n=u$, and so $ub=uu_1b=uv_1b_2=uu_2b_2=\cdots=uu_nb_n=uv_nb'=ub'$ as required.

Now suppose the condition holds for B and $x\otimes b=x\otimes b'$ in $S\otimes B$ $(x\in S, b, b'\in B)$. This means $xb=xb'$ in B and so, in $xS\otimes B$, $x\otimes b=xu\otimes b=x\otimes ub=x\otimes ub'=xu\otimes b'=x\otimes b'$, and thus B is principally weakly flat. □

Corollary 2.2. *If S is a left cancellative monoid, then a left S-set B is principally weakly flat if and only if it is torsion-free (see* [**13**]*).*

If ω denotes the additive monoid of non-negative integers then the ideal $(\omega\times\omega)\backslash\{(0,0)\}$ of the cancellative monoid $S=\omega\times\omega$ is principally weakly flat. However, this ideal is not weakly flat (see Proposition 2.5, for example).

It is apparent from Lemma 2.1 that if S is a regular monoid, every (left) S-set is principally weakly flat. (The converse is also true [**10**].) The condition in the following proposition appears in [**8**].

Proposition 2.3. *Let S be a regular monoid. Then a left S-set B is weakly flat if and only if, for every $b\in B$ and $x, y\in S$, if $xb=yb$ then there exists $z\in xS\cap yS$ such that $xb=yb=zb$.*

Proof. Suppose B is weakly flat and $xb=yb$. Then by (ii) of Proposition 1.2 there exist $z_1\in xS\cap yS$ and $b''\in B$ such that $xb=yb=z_1b''$. If $z'_1\in S$ satisfies $z_1z'_1z_1=z_1$, then $z=z_1z'_1x$ belongs to $xS\cap yS$ and satisfies $xb=yb=zb$.

Conversely suppose B satisfies the stated condition. Since we already know B is principally weakly flat, we show that (ii) of Proposition 1.2 holds. Suppose $xb=yb'$, where $x, y\in S$ and $b, b'\in B$. Let $xb=yb'=b''$ and let $x', y'\in S$ satisfy $xx'x=x$ and $yy'y=y$. Note that $b''=xx'b''=yy'b''$. Using the condition, this element is equal to zb'' for some $z\in xx'S\cap yy'S=xS\cap yS$, and the proof is complete. □

The following result of V. Fleischer [**8**] is now easily obtained. For $x, y\in S$, $\theta_L(x,y)$ $[\theta_R(x,y)]$ will denote the smallest left [right] congruence on S containing (x,y).

Corollary 2.4. *For any monoid S the following statements are equivalent:*

(i) S is weakly left absolutely flat.

(ii) S is regular and S satisfies

(L): for all $x, y\in S$ there exists $z\in xS\cap yS$ such that $(z,x)\in\theta_L(x,y)$.

Proposition 2.5. *Let S be a right PP monoid and let B be a left S-set. Then B is weakly flat if and only if, for all $x, y\in S$ and $b, b'\in B$, if $xb=yb'$ then there exist $b''\in B$, x_1, $y_1\in S$, and $u, v\in E(S)$ such that $xu=x$, $yv=y$, $ub=x_1b''$, $vb'=y_1b''$ and $xx_1=yy_1$.*

Proof. Suppose B is weakly flat and assume $xb=yb'$. By (ii) of Proposition 1.2 there exist $b''\in B$ and $z=xs=yt\in xS\cap yS$ such that $xb=yb'=zb''$. From $xb=x(sb'')$ it follows from Lemma 2.1 that there exists $u\in E(S)$ such that $xu=x$ and $ub=usb''$; similarly there exists $v\in E(S)$ such that $yv=y$ and $vb'=vtb''$. We may now take $x_1=us$ and $y_1=vt$.

Now assume the condition holds and suppose $x\otimes b=y\otimes b'$ in $S\otimes B$ (i.e. $xb=yb'$ in B). Then $b''\in B$, $x_1, y_1\in S$, and $u,v\in E(S)$ exist as described above, and in $(xS\cup yS)\otimes B$ we calculate $x\otimes b=xu\otimes b=x\otimes ub=x\otimes x_1b''=xx_1\otimes b''=yy_1\otimes b''=y\otimes b'$. Therefore B is weakly flat. □

In [8] V. Fleischer proved that, for any monoid S, if all left and right S-sets are weakly flat, then in fact they are all flat. Our paper [5] gives an alternative proof of this result. Using similar ideas we obtain the following:

Theorem 2.6. *Let S be any monoid. If*

(*i*) *S is right PP, and*

(*ii*) *S satisfies (R'): for all $u,v\in E(S)$ there exists $z\in Su\cap Sv$ such that $(z,u)\in\theta_R(u,v)$*

then every weakly flat left S-set is flat.

Proof. Let B denote any weakly flat left S-set, and choose an arbitrary right S-set, A. Suppose $a,a'\in A$ and $b,b'\in B$ are such that $a\otimes b=a'\otimes b'$ in $A\otimes B$. Then there exist $a_1,\dots,a_n\in A$, $b_2,\dots,b_n\in B$, $s_1,t_1,\dots,s_n$, $t_n\in S$ such that

$$\begin{array}{ll} a=a_1s_1 & \\ a_1t_1=a_2s_2 & s_1b=t_1b_2 \\ \vdots & \vdots \\ a_it_i=a_{i+1}s_{i+1} & s_ib_i=t_ib_{i+1} \\ \vdots & \vdots \\ a_nt_n=a' & s_nb_n=t_nb'. \end{array} \qquad (\Sigma)$$

By Lemma 2.2 of [2] we must show there exists a replacement scheme over $aS\cup a'S$ and B joining (a,b) to (a',b'). By Proposition 2.5, for each $i\in\{1,\dots,n\}$ there exist $b_i''\in B$, $x_i, y_i\in S$, and $u_i, v_i\in E(S)$ such that

$$s_iu_i=s_i,\ t_iv_i=t_i \qquad (1)$$

$$u_ib_i=x_ib_i'',\ v_ib_{i+1}=y_ib_i'' \qquad (2)$$

$$s_ix_i=t_iy_i. \qquad (3)$$

(In (2) we define $b_1 = b$ and $b_{n+1} = b'$.) By (R') there exist $q_i, r_i \in S$ $(1 \leqq i \leqq n-1)$ such that

$$q_i v_i = r_i u_{i+1} \tag{4}$$

and $(r_i u_{i+1}, u_{i+1}) \in \theta_R(u_{i+1}, v_i)$. Because the relation $\rho_i = \{(s,t) \in S \times S \mid a_{i+1} s_{i+1} s = a_{i+1} s_{i+1} t\}$ is a right congruence on S containing (u_{i+1}, v_i), we obtain

$$a_{i+1} s_{i+1} r_i u_{i+1} = a_{i+1} s_{i+1} \tag{5}$$

for $i \in \{1, \ldots, n-1\}$. We now proceed by induction on n.

If $n = 1$ it is easy to see that (using (1) to (3))

$$\begin{array}{ll}
a = au_1 & \\
ax_1 = a'y_1 & u_1 b = x_1 b_1'' \\
a'v_1 = a' & y_1 b_1'' = v_1 b'
\end{array}$$

is a suitable replacement scheme. Now consider a scheme (Σ) as above, where $n \geqq 2$. We first verify that

$$\begin{array}{cc}
a = au_1 & \\
ax_1 = (a_2 s_2) q_1 v_1 y_1 & u_1 b = x_1 b_1'' \\
(a_2 s_2) r_1 u_2 x_2 = (a_3 s_3) q_2 v_2 y_2 & q_1 v_1 y_1 b_1'' = r_1 u_2 x_2 b_2'' \\
\vdots & \vdots \\
(a_i s_i) r_{i-1} u_i x_i = (a_{i+1} s_{i+1}) q_i v_i y_i & q_{i-1} v_{i-1} y_{i-1} b_{i-1}'' = r_{i-1} u_i x_i b_i'' \\
\vdots & \vdots \\
(a_n s_n) r_{n-1} u_n x_n = a' y_n & q_{n-1} v_{n-1} y_{n-1} b_{n-1}'' = r_{n-1} u_n x_n b_n'' \\
a' v_n = a' & y_n b_n'' = v_n b'
\end{array}$$

is a scheme joining (a, b) to (a', b'). The reader may verify that the equalities above hold. For example,

$$\begin{aligned}
(a_i s_i) r_{i-1} u_i x_i &= a_i s_i x_i && \text{(by (5))} \\
&= a_i t_i y_i && \text{(by (3))} \\
&= a_{i+1} s_{i+1} y_i && \text{(from } (\Sigma)) \\
&= a_{i+1} s_{i+1} r_i u_{i+1} y_i && \text{(by (5))} \\
&= (a_{i+1} s_{i+1}) q_i v_i y_i && \text{(by (4)),}
\end{aligned}$$

and

$$
\begin{aligned}
q_{i-1}v_{i-1}y_{i-1}b''_{i-1} &= q_{i-1}v_{i-1}^2 b_i && \text{(by (2))} \\
&= q_{i-1}v_{i-1}b_i && \text{(since } v_{i-1} \in E(S)) \\
&= r_{i-1}u_i b_i && \text{(by (4))} \\
&= r_{i-1}u_i^2 b_i && \text{(since } u_i \in E(S)) \\
&= r_{i-1}u_i x_i b''_i && \text{(by (2)).}
\end{aligned}
$$

The proof is now completed as follows: the outlined subscheme above has length $n-1$ and so, using the inductive hypothesis, $ax_1 \otimes b''_1 = a'y_n \otimes b''_n$ in $(ax_1S \cup a'y_nS) \otimes B$, hence also in $(aS \cup a'S) \otimes B$. But using the initial and final portions of the scheme above it is immediate that $a \otimes b = ax_1 \otimes b''_1$ and $a'y_n \otimes b''_n = a' \otimes b'$, also in $(aS \cup a'S) \otimes B$. It follows that $a \otimes b = a' \otimes b'$ in $(aS \cup a'S) \otimes B$, as required. □

Corollary 2.7. *Let S be a monoid. Each of the following conditions implies every weakly flat S-set is flat.*

(i) S is weakly right absolutely flat.

(ii) S is right PP and its idempotents form a left regular band.

Proof.

(i) S is regular and satisfies (R), the dual of (L), by Corollary 2.4. Hence, S satisfies the hypotheses of Theorem 2.6.

(ii) $E(S)$ satisfies the identity $xyx = xy$. To verify (R') suppose $u, v \in E(S)$. Then $z = vuv = vu \in Su \cap Sv$ and $(z, u) \in \theta_R(u, v)$. □

The reader should note that the following monoids satisfy (ii) of Corollary 2.7:

(a) commutative PP monoids (characterized in **[12]**);

(b) right PP monoids with central idempotents (characterized in **[9]**) (This includes all left cancellative monoids, for example. For more information see **[4]**.)

(c) S^1 where S is any left generalized inverse semigroup (see **[15]**).

Corollary 2.8 (**[8]**). *If a monoid S is weakly absolutely flat then S is absolutely flat.*

The obvious one-sided analogue of Corollary 2.8 is not true. We consider the situation for bands, first noting the following:

Proposition 2.9. Let S be an idempotent monoid. Then S is weakly left absolutely flat if and only if S is right regular.

Proof. Suppose first that S is a right regular band. By Corollary 2.4 we need only verify that S satisfies (L). Let $x, y \in S$. Then $z = xyx = yx \in xS \cap yS$ and $(z, x) \in \theta_L(x, y)$ (as in the proof of Corollary 2.7(iii)).

Now assume S is weakly left absolutely flat, and let $S = \dot{\bigcup}_{\gamma \in \Gamma} S_\gamma$ where Γ is a semilattice and each S_γ is a rectangular band. We need to show that each S_γ is a right zero band, so let $x, y \in S_\gamma$ $(\gamma \in \Gamma)$. By (L) there exists $z = xz = yz \in xS \cap yS$ such that $(z, x) \in \theta_L(x, y)$, and thus there exist $u_1, \ldots, u_n$, $x_1, y_1, \ldots, x_n, y_n \in S$ with $\{x_i, y_i\} = \{x, y\}$ $(1 \leqq i \leqq n)$ for which

$$
\begin{aligned}
x &= u_1 x_1 \\
u_1 y_1 &= u_2 x_2 \\
&\vdots \\
u_n y_n &= z.
\end{aligned}
$$

It is easily seen that each of these $n+1$ elements and in particular, z, belongs to S_γ. Thus $xy = xyzy = xzy = yzy = y$ as required. $\square$

Example 2.10. Let S be any right normal band which does not have constant structure maps. Then S^1 is weakly left absolutely flat, by the preceding proposition, but not left absolutely flat (see [3, Proposition 2.3]). However, note that by the dual of Corollary 2.7(ii), every weakly flat right S^1-set is flat.

3. Other situations

In this section we examine further situations in which weakly flat S-sets are flat. First we consider some technical matters.

Lemma 3.1. *Suppose S is a monoid, A is a right S-set and B is a left S-set which is principally weakly flat. Assume $a, a_1 \in A$, $b, b_1 \in B$, $s_1 \in S$ are such that $a = a_1 s_1$ and $s_1 b = s_1 b_1$. Then $a \otimes b = a \otimes b_1$ in $aS \otimes B$.*

Proof. Since $s_1 b = s_1 b_1$, $s_1 \otimes b = s_1 \otimes b_1$ in $s_1 S \otimes B$ because B is principally weakly flat. Therefore $u_1, v_1 \ldots, u_n, v_n \in S$ and $b_2, \ldots, b_n \in B$ exist such that

$$
\begin{aligned}
s_1 &= s_1 u_1 & & \\
s_1 v_1 &= s_1 u_2 & u_1 b &= v_1 b_2 \\
&\vdots & &\vdots \\
s_1 v_n &= s_1 & u_n b_n &= v_n b_1.
\end{aligned}
$$

Then $a \otimes b = au_1 \otimes b = a \otimes u_1 b = a \otimes v_1 b_2 = av_1 \otimes b_2 = \cdots = av_n \otimes b_1 = a \otimes b_1$ in $aS \otimes B$. $\square$

Using Lemma 3.1 we obtain the following useful observation (see [8, Lemma 3]).

Lemma 3.2. *Suppose S is a monoid, A is a right S-set and B is a weakly flat left S-set. Assume $a, a' \in A$ and $b, b' \in B$ are such that $a \otimes b = a' \otimes b'$ in $A \otimes B$ via a scheme of length 1. Then $a \otimes b = a' \otimes b'$ in $(aS \cup a'S) \otimes B$.*

Proof. Suppose $a_1 \in A$, $s_1, t_1 \in S$ and

$$\begin{array}{ll} a = a_1 s_1 & \\ a_1 t_1 = a' & \qquad s_1 b = t_1 b' \end{array}$$

is a scheme of length 1 yielding $a \otimes b = a' \otimes b'$ in $A \otimes B$. By Proposition 1.2 there exist $\alpha_1, \beta_1 \in S$ and $b'' \in B$ such that $s_1\alpha_1 = t_1\beta_1$ and $s_1 b = t_1 b' = s_1\alpha_1 b'' = t_1\beta_1 b''$. By Lemma 3.1 it follows that $a \otimes b = a \otimes \alpha_1 b'' = a\alpha_1 \otimes b''$ in $aS \otimes B$ and similarly $a' \otimes b' = a'\beta_1 \otimes b''$ in $a'S \otimes B$. Furthermore $a\alpha_1 = a_1 s_1 \alpha_1 = a_1 t_1 \beta_1 = \alpha'\beta_1$ and so $a \otimes b = a' \otimes b'$ in $(aS \cup a'S) \otimes B$.

Remark 3.3. Suppose S is a monoid, A is a right S-set, and B is a weakly flat left S-set. Assume $a, a' \in A$ and $b, b' \in B$ are such that $a \otimes b = a' \otimes b'$ in $A \otimes B$ via a scheme

$$\begin{array}{ll} a = a_1 s_1 & \\ a_1 t_1 = a_2 s_2 & \qquad s_1 b = t_1 b_2 \\ \vdots & \qquad \vdots \\ a_i t_i = a_{i+1} s_{i+1} & \qquad s_i b_i = t_i b_{i+1} \\ \vdots & \qquad \vdots \\ a_n t_n = a' & \qquad s_n b_n = t_n b'. \end{array}$$

Our aim in the sequel will be to conclude, for particular monoids S, that $a \otimes b = a' \otimes b'$ in $(aS \cup a'S) \otimes B$. This conclusion is valid if $n = 1$ by Lemma 3.2. Suppose $n \geqq 2$ and that such a conclusion can be made for schemes of length $< n$. If $a_i t_i = a_{i+1} s_{i+1} \in aS \cup a'S$ for some $i \in \{1, \ldots, n-1\}$ then, using the inductive hypothesis twice, $a \otimes b = a_{i+1} s_{i+1} \otimes b_{i+1} = a_i t_i \otimes b_{i+1} = a' \otimes b'$ in $(aS \cup a'S) \otimes B$. In particular, if $t_1 \in s_1 S$ or $s_n \in t_n S$ this will be the case. Moreover, if any of $s_2, \ldots, s_n$, $t_1, \ldots, t_{n-1}$ is equal to 1, it is easy to see that the scheme may be replaced by one of shorter length, allowing the use of the inductive hypothesis.

Proposition 3.4. *Let S be a semigroup [with 0] which is the union of its [0−] minimal right ideals. Then every weakly flat left S^1-set is flat.*

Proof. Suppose A is a right S^1-set, B is a weakly flat left S^1-set, $a, a' \in A$, $b, b' \in B$ and $a \otimes b = a' \otimes b'$ in $A \otimes B$ via the scheme

$$\begin{array}{ll} a = a_1 s_1 & \\ a_1 t_1 = a_2 s_2 & s_1 b = t_1 b_2 \\ \vdots & \vdots \\ a_n t_n = a' & s_n b_n = t_n b'. \end{array}$$

Following the discussion in Remark 3.3 we may assume $n \geqq 2$, and none of s_1, t_1, s_n, t_n is 1.

Suppose S does not contain 0. Note that in this situation $xS = xS^1$ for every $x \in S$. Since $s_1 b = t_1 b_2$ Proposition 1.2(ii) shows that $s_1 S^1 \cap t_1 S^1 = s_1 S \cap t_1 S$ is not empty. Thus $s_1 S = t_1 S$, $t_1 \in s_1 S^1$, and so Remark 3.3 furnishes the desired conclusion.

Now assume S contains 0. If $s_1 S^1 \cap t_1 S^1 \neq \{0\}$ then $t_1 \in s_1 S^1$ and the proof is completed as above. A similar argument applies if $s_n S^1 \cap t_n S^1 \neq \{0\}$. Suppose then that $s_1 S^1 \cap t_1 S^1 = \{0\} = s_n S^1 \cap t_n S^1$. By Proposition 1.2(ii) there exist α_1, β_1, α_n, $\beta_n \in S^1$ and $\bar{b}, \hat{b} \in B$ such that $s_1 \alpha_1 = t_1 \beta_1 = 0 = s_n \alpha_n = t_n \beta_n$, $s_1 b = t_1 b_2 = 0\bar{b}$ and $s_n b_n = t_n b' = 0\hat{b}$. Because $a = a_1 s_1$ and $s_1 b = s_1 \alpha_1 \bar{b}$ Lemma 3.1 yields $a \otimes b = a \otimes \alpha_1 \bar{b}$ in $aS^1 \otimes B$. Similarly $a' \otimes b' = a' \otimes \beta_n \hat{b}$ in $a'S^1 \otimes B$. From the scheme it is clear that $0b = 0b'$ and hence $0\bar{b} = 0 s_1 b = 0b = 0b' = 0 t_n b' = 0\hat{b}$. Moreover $a\alpha_1 = a_1 s_1 \alpha_1 = a_1 0 = a0 = a'0 = a_n 0 = a_n t_n \beta_n = a'\beta_n$. Therefore, in $(aS^1 \cup a'S^1) \otimes B$, $a \otimes b = a \otimes \alpha_1 \bar{b} = a\alpha_1 \otimes \bar{b} = a0 \otimes \bar{b} = a \otimes 0\bar{b} = a \otimes 0\hat{b} = a0 \otimes b' = a'\beta_n \otimes \hat{b} = a' \otimes \beta_n \hat{b} = a' \otimes b'$. □

Proposition 3.4 applies to [0−] simple semigroups containing a [0−] minimal right ideal and in particular to completely [0−] simple semigroups and Croisot–Teissier semigroups (see [**6**, Chapter 8]).

We conclude with two partial results concerning the commutative situation.

Proposition 3.5. *Suppose S is a commutative monoid. Then every weakly flat* **cyclic** *S-set is flat.*

Proof. Suppose A and B are S-sets with $B = Sb$ cyclic, $a, a' \in A$, and $a \otimes b = a' \otimes b$ in $A \otimes Sb$. We need only show that this equality holds in $(aS \cup a'S) \otimes Sb$. By assumption there exist $a_1, \ldots, a_n \in A$, $s_1, t_1, \ldots, s_n$, $t_n \in S$ such that

$$\begin{array}{ll} a = a_1 s_1 & \\ a_1 t_1 = a_2 s_2 & s_1 b = t_1 b \\ \vdots & \vdots \\ a_n t_n = a' & s_n b = t_n b. \end{array}$$

For each i $(1 \leqq i \leqq n)$ there exist $\alpha_i, \beta_i \in S$ such that $s_i \alpha_i = t_i \beta_i$ and $s_i b = t_i b = s_i \alpha_i b$ by Proposition 1.2. For convenience let $\beta_0 = 1$, $s_{n+1} = 1$ and $a_{n+1} = a'$. We use induction on i to prove for each i $(1\leqq i\leqq n)$, $a_{i+1}s_{i+1}\beta_1\dots\beta_i\in aS$ and

$$a_is_i\beta_1\dots\beta_{i-1}\otimes b=a_{i+1}s_{i+1}\beta_1\dots\beta_i\otimes b \text{ in } aS\otimes Sb.$$

If $i=1$ then, since $a=a_1s_1$ and $s_1b=s_1(\alpha_1b)$, it follows from Lemma 3.1 that $a\otimes b=a\otimes\alpha_1b$ in $aS\otimes Sb$. Hence, $a_1s_1\beta_0\otimes b=a\otimes b=a\alpha_1\otimes b=a_1s_1\alpha_1\otimes b=a_1t_1\beta_1\otimes b=a_2s_2\beta_1\otimes b$ in $aS\otimes Sb$ as required. Now assume the result is true for a particular i $(1\leqq i\leqq n)$. Since $a_{i+1}s_{i+1}\beta_1\dots\beta_i=a_{i+1}(s_{i+1}\beta_1\dots\beta_i)$ and $(s_{i+1}\beta_1\dots\beta_i)b=(s_{i+1}\beta_1\dots\beta_i)(\alpha_{i+1}b)$, it again follows from Lemma 3.1 that $a_{i+1}s_{i+1}\beta_1\dots\beta_i\otimes b=a_{i+1}s_{i+1}\beta_1\dots\beta_i\otimes\alpha_{i+1}b$ in $a_{i+1}s_{i+1}\beta_1\dots\beta_iS\otimes Sb$, hence, in $aS\otimes Sb$. Furthermore, $a_{i+1}s_{i+1}\beta_1\dots\beta_i\alpha_{i+1}=a_{i+1}s_{i+1}\alpha_{i+1}\beta_1\dots\beta_i=a_{i+1}t_{i+1}\beta_{i+1}\beta_1\dots\beta_i=a_{i+2}s_{i+2}\beta_{i+1}\beta_1\dots\beta_i$ and the conclusion follows easily.

Now, calculating in $aS\otimes Sb$, $a\otimes b=a_2s_2\beta_1\otimes b=\dots=a_ns_n\beta_1\dots\beta_{n-1}\otimes b=a_{n+1}s_{n+1}\beta_1\dots\beta_n\otimes b=a'\beta_1\dots\beta_n\otimes b$. Dually, $a'\otimes b=a\alpha_1\dots\alpha_n\otimes b$ in $a'S\otimes Sb$. Moreover, $a\alpha_1\dots\alpha_n=a_1s_1\alpha_1\dots\alpha_n=a_1t_1\beta_1\alpha_2\dots\alpha_n=a_2s_2\beta_1\alpha_2\dots\alpha_n=a_2s_2\alpha_2\dots\alpha_n\beta_1=\dots=a_ns_n\alpha_n\beta_1\dots\beta_{n-1}=a_nt_n\beta_n\beta_1\dots\beta_{n-1}=a'\beta_1\dots\beta_n$. Therefore $a\otimes b=a'\otimes b$ in $(aS\cup a'S)\otimes Sb$. $\square$

Proposition 3.6. *Let S be a commutative monoid whose (principal) ideals form a chain. Then every weakly flat S-set is flat.*

Proof. Suppose A is an S-set, B is a weakly flat S-set, $a,a'\in A$, $b,b'\in B$, and $a\otimes b=a'\otimes b'$ in $A\otimes B$. Then there exist $a_1,\dots,a_n\in A$, $b_2,\dots,b_n\in B$, $s_1,t_1,\dots,s_n$, $t_n\in S$ such that

$$\begin{array}{ll}
a=a_1s_1 & \\
a_1t_1=a_2s_2 & s_1b=t_1b_2\\
\vdots & \vdots\\
a_{i-1}t_{i-1}=a_is_i & s_{i-1}b_{i-1}=t_{i-1}b_i\\
a_it_i=a_{i+1}s_{i+1} & s_ib_i=t_ib_{i+1}\\
a_{i+1}t_{i+1}=a_{i+2}s_{i+2} & s_{i+1}b_{i+1}=t_{i+1}b_{i+2}\\
\vdots & \vdots\\
a_nt_n=a' & s_nb_n=t_nb'.
\end{array}\qquad(\Sigma)$$

If $n=1$, $a\otimes b=a'\otimes b'$ in $(aS\cup a'S)\otimes B$ by Remark 3.3. Now assume $n\geqq 2$ and (Σ) is a scheme of minimum length joining (a,b) to (a',b') over A and B. For convenience we let $a_0=a$, $a_{n+1}=a'$, $b_1=b$, $b_{n+1}=b'$, $t_0=1$, and $s_{n+1}=1$. Note that for $1\leqq i\leqq n-1$, $s_i\in t_iS$ implies $t_i\in s_{i+1}S$, for otherwise, there exist x, $y\in S$ such that $s_i=t_ix$ and $s_{i+1}=t_iy$ (the ideals form a chain). But then the boxed portion of (Σ) may be replaced by

$$a_{i-1}t_{i-1}=a_{i+1}s_{i+1}x$$

$$a_{i+1}t_{i+1}=a_{i+2}s_{i+2} \qquad s_{i+1}xb_i=t_{i+1}b_{i+2}$$

contradicting the assumption that the length of (Σ) is minimal. Similarly, for $1 \leqq i \leqq n-1$, $t_i \in s_{i+1}S$ implies $s_{i+1} \in t_{i+1}S$.

By Remark 3.3 we may assume $s_1 \in t_1S$. From the observations above, $s_1 \in t_1S$ implies $t_1 \in s_2S$ implies $s_2 \in t_2S$ implies ... implies $s_n \in t_nS$. Because $s_n \in t_nS$ the proof is complete, again using Remark 3.3. $\square$

REFERENCES

1. N. Bourbaki, *Commutative Algebra* (Addison-Wesley, Reading, Mass., 1972).

2. S. Bulman-Fleming and K. McDowell, Absolutely flat semigroups, *Pacific J. Math.* **107** (1983), 319–333.

3. S. Bulman-Fleming and K. McDowell, Left absolutely flat generalized inverse semigroups, *Proc. Amer. Math. Soc.* **94** (1985), 553–561.

4. S. Bulman-Fleming and K. McDowell, A characterization of left cancellative monoids by flatness properties, *Semigroup Forum* **40** (1990), 109–112.

5. S. Bulman-Fleming and K. McDowell, On V. Fleischer's characterization of absolutely flat monoids, *Algebra Universalis* **25** (1988), 394–399.

6. A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*; vol. II (*Mathematical Surveys of the American Math. Soc.* **7**, Providence, R.I., 1967).

7. J. M. Howie, *An Introduction to Semigroup Theory* (Academic Press, London, 1976).

8. V. Fleischer, Completely flat monoids, *Uč̌h. Zap. Tartu Un-ta* **610** (1982), 38–52 (Russian). (MR85:20178). (English translation: *Amer. Math. Soc. Transl.* (2) **142** (1989), 19–31.)

9. J. Fountain, Right PP monoids with central idempotents, *Semigroup Forum* **13** (1977), 229–237.

10. M. Kilp, On completely flat monoids, *Tartu Riikl. Ül. Toimetised* **700** (1985), 32–37.

11. M. Kilp, Strong flatness of flat cyclic left acts, *Tartu Riikl. Ül. Toimetised* **700** (1985), 38–41.

12. M. Kilp, Commutative monoids all of whose principal ideals are projective, *Semigroup Forum* **6** (1973), 334–339.

13. U. Knauer and M. Petrich, Characterization of monoids by torsion-free, flat, projective and free acts, *Arch. Math.* **36** (1981), 289–294.

14. P. Normak, On equalizer-flat and pullback-flat acts, *Semigroup Forum* **36** (1987), 293–313.

15. M. Yamada, Regular semigroups whose idempotents satisfy permutation identities, *Pacific J. Math.* **21** (1967), 371–392.

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