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Abelian Subalgebras of Maximal Dimension in Euclidean Lie Algebras

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ABELIAN SUBALGEBRAS OF MAXIMAL DIMENSION IN EUCLIDEAN LIE ALGBEBRAS

by

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1. Abstract

In this paper we define, discuss and prove the uniqueness of the abelian subalgebra of maximal dimension of the Euclidean Lie algebra $\mathcal{E}(n)$ for any positive integer n. We also construct a family of maximal abelian subalgebras and prove that they are maximal.

2. Acknowledgements

I would like to thank my supervisor Dr. Kaiming Zhao for his support, encouragement, and kindness throughout my entire graduate studies at Wilfrid Laurier University. I would also like to thank the Mathematics department as a whole for their warmth and inclusivity; in this regard and in many other ways Laurier is an exceptional school.

3. Dedication

This thesis is dedicated to my deceased grandmother Maria Romualdi who passed away during my graduate studies.

Contents

4. Introduction

The study of matrix analysis, and Lie Theory is important for both applied and theoretical endeavours. Results from Linear Algebra are used in almost every field in mathematics, including algebra, graph theory, and analysis. Lie Theory has many applications in fields such as applied mathematics, differential equations, statistical analysis, machine learning, economics, physics and more. On the theoretical side of Lie Theory, there has been work done on the characterization of types of Lie algebras. Finite dimensional semisimple Lie algebras over algebraically closed field of characteristic 0 have been fully characterized. Simple Lie algebras form four families A_n, B_n, C_n, D_n, and five exceptions, the Lie algebras E_6, E_7, E_8, F_4, G_2. These classifications were given by Killing, Cartan, and others. Every finite dimensional semisimple Lie algebra can be expressed as a direct sum of simple Lie algebras. There are still open problems in classifying Lie algebras that do not fit into these categories, such as infinite dimensional simple Lie algebras, finite dimensional nilpotent or solvable Lie algebra.

In this paper we will consider the Euclidean Lie algebra $\mathcal{E}(n)$ for any positive integer n, and specifically prove the abelian subalgebra of maximal dimension is unique. Also we will be examining their maximal abelian subalgebras. There has been similar work done in characterizing the abelian subalgebra of maximal dimension for Lie algebras. Every simple Lie algebra's abelian subalgebras of maximal dimension has been characterized and determined. Also there has been several papers that characterize maximal abelian subalgebras of Lie algebras including the orthogonal Lie algebra and the pseudo orthogonal Lie algebras.

In this paper we will give some historical context to the subject by discussing past contibutions from mathematicians that produced profound results in the field of commuting matrices, including Schur, Jacobson, and Malcev. Then we shall define useful terms and give helpful lemmas, as a preliminary to the thesis. After this we will prove that the abelian subalgebra of maximal dimension for $\mathcal{E}(n)$ is unique. At the end we give a bibliography of all the papers used and referenced.

It should be noted that throughout this thesis, all the Lie algebras, and vector spaces are taken over the algebraically closed field $\mathbb{C}$, but the results work for any algebraically closed field with characteristic 0.

5. Historical Context

The study of Lie algebras was first introduced by Sophus Lie who was studying differential equations, and transformation groups. His goal was to construct continuous groups, which were supposed to be analogous to discrete groups. From Lie groups Lie algebras can be constructed, and a wealth of theory of Lie algebras were developed by Lie, Cartan, Killing and others. From representation theory we know that every finite dimensional Lie algebra has a homomorphism into the general linear Lie algebra, the Lie algebra of all $n \times n$ matrices. Thus the study of matrices is central to the study of Lie algebras. In this way examining the Lie algebras of matrices became the focus of many papers. Since the the canonical Lie bracket for matrices is $[A, B] = AB - BA$, matrices commute if $[A, B] = 0$. The question from here is to find the abelian Lie sublagebra of maximal dimension and maximal abelian subalgebras. Specifically for this paper we will be examining the Euclidean Lie algebra $\mathcal{E}(n)$.

The first person to tackle this problem was Schur [9], who in 1905 proved that the dimension of the abelian subalgebra of maximal dimension was bounded above by $[\frac{n^2}{4}]+1$ (where $[x]$ refers to the integral part of x) for the Lie algebra of all $n \times n$ matrices over an algebraically closed field. Afterwards Jacobson [8] in 1944 gave a simpler proof, and proved that the theorem still holds for any arbitrary field. He also proved that there is an abelian subalgebra in $\mathfrak{gl}_n$ that has such a dimension.

By the classification results for finite dimensional simple Lie algebras: which are Lie algebras whose only ideals are 0 and itself, we see they are isomorphic to one of a finite set of classes of Lie algebras. We also know that every semisimple Lie algebra can be written as the direct sum of simple Lie algebras. By Malcev's paper [16] the problem of determining the maximal commutative subalgebra can be trivially reduced to the determination of subalgebras with nilpotent elements. He was able to determine the abelian subalgebras of maximal dimension of simple and semisimple Lie algebras in 1945, as listed in the table below.

Type of $\mathfrak{g}$	E_6	E_7	E_8	F_4	G_2	$A_l(l \geq 1)$	$B_l(l \geq 2)$	$C_l(l \geq 3)$	$D_l(l \geq 4)$
$\dim(\mathfrak{g})$	78	133	248	52	14	l^2+2l	$2l^2+l$	$2l^2+l$	$2l^2-l$
$\mathcal{M}(\mathfrak{g})$	16	27	36	9	3	$\lfloor \frac{(l+1)^2}{4} \rfloor$	$\frac{l(l-1)}{2}+1$	$\frac{l(l+1)}{2}$	$\frac{l(l-1)}{2}$

This table lists the types of simple Lie algebras, their dimension, and the dimension of their maximal abelian subalgebra.

More recently, Ceballos, Nunez, and Tenorio [3] devised an algorithm that generates the abelian subalgebras of maximal dimension for the Lie algebras of upper and strictly upper triangular matrices. Not only is the abelian subalgebra of maximal dimension a topic of interest, but so are maximal abelian subalgebras, that is, abelian subalgebras not contained in any other abelian subalgebra. It is important to note that the abelian subalgebra of maximal dimension need not be unique. For example the Lie algebra $\mathfrak{sl}_{2n}$ contains many abelian subalgebras of maximal dimension.

There are several papers dedicated to characterizing specific Lie algebras, and specifically the maximal abelian subalgebras, including a paper by Hussin, Winternitz and Zassenhaus [13] where they discuss the maximal abelian subalgebras of the complex Lie algebra $\mathfrak{o}(n)$, which is the Lie algebra isomorphic to the Lie algebra of all skew symmetric matrices. In this paper they take a more general definition of of $\mathfrak{o}(\mathfrak{n})$ which we discuss in the next chapter. The paper discusses *maximal abelian subalgebras* ($MASA$) and *maximal nilpotent subalgebras* ($MANS$) over the complex field. They also discuss the concept of decomposition, that is an element can be *orthogonally indecomposable* (OID) or *orthogonally decomposable* (OD). Also an element can be *decomposable*(D) or *indecomposable*(ID). An element can be OID but D (only when the dimension is even) or indecomposable (OID and ID). A theorem from this paper then says, OID and ID $MASAs$ of $\mathfrak{o}(\mathfrak{n})$ exist for all $n \geqslant 2$, and they are all $MANS$ of $\mathfrak{o}(\mathfrak{n})$. In this way, like Malcev's paper, the problem of $MASA$s is reduced to the problem of $MANS$.

In the vein of discussing specific Lie algebras there are several papers, we discuss two here. First there is a paper by Dobrev, Doenhner, and Mrugalla [14] which discusses the Schrodinger algebra and its lowest weight representation. In this paper they discuss how non-semisimple Lie algebras play an important role in applications in physics. There is less theory on non-semisimple Lie algebras compared to semisimple Lie algebras, and thus makes them harder to study. Also we have the paper by Barannyk [12] which finds a classification of the subalgebras the Galilei Lie algebra. Here the Galilei Lie algebra plays a vital role in solving differential equations such as the heat equation. We looked to these and several other papers for inspiration for our construction of our definition of the Euclidean Lie algebra $\mathcal{E}(n)$.

6. Preliminaries

Denote by $\mathbb{Z}, \mathbb{N}, \mathbb{C}$ the sets of integers, positive integers, complex numbers respectively. Throughout this paper, we assume that all algebras and vector spaces are over the complex numbers $\mathbb{C}$.

A Lie algebra L is a vector space over a field $\mathbb{F}$, with a multiplication called Lie brackets,

$$[\cdot,\cdot] : L \times L \to L$$

$$(x, y) \to [x, y]$$

With the following properties:
$(x, y) \to [x, y]$ is bilinear,
$[x, x] = 0$ for all $x \in L$,
and the Jacobi identity, $[x, [y, z]] = [[x, y], z] + [y, [x, z]], \forall x, y, z \in L$

Proposition 6.1.

For any x, y in a Lie algebra L $[x, y] = -[y, x]$.

Proof.

$$\begin{aligned} \text{Since } 0 &= [x + y, x + y] \\ &= [x, x] + [x, y] + [y, x] + [y, y] \\ &= [x, y] + [y, x] \end{aligned}$$

□

Let A be an associative algebra over a field $\mathbb{F}$, and so we have a multiplication

$$A \times A \to A$$

$$(x, y) \to xy$$

with the property of associativity: $x(yz) = (xy)z$ for all $x, y, z \in A$. We can then construct a Lie algebra on A by defining the bracket as

$$[x, y] = xy - yx, \forall x, y \in A.$$

It is clear that this will satisfy the three axioms.

Example 6.1.

An example of a Lie Algebra is the vector space $\mathbb{R}^3$ over either $\mathbb{R}$ or $\mathbb{C}$, and the Lie multiplication the cross product. It is clear that this satisfies the first two axioms. It can be shown that the cross product satisfies the Jacobian Identity.

Definition 6.2.

Let S, T be subspaces of a Lie algebra L. Define $[S, T] = \{[s, t] : s \in S,\ t \in T\}$

A subspace S of a Lie algebra L is a subalgebra if $[S, S] \subseteq S$.

A subspace I of a Lie algebra L is an ideal of L if $[I, L] \subseteq I$.

A subalgebra S of L is abelian if $[S, S] = 0$.

Proposition 6.3.

Let L be a Lie algebra, and $H, K \subseteq L$. Then $[H, K] = [K, H]$.

Proof. Let $h \in H$ and $k \in K$. Then $[h,k] \in [H,K]$, but $[h,k] = [-k,h] \in [K,H]$. So $[H,K] \subseteq [K,H]$. Similarly then the argument may work in reverse, and so $[H,K] = [K,H]$.

□

Definition 6.4.

Let L be a Lie algebra, and let I be an ideal, then the quotient space L/I can be made into a Lie algebra by defining the multiplication as $[I+x, I+y] = I + [x,y] \; \forall x, y \in L$.

Definition 6.5.

Let L_1 and L_2 be two Lie algebras over the same field $\mathbb{F}$. Then we define a Lie homomorphism to be a function $\theta : L_1 \to L_2$ that is linear and that has the property that $\theta([x,y]) = [\theta(x), \theta(y)]$, for every $x, y \in L_1$.

A Lie isomorphism is a Lie homomorphism that is both injective and surjective.

Definition 6.6.

Let $\mathbb{C}^{n\times n}$ denote the associative algebra of all $n \times n$ matrices over $\mathbb{C}$. Using the construction above we define $[A,B] = AB - BA$ for all $A, B \in \mathbb{C}^{n\times n}$, then we have the general linear Lie algebra $\mathfrak{gl}_n = (\mathbb{C}^{n\times n}, [\cdot,\cdot])$.

The Lie subalgebra consisting of all matrices that have trace zero with the same Lie product as $\mathfrak{gl}_n$ is called the special linear Lie algebra $\mathfrak{sl}_n$.

A maximal abelian subalgebra is an abelian subalgebra which is not contained in any other abelian subalgebra.

An abelian subalgebra of maximal dimension is an abelian subalgebra whose dimension is maximal amongst all abelian subalgebras.

The subspace of all skew symmetric matrices in $\mathbb{C}^{n\times n}$ forms a Lie algebra with the same Lie product as $\mathfrak{gl}_n$.

Definition 6.7. Let $l \in \mathbb{N}$ and $K = \begin{pmatrix} 0 & I_l \\ I_l & 0 \end{pmatrix}$ then we define the Lie algebra

$$\mathfrak{o}(2l) = \left\{X \in \mathfrak{gl}_{2l} : XK + KX^T = 0\right\}$$

.

Matrices in $\mathfrak{o}(2l)$ have the form $\begin{pmatrix} m & n \\ p & -m^T \end{pmatrix}$ with $n = -n^T$ and $p = -p^T$.

In order to define $\mathfrak{o}(2l+1)$, let $K = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_l \\ 0 & I_l & 0 \end{pmatrix}$ then we define

$$\mathfrak{o}(2l+1) = \left\{X \in \mathfrak{gl}_{2l+1} : XK + KX^T = 0\right\}$$

.

Matrices in $\mathfrak{o}(2l+1)$ have the form

$$\begin{pmatrix} 0 & a & b \\ -b^T & m & n \\ -a^T & p & -m^T \end{pmatrix}$$

with $n = -n^T$ and $p = -p^T$.

For this paper, we will use an isomorphic definition of $\mathfrak{o}(n)$ that is the Lie algebra of all skew symmetric matrices, all matrices satisfying $A = -A^T$. We mention the alternative definition because several of the important results for $\mathfrak{o}(n)$ fall from this definition, mainly that the abelian subalgebra of maximal dimension has dimension $\frac{\ell^2}{4}$ when $n = 2\ell$.

For $n \geqslant 5$ then $\mathfrak{o}(n)$ is simple since when $n = 2l > 4$ the Lie algebra $\mathfrak{o}(n)$ isomorphic to D_l, and when $n = 2l+1 > 4$ the Lie algebra $\mathfrak{o}(n)$ is isomorphic to B_l.

Lemma 6.8. *The Lie algebra* $\mathfrak{o}(3)$ *is simple.*

Proof. Let $E_1 = e_{1,2} - e_{2,1},\ E_2 = e_{1,3} - e_{3,1}$ and let $E_3 = e_{2,3} - e_{3,2}$.

Then $[E_1, E_2] = -E_3$, $[E_2, E_3] = -E_1$ and $[E_1, E_3] = -E_2$.

Let I be an ideal of $\mathfrak{o}(3)$, whose elements are of the form $aE_1 + bE_2 + cE_3$ where $a, b, c \in \mathbb{C}$ and $I \neq 0$.

Let's examine
$$[E_1, aE_1 + bE_2 + cE_3] = b[E_1, E_2] + c[E_1, E_3]$$
$$= -bE_3 - cE_2.$$

Now $[E_2, -bE_3 - cE_2] = -b[E_2, E_3] = bE_1$.

Therefore $E_1 \in I$. Applying the same strategy we would see that all the basis elements of $\mathfrak{o}(3)$ are in I, hence $I = \mathfrak{o}(3)$, and so the only ideals of $\mathfrak{o}(3)$ are 0 and itself. Therefore $\mathfrak{o}(3)$ is simple.

□

Lemma 6.9. *Every element $A \in \mathfrak{o}(n)$ has trace zero.*

Proof.

Let $A \in \mathfrak{o}(n)$, then $A = -A^t$. Then we have $A_{i,i} = -A_{i,i}$, and so $2A_{i,i} = 0$, and hence $A_{i,i} = 0$. Therefore $tr(A) = 0$.

□

Definition 6.10.

Let L be a Lie algebra and $H \subset L$. We define the normalizer of H, $N(H) = \{x \in L : [h, x] \in H \ \forall h \in H\}$. That is $N(H)$ largest subalgebra of L which contains H as an ideal.

From this we define the Cartan subalgebra. A Cartan subalgebra of a Lie algebra is a subalgebra H such that H is nilpotent and $H = N(H)$.

Proposition 6.11.

From [13] a Cartan subalgebra of $\mathfrak{o}(n)$ when $n = 2\ell$, consists of matrices

$$\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_\ell \end{pmatrix}$$

where $A_i = a_i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in \mathfrak{gl}_2$ and $a_i \in \mathbb{C}$. When $n = 2\ell + 1$,

$$\begin{pmatrix} A_1 & & & & \\ & A_2 & & & \\ & & \ddots & & \\ & & & A_\ell & \\ & & & & 0 \end{pmatrix}$$

where $0 \in \mathbb{C}$.

The dimension of a Cartan subalgebra of $\mathfrak{o}(2\ell)$ and $\mathfrak{o}(2\ell+1)$ is ℓ.

Definition 6.12.

We define the powers of a Lie algebra by $L^1 = L$, and $L^n = [L, L^{n-1}]$ for any $n \geqslant 1$ and $n \in \mathbb{N}$.

We define the derived powers of a Lie algebra to be $L^{(0)} = L$, $L^{(1)} = [L^{(0)}, L^{(0)}] = [L, L]$, and $L^{(n+1)} = [L^{(n)}, L^{(n)}]$ for $n \geqslant 0$ and $n \in \mathbb{N}$.

A Lie algebra is called nilpotent if there is some $n \in \mathbb{N}$ such that $L^n = 0$.

A Lie algebra is called solvable if there exists some $n \in \mathbb{N}$ such that $L^{(n)} = 0$.

Proposition 6.13. *Every finite dimensional Lie algebra L contains a unique maximal solvable ideal R.*

We call this maximal solvable ideal R the solvable radical.

It can be shown that every nilpotent Lie algebra is solvable, and that $L^{(n)} \subseteq L^n$ for all $n \geqslant 1$.

An L-module is a vector space V over a field $\mathbb{F}$ together with a multiplication

$$L \times V \to V$$

$$(x, v) \to xv$$

satisfying the axioms:

(x, v) is bilinear in L and V,

$[x, y]v = x(yv) - y(xv)$ for all $x, y \in L$ and $v \in V$.

A submodule of V is a subspace U of V such that $LU \subset U$.

An L-module V is called irreducible if it has no proper submodules.

We define the adjoint module as

$$\mathrm{ad}_x : L \to L$$

$$\mathrm{ad}_x(y) = [x, y] \; for \; x, y \in L$$

.

Every ideal I of a Lie algebra L is also be an $L-$module since the multiplication is defined by the Lie brackets and $[L, I] \subseteq I$.

$$L \times I \to I$$

$$(\ell, i) \to [\ell, i]$$

where $i \in I$ and $\ell \in L$.

Here we list some important theorems from the theory of Lie algebras.

Proposition 6.14. *Let L be a Lie algebra, and I an ideal. If L is solvable, then so is L/I.*

Theorem 6.15 (Lie's Theorem)**.** *Let L be a solvable Lie algebra and let V be a finite dimensional irreducible L-module. Then $dim(V) = 1$.*

Theorem 6.16 (Engel's Theorem)**.** *A finite dimensional Lie algebra L is nilpotent if and only if $ad_x : L \to L$ is nilpotent for any $x \in L$.*

Definition 6.17. Let L be a Lie algebra, and let $x \in L$. Let $L_{0,x}$ be the generalized eigenspace of ad_x with eigenvalue 0. Then

$$L_{0,x} = \{y \in L | \exists n \text{ such that } (ad_x(y))^n = 0\}.$$

An element $x \in L$ is regular then if the dimension $L_{0,x}$ is as small as possible compared with the dimension for all other choices of x.

Theorem 6.18. *Let $x \in L$ be a regular element, then $L_{0,x}$ is a Cartan subalgebra of L.*

In this way, regular elements are connected to Cartan subalgebras, and conversely for any Cartan subalgebra, we can find a regular element within it.

Theorem 6.19. *Any two Cartan subalgebras of L are conjugate.*

7. Euclidean Lie algebras

Definition 7.1.

For $n \geq 2$, the Euclidean Lie algebra $\mathcal{E}(n)$ is the Lie subalgebra of $\mathfrak{gl}_{2n}$ that consists of all the following matrices

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathfrak{gl}_{2n}, \text{ where } A, B, C \in \mathfrak{gl}_n \text{ with } A^T = -A, C^T = -C.$$

We should notice that $\mathcal{E}(n) \subset \mathfrak{sl}_{2n}$ since from Lemma 6.9 we see that the trace of any matrix in $\mathcal{E}(n)$ is zero.

Lemma 7.2.

The Lie algebra $\mathcal{E}(n)$ is not simple or semisimple.

Proof.

We know that the subalgebra of the matrices $\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix}$ is an ideal since given any matrix $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathcal{E}(n)$ we have

$$\left[\begin{pmatrix} 0 & X \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}\right] = \begin{pmatrix} 0 & XC - AX \\ 0 & 0 \end{pmatrix}.$$

Hence the subalgebra is an ideal. So this ideal is nilpotent and therefore a solvable ideal. Hence the solvable radical of $\mathcal{E}(n)$ is non zero. □

Let

$$M = \left\{ \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \in \mathfrak{gl}_{2n} : A \in \mathfrak{gl}_n \right\}.$$

.

Let $n = 2$ then every matrix in $\mathcal{E}(2)$ is of the form $\begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$ with $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix} \in \mathfrak{o}(2)$, and $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \in \mathfrak{o}(2)$.

Now let $P = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}, Q = \begin{pmatrix} C & Y \\ 0 & D \end{pmatrix} \in \mathcal{E}(2)$ then

$$[P, Q] = PQ - QP.$$

So We have

$$[P, Q] = \begin{pmatrix} AC - CA & AY + XD - CX + YB \\ 0 & BD - DB \end{pmatrix}.$$

Notice that $AC = \begin{pmatrix} -ac & 0 \\ 0 & -ac \end{pmatrix}$, and similarly $CA = \begin{pmatrix} -ac & 0 \\ 0 & -ac \end{pmatrix}$. Therefore $AC - CA = 0$. Also, $BD = DB$. So $\mathcal{E}(2)^2 = [\mathcal{E}(2), \mathcal{E}(2)] \subseteq M$ where Notice though that $\mathcal{E}(2)$ is not nilpotent since

$$\mathcal{E}(2)^3 = [\mathcal{E}(2)^2, \mathcal{E}(2)] \subseteq [M, \mathcal{E}(2)] \subseteq M,$$

and by induction,

$$\mathcal{E}(2)^k = [\mathcal{E}(2)^{k-1}, \mathcal{E}(2)] \subseteq [M, \mathcal{E}(2)] \subseteq M$$

. It is still possible that $\mathcal{E}(2)^3 = 0$, and in general $\mathcal{E}(2)^k = 0$, so we must show that there is a non zero element in $\mathcal{E}(2)^3$.

Let $P = \begin{pmatrix} A & I_2 \\ 0 & C \end{pmatrix}, Q = \begin{pmatrix} A & X \\ 0 & A \end{pmatrix}, R = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \in \mathcal{E}(2)$ with I_2 being the identity matrix, $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, and X being the matrix of all ones. Then $[[P,Q],R] = \begin{pmatrix} 0 & Y \\ 0 & 0 \end{pmatrix} \in \mathcal{E}(n)^3$, where $Y = \begin{pmatrix} 0 & -4 \\ -4 & 0 \end{pmatrix}$. Thus $\mathcal{E}(2)^3$ is not empty. We can use induction to show that $\mathcal{E}(2)^k$ is not empty for any $k \in \mathbb{N}$.

On the other hand, it is clear that $\mathcal{E}(2)$ is solvable since $\mathcal{E}(2)^{(1)} = [\mathcal{E}(2), \mathcal{E}(2)] \subseteq M$, and therefore $\mathcal{E}(2)^{(2)} = [\mathcal{E}(2)^{(1)}, \mathcal{E}(2)^{(1)}] \subseteq [M, M] = 0$.

For every other Euclidean algebra where $n \geqslant 5$, we have $\mathfrak{o}(n)$ being a simple Lie algebra and thus not nilpotent or solvable.

Take the subalgebra M_1 of $\mathcal{E}(n)$ whose matrices are of the form $\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$. Then for any $M_1^k \neq 0$ and $M_1^{(k)} \neq 0$, since $\mathfrak{o}(n)$ is simple.

Notice though that $\mathfrak{o}(3)$ is simple from lemma 6.8 and therefore the Euclidean Lie algebra $\mathcal{E}(3)$ is neither solvable nor nilpotent.

8. Abelian subalgebras of maximal dimension of Euclidean Lie algebras

We denote the Lie subalgebra of $\mathfrak{gl}_n$ that consists of all upper-triangular matrices by T_n and the Lie subalgebra of $\mathfrak{gl}_n$ that consists of all strictly upper-triangular matrices by S_n.

From [8] we know that the maximal dimension of a abelian subalgebras of $\mathfrak{gl}_n$ or T_n is $\frac{n^2}{4}+1$ if n is even, and $\frac{(n+1)(n-1)}{4}+1$ if n is odd; while the maximal dimension of abelian subalgebras of S_n is $\frac{n^2}{4}$ if n is even, and $\frac{(n+1)(n-1)}{4}$ if n is odd.

Proposition 8.1. *Let J_m and J_n be the nilpotent Jordan blocks of size m and n,*

$$V = \{X \in \mathbb{C}^{m\times n} : J_m X = X J_n\}.$$

Then $\dim V = \min\{m, n\}$.

Proof. First for simplicity suppose that $m > n$.

We have $J_m = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & & & & 1 \\ 0 & \cdots & & & 0 \end{pmatrix}_{m\times m}$ and

$$J_n = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \\ 0 & & & & 1 \\ 0 & \cdots & & & 0 \end{pmatrix}_{n\times n}$$

and finally let $X = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots & x_{2,n} \\ \vdots & & \ddots & & \\ x_{m,1} & x_{m,2} & \cdots & x_{m,n-1} & x_{m,n} \end{pmatrix}_{m\times n}$.

We see that $J_mX = \begin{pmatrix} x_{2,1} & x_{2,2} & x_{2,3} & \cdots & x_{2,n} \\ x_{3,1} & x_{3,2} & x_{3,3} & \cdots & x_{3,n} \\ \vdots & & \ddots & & \\ x_{m,1} & x_{m,2} & \cdots & x_{m,n-1} & x_{m,n} \\ 0 & & \cdots & & 0 \end{pmatrix}_{m\times n}$

and

$$XJ_n = \begin{pmatrix} 0 & x_{1,1} & x_{1,2} & \cdots & x_{1,n-1} \\ 0 & x_{2,1} & x_{2,2} & \cdots & x_{2,n-1} \\ \vdots & & \ddots & & \\ 0 & x_{m,1} & x_{m,2} & \cdots & x_{m,n-1} \end{pmatrix}_{m\times n}.$$

Thus if we want $J_mX = XJ_n$, we must have

$$\begin{pmatrix} x_{2,1} & x_{2,2} & x_{2,3} & \cdots & x_{2,n} \\ x_{3,1} & x_{3,2} & x_{3,3} & \cdots & x_{3,n} \\ \vdots & & \ddots & & \\ x_{m,1} & x_{m,2} & \cdots & x_{m,n-1} & x_{m,n} \\ 0 & & \cdots & & 0 \end{pmatrix}_{m\times n} = \begin{pmatrix} 0 & x_{1,1} & x_{1,2} & \cdots & x_{1,n-1} \\ 0 & x_{2,1} & x_{2,2} & \cdots & x_{2,n-1} \\ \vdots & & \ddots & & \\ 0 & x_{m,1} & x_{m,2} & \cdots & x_{m,n-1} \end{pmatrix}_{m\times n}$$

.

Notice that $x_{1,n}$ is free. Also $x_{2,1} = ... = x_{m,1} = 0$, and $x_{m,1} = ... = x_{m,n-1} = 0$.

Also, notice that $x_{1,1} = x_{2,2} = ... = x_{n,n}$, and $x_{1,n-1} = x_{2,n}$. In general for the non-main diagonal $x_{k-1,k} = x_{k,k+1}$, where $2 \leqslant k \leqslant n-1$ and zero everywhere else.

$$\text{Thus when } m > n,\ X = \begin{pmatrix} x_1 & x_2 & \cdots & x_{n-1} & x_n \\ 0 & x_1 & x_2 & \cdots & x_{n-1} \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & & & x_1 & x_2 \\ 0 & & & & x_1 \\ 0 & & \cdots & & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & & \cdots & & 0 \end{pmatrix}$$

Clearly then the dimension of V would be $\min\{n, m\}$.

In the case where $n > m$, i.e. the number of columns is greater than the number of rows, we get a similar matrix of the form

$$X = \begin{pmatrix} 0 & \cdots & 0 & x_1 & x_2 & \cdots & x_m \\ 0 & \cdots & 0 & 0 & x_1 & \cdots & x_{n-1} \\ \vdots & \ddots & \vdots & & & \ddots & \\ 0 & \cdots & 0 & & & & x_1 \end{pmatrix}$$

□

Theorem 8.2. *Let $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{m\times m}$, and*

$$V = \{X \in \mathbb{C}^{n\times m} : AX = XB\}.$$

We will determine the dimension of V.

Proof. Let P, Q be non-singular matrices of appropriate sizes such that $PAP^{-1} = J_A$, and $QBQ^{-1} = J_B$ where J_A, J_B are the Jordan Canonical Forms of A and B. So $PAXQ^{-1} = PXBQ^{-1}$ then $PAP^{-1}PXQ^{-1} =$

$PXQ^{-1}QBQ^{-1}$. Therefore $J_A PXQ^{-1} = PXQ^{-1}J_B$. Thus the dimension of V is unchanged when we replace A and B by their Jordan Canonical Forms. Therefore we can assume A and B are in Jordan Canonical Form.

First let us prove that if $\text{rank}(A) = m$ and $\text{rank}(B) = n$, then $V = 0$. So let $\text{rank}(A) = m$, and $\text{rank}(B) = n$.

All we need is to have A and B be Jordan blocks of full rank, and thus their eigenvalues are non-zero from lemma 1. So let

$$A = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots 0 & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{pmatrix}_{m\times m}$$

and let $B = \begin{pmatrix} \mu & 1 & 0 & \cdots & 0 \\ 0 & \mu & 1 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & \cdots 0 & \mu & 1 \\ 0 & 0 & \cdots & 0 & \mu \end{pmatrix}_{n\times n}$.

Finally let $X = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} & \cdots & x_{1,n} \\ x_{2,1} & x_{2,2} & x_{2,3} & \cdots & x_{2,n} \\ \vdots & & \ddots & & \vdots \\ x_{m-1,1} & x_{m-1,2} & \cdots & x_{m-1,n-1} & x_{m-1,n} \\ x_{m,1} & x_{m,2} & \cdots & x_{m,n-1} & x_{m,n} \end{pmatrix}_{m\times n}$

We then have

$$AX = \begin{pmatrix} \lambda x_{1,1} + x_{2,1} & \lambda x_{1,2} + x_{2,2} & \cdots & \lambda x_{1,n} + x_{2,n} \\ \lambda x_{2,1} + x_{3,1} & \lambda x_{2,2} + x_{3,2} & \cdots & \lambda x_{2,n} + x_{3,n} \\ \vdots & & \ddots & \vdots \\ \lambda x_{m-1,1} + x_{m,1} & \lambda x_{m-1,2} + x_{m,2} & \cdots & \lambda x_{m-1,n} + x_{m,n} \\ \lambda x_{m,1} & \lambda x_{m,2} & \cdots & \lambda x_{m,n} \end{pmatrix}_{m\times n}$$

and

$$XB = \begin{pmatrix} \mu x_{1,1} & x_{1,1} + \mu x_{1,2} & \cdots & x_{1,n-1} + \mu x_{1,n} \\ \mu x_{2,1} & x_{2,1} + \mu x_{2,2} & \cdots & x_{2,n-1} + \mu x_{2,n} \\ \vdots & & \ddots & \vdots \\ \mu x_{m-1,1} & x_{m-1,1} + \mu x_{m-1,2} & \cdots & x_{m-1,n-1} + \mu x_{m-1,n} \\ \mu x_{m,1} & x_{m,1} + \mu x_{m,2} & \cdots & x_{m,n-1} + \mu x_{m,n} \end{pmatrix}_{m\times n}$$

So we have

$$0 = AX - XB =$$

$$\begin{pmatrix} (\lambda-\mu)x_{1,1} + x_{2,1} & (\lambda-\mu)x_{1,2} + x_{2,2} - x_{1,1} & \cdots & (\lambda-\mu)x_{1,n} + x_{2,n} - x_{1,n-1} \\ (\lambda-\mu)x_{2,1} + x_{3,1} & (\lambda-\mu)x_{2,2} + x_{3,2} - x_{2,1} & \cdots & (\lambda-\mu)x_{2,n} + x_{3,n} - x_{2,n-1} \\ \vdots & & \ddots & \vdots \\ (\lambda-\mu)x_{m-1,1} + x_{m,1} & (\lambda-\mu)x_{m-1,2} + x_{m,2} - x_{m-1,1} & \cdots & (\lambda-\mu)x_{m-1,n} + x_{m,n} - x_{m-1,n-1} \\ (\lambda-\mu)x_{m,1} & (\lambda-\mu)x_{m,2} - x_{m,1} & \cdots & (\lambda-\mu)x_{m,n} - x_{m,n-1} \end{pmatrix}_{m\times n}$$

Starting from the bottom left corner we see that $x_{m,1} = 0$, and moving outwards see that every diagonal line is zero. Therefore $X = 0$. Alternatively we can do induction on m to show $X = 0$.

Let J_k be the $k \times k$ matrix with zeros everywhere except ones on the superdiagonal.

Notice that $(\lambda I_n + J_n)X = X(\mu I_m + J_m)$, and so if $\lambda = \mu$ then $J_m X = X J_n$. So we can use the previous proposition, and we get that $\dim(V) = \min\{m, n\}$.

So let $A = \begin{pmatrix} J_{A_1} & & & 0 \\ & J_{A_2} & & \\ & & \ddots & \\ 0 & & & J_{A_p} \end{pmatrix}$ and $B = \begin{pmatrix} J_{B_1} & & & 0 \\ & J_{B_2} & & \\ & & \ddots & \\ 0 & & & J_{B_q} \end{pmatrix}$ where J_{A_i}, J_{B_j} are Jordan blocks of size $r_{A_i} \times r_{A_i}$ and $r_{B_j} \times r_{B_j}$ with eigenvalues λ_i and μ_j respectively.

Let $X = \begin{pmatrix} X_{1,1} & X_{1,2} & \cdots & X_{1,q} \\ X_{1,1} & X_{1,2} & \cdots & X_{1,q} \\ \vdots & & \ddots & \vdots \\ X_{p,1} & X_{p,2} & \cdots & X_{p,q} \end{pmatrix}$

Where $J_{A_i}(X_{ij}) = (X_{ij})J_{B_j}$.

So

$$\dim(\{X_{i,j}\})$$

$$= \begin{cases} min\{r_{A_i}, r_{B_j}\} & if\ rank(J_{A_i}) < r_{A_i}\ and\ rank(J_{B_j}) < r_{B_j}\ or \lambda_i = \mu_j \\ 0 & if\ rank(J_{A_i}) = r_{A_i}\ or\ rank(J_{B_j}) = s_{B_j}\ and\ \lambda_i \neq \mu_j \end{cases}$$

Thus

$$\dim(V) = \sum_{i=1}^{p} \sum_{j=1}^{q} \dim(\{X_{i,j}\}).$$

□

Consider the block matrices $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathfrak{gl}_{2n}$ where $A, B, C \in \mathfrak{gl}_n$ and $A^T = -A$ and $C^T = -C$. Let

$$M = \left\{ \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \in \mathfrak{gl}_{2n} : A \in \mathfrak{gl}_n \right\},$$

$$M_1 = \left\{ \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{gl}_{2n} : A^T = -A \right\},$$

$$M_2 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix} \in \mathfrak{gl}_{2n} : A^T = -A \right\}.$$

We see that

$$\mathcal{E}(n) = M_1 \oplus M_2 \oplus M.$$

By easy computations we see that

$$\dim M_1 = \dim M_2 = \frac{n(n-1)}{2}$$

$$\dim(M') = \dim(M) = n^2.$$

$$\dim(\mathcal{E}(n)) = 2n^2 - n$$

Lemma 8.3.

M is an abelian subalgebra of maximal dimension.

Proof.

We know that M is a maximal abelian subalgebra of $\mathcal{E}(n)$, since $\mathcal{E}(n) \subset \mathfrak{sl}_{2n}$ and so the dimension of the abelian subalgebra of maximal dimension for $\mathcal{E}(n)$ can be at most the dimension of the abelian subalgebras of maximal dimension of $\mathfrak{sl}_{2n}$. Since M is an abelian subalgebra of maximal dimension in $\mathfrak{sl}_{2n}$ and is an abelian subalgebra of $\mathcal{E}(n)$, then M is an abelian subalgebra of maximal dimension of $\mathcal{E}(n)$. □

We have the projections

$\rho : \mathcal{E}(n) \to M; x + y + z \mapsto z, \forall x \in M_1, y \in M_2, z \in M;$

$\rho_1 : \mathcal{E}(n) \to M_1; x + y + z \mapsto x, \forall x \in M_1, y \in M_2, z \in M;$

$\rho_2 : \mathcal{E}(n) \to M_2; x + y + z \mapsto y, \forall x \in M_1, y \in M_2, z \in M.$

$\rho_0 : \mathcal{E}(n) \to M_1 + M_2; x + y + z \mapsto x + y, \forall x \in M_1, y \in M_2, z \in M$

Given a subalgebra M' of $\mathcal{E}(n)$, in general, $\rho_0(M') \neq \rho_1(M')+\rho_2(M')$.

Suppose $M' \neq M$ is another abelian subalgebra of maximal dimension in $\mathcal{E}(n)$. We know that $\dim M' = n^2$ and that $M'^2 = 0$.

Let $M'_{12} = (\rho_1 + \rho_2)(M'), M'' = M' \cap M$.

Lemma 8.4. *Let* $X, Y \in \mathcal{E}(n)$ *then* $\rho_0(aX) = a\rho(X)$ *for any* $a \in \mathbb{C}$, $\rho_0(X + Y) = \rho_0(X) + \rho_0(Y)$, *and* $\rho_0(XY) = \rho_0(X)\rho_0(Y)$ *and* $\rho_0([X, Y]) = [\rho_0(X), \rho_0(Y)]$.

Proof. By definition

$$M'_{12} = \{\ m_1 + m_2 \mid \exists m \in M\ s.t\ m_1 + m_2 + m \in M'\}.$$

Let $X = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathcal{E}(n)$, then $aX = \begin{pmatrix} aA & aB \\ 0 & aC \end{pmatrix}$. So $\rho_0(aX) = \begin{pmatrix} aA & 0 \\ 0 & aC \end{pmatrix} = a\rho_0(X)$.

Let $A, B \in M'_{12}$, so $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$

$B = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$ so there are $m_A, m_B \in M$ such that $X = A + m_A$ and $Y = B + m_B$ are both in $\mathcal{E}(n)$. Then $\rho_0(X+Y) = \begin{pmatrix} A_1 + B_1 & 0 \\ 0 & A_2 + B_2 \end{pmatrix}$, but also $\rho_0(X) + \rho_0(Y) = \begin{pmatrix} A_1 + B_1 & 0 \\ 0 & A_2 + B_2 \end{pmatrix}$. So $\rho_0(X + Y) = \rho_0(X) + \rho_0(Y)$.

Similarly $\rho_0(XY) = \begin{pmatrix} A_1B_1 & 0 \\ 0 & A_2B_2 \end{pmatrix} = \rho_0(X)\rho_0(Y)$.

Finally from the above two results, for any $X, Y \in \mathcal{E}(n)$,

$$\begin{aligned} \rho_0([X, Y]) &= \rho_0(XY - YX) \\ &= \rho_0(XY) - \rho_0(YX) \\ &= \rho_0(X)\rho_0(Y) - \rho_0(Y)\rho_0(X) \\ &= [\rho_0(X), \rho_0(Y)] \end{aligned}$$

□

Theorem 8.5. (a). *$\dim(M'') \geq n$.*

(b). *M'_{12} is an abelian subalgebra in $\mathcal{E}(n)$ and $\dim M'_{12} \neq 0$.*

(c). *The subspace $M'_{12} \oplus M''$ is a abelian subalgebra of dimension n^2 in $\mathcal{E}(n)$.*

Proof. (*a*). Since $M + M'$ is a subspace of $\mathcal{E}(n)$, then $\dim(M' + M) \leq \dim \mathcal{E}(n) = n(2n - 1)$. So

$$\begin{aligned} \dim(M'') =& \dim(M' \cap M) = \dim(M') + \dim(M) - \dim(M' + M) \\ \geq& 2n^2 - n(2n - 1) = n. \end{aligned}$$

(b) We know that M'_{12} is not $\{0\}$ because otherwise $M' \subseteq M$, which is a contradiction.

Let $X, Y \in M'$. We know $[X, Y] = 0$. So $0 = \rho_0([X, Y])$, and thus from the above lemma, $[\rho_0(X), \rho_0(Y)] = 0$.

So M'_{12} is a non-trivial abelian Lie algebra, and it should be clear that $M'_{12} \subset M_1 + M_2 \subset \mathcal{E}(n)$.

(c) Let us examine the function $\rho_1 + \rho_2 : M' \to M'_{12}$. First lets denote $f := \rho_1 + \rho_2$. with $m_1 + m_2 + m \to m_1 + m_2$ with $m_1 \in M_1,\ m_2 \in M_2,\ m \in M$.

Given any matrix $m_1 + m_2 \in M'_{12}$ with $m_1 \in M_1$ and $m_2 \in M_2$, then I can find a matrix $m \in M$ such that $m_1 + m_2 + m \in M'$. Thus the function is onto. Notice the function may not be one-to-one.

Now using the Rank Nullity theorem, we have $dim(M') = Rank(f) + Nullity(f)$. Now we have $Rank(f) = dim(R(f))$. By definition $R(f) = \{f(X) | X \in M'\}$ and since this function is onto we have $R(f) = M'_{12}$. We also have $Nullity(f) = dim(N(f))$. we have $N(f) = \{X \in M' | f(X) = 0\}$ it is also clear that all of these matrices with also be in M, so we have $N(f) = M \cap M'$. so $n^2 = dim(M'_{12}) + dim(M'')$, and thus $M'_{12} \oplus M''$ is a subspace of dimension n^2.

Now let us prove that $M'_{12} \oplus M''$ is an abelian subalgebra.

Well $[M'_{12} \oplus M'', M'_{12} \oplus M''] = [M'_{12}, M'_{12}] + [M'_{12}, M''] + [M'', M'_{12}] + [M'', M''] \subseteq [M'_{12}, M'']$

Hence if we can prove that the matrices in M'_{12} commute with the matrices in M'' then we have that the matrices in $M'_{12} \oplus M''$ are commute.

Let $P = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \in M'_{12}$, and let $R = \begin{pmatrix} 0 & m \\ 0 & 0 \end{pmatrix} \in M$ such that $P + R = \begin{pmatrix} A & m \\ 0 & C \end{pmatrix} \in M'$ and let $Q = \begin{pmatrix} 0 & C \\ 0 & 0 \end{pmatrix} \in M''$. So $[P, Q] = [P + R - R, Q] = [P + R, Q] - [R, Q]$.

Then $[P + R, Q] = 0$ since $P + R, Q \in M'$ and M' by definition is abelian. Also, $[R, Q] = 0$ since both $R, Q \in M$ and clearly $[M, M] = 0$. So, $[P, Q] = 0$, and therefore $M'_{12} \oplus M''$ is an abelian subalgebra.

Therefore $M'' \oplus M'_{12}$ is an abelian subalgebra of maximal dimension.

□

Theorem 8.6. *The abelian subalgebra of maximal dimension of $\mathcal{E}(n)$ is unique.*

Suppose we have another abelian Lie subalgebra M' of $\mathcal{E}(n)$ with dimension n^2 not necessarily distinct from M. Since M' is a subalgebra of $\mathcal{E}(n)$, we can then write M' as

$$M' = \left\{ \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathfrak{gl}_{2n} : A^T = -A,\ C^T = -C,\ and\ B \in \mathfrak{gl}_n \right\}$$

and $dim(M') = n^2$ with $[M', M'] = 0$.

Let $X = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ be any matrix in M', then I can find a non-singular matrix $R = \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix}$, such that

$$RXR^{-1} = \begin{pmatrix} PAP^{-1} & PBQ^{-1} \\ 0 & QCQ^{-1} \end{pmatrix}$$

is an upper triangular matrix. So $RXR^{-1} \in T(2n)$. Since X is in M', then RXR^{-1} must be in a Lie subalgebra of dimension n^2 in $T(2n)$. We know that each abelian Lie subalgebra of $T(2n)$ of dimension n^2 is a subalgebra of the abelian lie algebra of maximal dimension of $T(2n)$. Every matrix in this abelian subalgebra are of the form $\begin{pmatrix} aI_n & A \\ 0 & aI_n \end{pmatrix}$, where $a\mathbb{C}$, $I_n, A \in \mathfrak{gl}_n$, with I_n being the multiplicative identity matrix. On the other hand, $tr(RXR^{-1}) = tr(PAP^{-1} + QCQ^{-1})$

$$= tr(PAP^{-1}) + tr(QCQ^{-1})$$

$$= tr(P^{-1}PA) + tr(Q^{-1}QC)$$

$$= tr(A) + tr(C)$$

$$= 0.$$

Therefore we have $RXR^{-1} = \begin{pmatrix} PAP^{-1} & PBQ^{-1} \\ 0 & QCQ^{-1} \end{pmatrix} = \begin{pmatrix} 0 & PBQ^{-1} \\ 0 & 0 \end{pmatrix}$. Hence $PAP^{-1} = 0$, and therefore $A = 0$. Similarly $C = 0$.

Thus we have $X \in M$, but X was an arbitrary matrix in M', thus we have $M' \subseteq M$, and since their dimensions are the same we have that $M' = M$.

Therefore the abelian subalgebra of maximal dimension for $\mathcal{E}(n)$ is unique.

Corollary 8.7. *The abelian subalgebra of maximal dimension M is an ideal of $\mathcal{E}(n)$.*

Proof. Let $X = \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \in \mathcal{E}(n)$ and let $Y = \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \in M$. Then

$$[X, Y] = XY - YX$$
$$= \begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & D \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$$
$$= \begin{pmatrix} 0 & AD - DC \\ 0 & 0 \end{pmatrix} \in M.$$

Therefore M is an ideal of $\mathcal{E}(n)$. □

9. Maximal abelian subalgebras of The Euclidean Lie algebra

In this section we construct a maximal abelian subalgebra distinct from the abelian sublagebra of maximal dimension discussed in the previous section. We know from [13] that an abelian Lie subalgebra of $\mathfrak{o}(n)$ when $n = 2\ell$ has matrices of the form

$$\begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_\ell \end{pmatrix}$$

where $A_i = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$ with $a \in \mathbb{C}$.

When $n = 2\ell + 1$ we have

$$\begin{pmatrix} A_1 & & & & \\ & A_2 & & & \\ & & \ddots & & \\ & & & A_\ell & \\ & & & & 0 \end{pmatrix}$$

with $0 \in \mathbb{C}$.

These elements are regular elements in the Lie algebra $\mathfrak{o}(n)$, and they are elements in its Cartan subalgebra.

Lemma 9.1. *If* $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$*, and* $B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix}$ *with* $a \neq b$*, and* $a \neq -b$*, then the only* $X \in \mathfrak{gl}_2$ *that satisfies* $AX = XB$ *is* $X = 0$*.*

Proof. Let $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, so we have

$a \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = b \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Thus we get

$a \begin{pmatrix} x_3 & x_4 \\ -x_1 & -x_2 \end{pmatrix} = b \begin{pmatrix} -x_2 & x_1 \\ -x_4 & x_3 \end{pmatrix}$.

From this we see that $ax_3 = -bx_2$, and $-ax_2 = bx_3$. Thus $a^2x_2 = b^2x_2$, and since $a \neq b$ and $a \neq -b$, then we see that the only solution the the previous equation is that $x_2 = 0$, thus $x_3 = 0$. Similarly we can see that $x_1 = x_4 = 0$. Hence $X = 0$. □

Lemma 9.2. *If* $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$*,* $X \in \mathfrak{gl}_2$ *and* $AX = -XA$*, then* $X = \begin{pmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{pmatrix}$ *where* $x_1, x_2 \in \mathbb{C}$*.*

Proof. As before let $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ and notice that since we want $AX = -XA$, which can be rewritten as

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ since we can factor out a from both sides.

So $\begin{pmatrix} x_3 & x_4 \\ -x_1 & -x_2 \end{pmatrix} = \begin{pmatrix} x_2 & -x_1 \\ x_4 & -x_3 \end{pmatrix}$. Thus we have $x_2 = x_3$, and $x_1 = -x_4$. So we have $Z = \begin{pmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{pmatrix}$. □

Lemma 9.3. *If* $A = \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \in \mathbb{C}^{2\times 2}$, $X \in \mathfrak{gl}_2$ *and* $AX = XA$, *then* $X = \begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix}$.

Proof. Again let $X = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$ and notice that $AX = XA$ is the same as $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

So we get $\begin{pmatrix} x_3 & x_4 \\ -x_1 & -x_2 \end{pmatrix} = \begin{pmatrix} -x_2 & x_1 \\ -x_4 & x_3 \end{pmatrix}$, and thus $x_2 = -x_3$, and $x_1 = x_4$.

Therefore $\begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix}$. □

Definition 9.4. Let n be even. Let $\mathcal{A}(n) \subset \mathcal{E}(n)$ be the Lie algebra of $2n\times 2n$ matrices of the form $\begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$ with $A = a \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_{\frac{n}{2}} \end{pmatrix} \in \mathfrak{gl}_n$, and $B = a \begin{pmatrix} \beta_1 & & & \\ & \beta_2 & & \\ & & \ddots & \\ & & & \beta_{\frac{n}{2}} \end{pmatrix} \in \mathfrak{gl}_n$, where α_i, β_j can be either

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. The choices of α_i and β_j are fixed. We construct

$$X = \begin{pmatrix} X_{1,1} & X_{1,2} & \cdots & X_{1,\frac{n}{2}} \\ X_{2,1} & X_{2,2} & \cdots & X_{2,\frac{n}{2}} \\ \vdots & & \ddots & \vdots \\ X_{\frac{n}{2},1} & X_{\frac{n}{2},2} & \cdots & X_{\frac{n}{2},\frac{n}{2}} \end{pmatrix}$$

where $X_{i,j} \in \mathfrak{gl}_2$ and satisfies $A_i(X_{i,j}) = (X_{i,j})B_j$.

Theorem 9.5. *The Lie subalgebra $\mathcal{A}(n)$ is abelian, and has dimension $1 + \frac{n^2}{2}$ and is a maximal abelian sublagebra of $\mathcal{E}(n)$.*

Proof. Let $P = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}, Q = \begin{pmatrix} C & Y \\ 0 & D \end{pmatrix} \in \mathcal{A}(n)$. Then

$$[P, Q] = \begin{pmatrix} AC - CA & AY - YB + XD - CX \\ 0 & BD - DB \end{pmatrix}$$

. Since A, B, C, D are elements of a Cartan subalgebras of $\mathfrak{o}(n)$, we see from proposition 6.11 that these elements commute.

From our construction from the above definition, we get $A_i(Y_{i,j}) = (Y_{i,j})B_j$, and $C_i(X_{i,j}) = (X_{i,j})D_j$ for all $i, j \in \overline{1 : \frac{n}{2}}$.

We can see from the definition of $\mathcal{A}(n)$ and the three lemmas above that each $(X_{i,j})$ is either $\begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix}$ or $\begin{pmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{pmatrix}$ depending on the signs of α_i and β_j. Hence $\dim((X_{i,j})) = 2$. It is clear that the subalgebra of all matrices of the form $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ has dimension one, therefore

$\dim(\mathcal{A}(n)) = 1 + \frac{n^2}{2}$.

Now we must prove that $\mathcal{A}(n)$ is a maximal abelian subalgebra of $\mathcal{E}(n)$. Assume not. Suppose that $\mathcal{A}(n) \subseteq \mathcal{A}'(n)$, where $\mathcal{A}'(n)$ is a maximal abelian subalgebra of $\mathcal{E}(n)$.

For simplicity of the argument Let all elements of $\mathcal{A}(n)$ have the form $\begin{pmatrix} A & X \\ 0 & A \end{pmatrix}$, and let $A = \begin{pmatrix} \alpha & & & \\ & \alpha & & \\ & & \ddots & \\ & & & \alpha \end{pmatrix}$ with $\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

Thus

$$X = \begin{pmatrix} X_{1,1} & X_{1,2} & \cdots & X_{1,\frac{n}{2}} \\ X_{2,1} & X_{2,2} & \cdots & X_{2,\frac{n}{2}} \\ \vdots & & \ddots & \vdots \\ X_{\frac{n}{2},1} & X_{\frac{n}{2},2} & \cdots & X_{\frac{n}{2},\frac{n}{2}} \end{pmatrix}$$

with $\alpha(X_{i,j}) = (X_{i,j})\alpha$ we get $X_{i,j} = \begin{pmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{pmatrix}$ with $x_1, x_2 \in \mathbb{C}$ from lemma 9.3.

Let $P = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \in \mathcal{A}(n)$ and $R = \begin{pmatrix} A' & B' \\ 0 & C' \end{pmatrix} \in \mathcal{A}'(n) \setminus \mathcal{A}(n)$, then $[P, R]$ yields $AA' = A'A$ $AB' = B'A$ and $AC' = C'A$ from this

$$B = \begin{pmatrix} B_{1,1} & B_{1,2} & \cdots & B_{1,\frac{n}{2}} \\ B_{2,1} & B_{2,2} & \cdots & B_{2,\frac{n}{2}} \\ \vdots & & \ddots & \vdots \\ B_{\frac{n}{2},1} & B_{\frac{n}{2},2} & \cdots & B_{1,\frac{n}{2}} \end{pmatrix}$$

. Similarly

$$A' = \begin{pmatrix} A'_{1,1} & A'_{1,2} & \cdots & A'_{1,\frac{n}{2}} \\ -{A'_{1,2}}^t & A'_{2,2} & \cdots & A'_{2,\frac{n}{2}} \\ \vdots & & \ddots & \vdots \\ -{A'_{\frac{n}{2},1}}^t & -{A'_{\frac{n}{2},2}}^t & \cdots & A'_{1,\frac{n}{2}} \end{pmatrix}$$

and

$$C' = \begin{pmatrix} C'_{1,1} & C'_{1,2} & \cdots & C'_{1,\frac{n}{2}} \\ -{C'_{1,2}}^t & C'_{2,2} & \cdots & C'_{2,\frac{n}{2}} \\ \vdots & & \ddots & \vdots \\ -{C'_{\frac{n}{2},1}}^t & -{C'_{\frac{n}{2},2}}^t & \cdots & C'_{1,\frac{n}{2}} \end{pmatrix}$$

where $A'_{i,i} = -{A'_{i,i}}^t$ and $C'_{i,i} = -{C'_{i,i}}^t$.

Let $Y = \begin{pmatrix} Y_{1,1} & Y_{1,2} & \cdots & Y_{1,\frac{n}{2}} \\ Y_{2,1} & Y_{2,2} & \cdots & Y_{2,\frac{n}{2}} \\ \vdots & & \ddots & \vdots \\ Y_{\frac{n}{2},1} & Y_{\frac{n}{2},2} & \cdots & Y_{\frac{n}{2},\frac{n}{2}} \end{pmatrix}$. We call

$$\overline{Y_{i,j}} = \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & & \\ 0 & \cdots & Y_{i,j} & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} \in \mathcal{A}(n)$$

where $Y_{i,j}$ is the same 2 block of Y.

Let $Q = \begin{pmatrix} 0 & \overline{Y_{i,j}} \\ 0 & 0 \end{pmatrix}$.

Then $[R,Q]=0$, and so $A'\overline{Y_{i,j}} = \overline{Y_{i,j}}C'$. Thus

$$A'\overline{Y_{i,j}} = \begin{pmatrix} 0 & \cdots & A'_{1,j}Y_{i,j} & \cdots & 0 \\ 0 & \cdots & A'_{2,j}Y_{i,j} & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & A'_{\frac{n}{2}-1,j}Y_{i,j} & \cdots & 0 \\ 0 & \cdots & A'_{\frac{n}{2},j}Y_{i,j} & \cdots & 0 \end{pmatrix}$$

and

$$\overline{Y_{i,j}}C' = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ Y_{i,j}C'_{j,1} & Y_{i,j}C'_{j,2} & \cdots & Y_{i,j}C'_{j,\frac{n}{2}} \\ 0 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

where $A'_{k,j} = -{A'_{j,k}}^t$ and $C'_{i,\ell} = -{C'_{\ell,i}}^t$.

where the non zero column of $A'\overline{Y_{i,j}}$ equal the non zero row of $\overline{Y_{i,j}}C'$ we have several possible cases. Notice though, that for any i,j chosen, the 2×2 block from A', and from C' will always be a block that lies on the diagonal. In general we get $A'_{k,j}Y_{i,j} = Y_{i,j}C'_{i,\ell}$. We get $\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{pmatrix} \begin{pmatrix} 0 & c \\ -c & 0 \end{pmatrix}$. Hence $\begin{pmatrix} -ay_2 & ay_1 \\ -ay_1 & -ay_2 \end{pmatrix} = \begin{pmatrix} -cy_2 & cy_1 \\ -cy_1 & -cy_2 \end{pmatrix}$ therefore we must have $a=c$.

In every other location we have every other block equalling zero. Thus $A'_{k,j}Y_{i,j} = 0$. So $\begin{pmatrix} a_1 & a_2 \\ -a_2 & a_1 \end{pmatrix} \begin{pmatrix} y_1 & y_2 \\ -y_2 & y_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. So

$$\begin{pmatrix} a_1y_1 - a_2y_2 & a_1y_2 + a_2y_1 \\ -a_2y_1 - a_1y_2 & -a_2y_2 + a_1y_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

. We get two equations. $a_1y_1 - a_2y_2 = 0$ and $a_1y_2 + a_2y_1 = 0$. This is possible only if $a_1 = a_2 = 0$. We derive a very similar solution with $Y_{i,j}C'_{i,\ell} = 0$.

Thus if we do this for all choices of i and j, then we can see that $A' = C'$, every diagonal block of A' are equal, and any 2×2 block not on the diagonal are zero. Thus $\mathcal{A}'(n) = \mathcal{A}(n)$. Therefore $\mathcal{A}(n)$ is a maximal abelian subalgebra. □

This argument can be expanded to include our original definition of $\mathcal{A}(n)$. Thus we have a family of maximal abelian subalgebras.

For $n = 2\ell + 1$, we have a similar algebra but the matrices in $\mathcal{A}(n)$ are of the form $\begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$

where

$$A = a \begin{pmatrix} \alpha_1 & & & & \\ & \alpha_2 & & & \\ & & \ddots & & \\ & & & \alpha_\ell & \\ & & & & 0 \end{pmatrix}$$

and

$$B = a \begin{pmatrix} \beta_1 & & & & \\ & \beta_2 & & & \\ & & \ddots & & \\ & & & \beta_\ell & \\ & & & & 0 \end{pmatrix}$$

and X is a block matrix which satisfies $\alpha_i(X_{ij}) = (X_{ij})\beta_j$ for every i, j. This definition is very similar to the definition as above with the notable exception that the final row and column of X are all zeros except for $x_{2\ell+1,2\ell+1}$ which is free. Thus the dimension is $2 + \frac{\ell^2}{2}$

10. Conclusion

In this thesis we have given an abstract to the topic of abelian subalgebras of maximal dimension and maximal abelian subalgebras. We also discussed the historical significance to the topic. We then gave preliminaries to the topic of Lie algebra, proved some useful results and gave some major theorems in the field of Lie algebras. Afterwards we constructed the Euclidean Lie algebras $\mathcal{E}(n)$ and saw how $\mathcal{E}(n)$ behaved for low values of n. We gave the abelian subalgebra of maximal dimension, and constructed a unique way of viewing $\mathcal{E}(n)$ as the direct sum of three subalgebras. After we proved that the abelian sublagebra of maximal dimension was unique. Finally we gave an example of a maximal abelian subalgebra that is not the abelian subalgebra of maximal dimension.

It is an open problem how to determine all maximal abelian subalgebras of the Euclidean Lie algebras $\mathcal{E}(n)$.

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