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Contents

Abstract

Many authors have used a time-changed Brownian motion as a model of log-stock returns. Using a Lévy process as a stochastic time change, one obtains well known asset price models such as the variance gamma (VG) and normal inverse Gaussian (NIG) models. Following on the heels of these asset price models, it is natural to extend structural credit models by using a time-changed geometric Brownian motion and other jump-diffusion processes to model the value of a firm. To avoid the difficulties that arise in computing the associated first passage time distribution and in analogy to the time-changed Markov chain models, where the default state is an absorbing state, we propose a specific variation of the first passage time applicable to time-changed Brownian motions, but not to general jump diffusions.

This thesis deals with a time-changed bivariate Brownian motion (TCBBM) to model default in credit risk. In particular, we use a gamma process as a stochastic time change. The time of default is modelled as the first-hitting time of a default state. Analytical expressions of the probability of default for a single firm are obtained. We develop the formulas for the probability of multiple default for the case with two firms as well. The Monte Carlo method is also presented to compute the default probability under the TCBBM model in a general case.

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CHAPTER 1

Introduction

The computation of the probability of default is one of the primary objectives of credit risk analysis, which deals with distributions of financial losses due to unexpected changes in the credit quality of a counterparty in a financial agreement. Over the last three decades, different credit risk models have been set up to quantify this credit risk and to price bonds and credit derivatives. The vast majority of all modern credit risk models is based on either one of the two following principles: the structural approach or the reduced-form approach. The structural approach to credit modelling, beginning with the works of Merton (1974) and Black & Cox (1976), treats debt and equity as contingent claims on the firm's asset value process. While this unification of debt with equity is conceptually satisfying, the approach often leads to inconsistencies with intuition and observation, such as the zero short-spread property and time inconsistency in Merton type models. Furthermore, it leads to technical difficulties when pushed to provide realistic correlations between different firms' defaults and with other market observables. Formulas in structural models tend to be either tractable but inflexible, or flexible but computationally intractable.

Reduced-form modelling, introduced by Jarrow & Turnbull (1995), has been highly successful in providing remedies for these problematic aspects. It treats default as locally unpredictable, with an instantaneous hazard rate, but does away with the connection between default and the firm's asset value process.

Following on the heels of these models, Jarrow, Lando and Turnbull (1998) bridged the gap between reduced form and structural models by proposing a continuous time Markov chain to replace the firm value process as a determinant of credit quality.

The purpose of this thesis is to develop and test some new approaches for determining the probability of default. We have used a stochastic time-changed Brownian motion as the model of log-prices. Using an independent Lévy process as the stochastic time change process, one can recover well-known models such as the variance gamma model of Madan and Seneta (1990), which assumes that the log-price obeys a pure jump Lévy process with stationary increments that follow the gamma distribution. The time of default is the first-hitting time of a default state which is an absorbing state of the asset price process. The analytical formulas of the probability of default for a single firm are obtained, and we develop the formula of the probability of multiple joint default for the case with two firms as well. The Monte Carlo method is developed to compute the default probability under this model in the general case. Moreover, numerical results are presented.

The outline of this thesis is as follow. In Chapter 1, we present preliminaries related to the theory of credit risk modelling and stochastic processes. In Chapter 2, we derive the formula of probability of default for a single firm, when the stochastic time change is a gamma process. In Chapter 3, we develop the formula for the probability of multiple default for the case with two firms, and applications of the model are presented. We also study the calculation of default probabilities using the Monte Carlo method. In Chapter 4, we present numerical results. We conclude the thesis in Chapter 5.

1.1. Credit Risk Modelling

1.1.1. Credit Risk. Before we understand why *Credit Risk* is an important issue in the current environment, we define *a default risk*, as a possibility that a counterparty in a financial contract will not fulfill a contractual commitment to meet her / his obligations in the contract. If this actually happens, we say that the party defaults or that a default event occurs.

Credit Risk is a risk associated with any kind of credit-linked events, such as changes in the credit quality, variations of credit spreads, and the default event.

Credit risk can happen if someone buys a good or service without paying immediately for it. For example, individuals buy a house through borrowing money from banks whereas companies borrow to grow their share value. Therefore, everyone in the world is subject to a credit risk.

Suppose one company has decided to take collaterals such as lands, farms or buildings in a small town. As we know, it takes too much time to liquidate those collaterals in the case of default. Banks will not give credits to those companies which we can predict to be as "unreliable".

Moreover, suppose a big multinational bank wants to sell a European call option both to a small local bank and another big multinational bank. Does it charge them with the same premium? How can we determine a logical premimum according to the company and the current market situation.

Answers to all above questions may be found by using credit risk models, which have developed rapidly over the past few years to become a key component in risk management systems.

1.1.2. Credit Risk Modelling. Credit risk models play an important role in risk management and performance measurement processes. Such models are intended to aid banks and other financial institutions in quantifying, aggregating and managing risk.

There are three main quantitative approaches to analyzing credit risk. In the *structural* approach, we make explicit assumptions about the dynamics of a firm's assets, its capital structure, its debt, and share holders. A firm defaults if its assets are insufficient according to some measure. In this situation a corporate liability can be characterized as an option on the firm's assets. The *reduced form* approach is silent about why a firm defaults. Instead, the dynamics of default are exogenously given through a default rate, intensity or hazard rate. In this approach, prices of credit sensitive securities can be calculated as if they were default free using an interest rate that is a riskfree rate adjusted by the intensity. The *incomplete information* approach combines the structural and reduced form models. While avoiding their difficulties, it picks the best features of both approaches: the economic and intuitive appeal of the structural approach and the tractability and empirical fit of the reduced form approach.

1.1.2.1. *Univariate Structural Default Models.* Structural default models aim to explain the economic cause of credit default of a company. More precisely, default is assumed to be the consequence of insufficient financial strength of a company. Solvency is linked to the ratio of the firm's assets and liabilities via the assumption that default is triggered when the value of the firm falls below a certain threshold. Consequently, the model of the firm-value process implicitly specifies the term structure of default probabilities. Therefore, this process plays the pivotal role in structural default models. Corporate bonds and credit derivatives are then priced based on this implied term structure of default probabilities.

A natural criterion to distinguish structural default models is to classify them according to the underlying firm-value process. This classification is closely related to the historical development, as the model of the firm-value process has been generalized over the years. The first structural default model was published by Black and Scholes (1973), it relies on a geometric Brownian motion for the firm-value process. Originally, this model was designed to describe stock prices rather than the value of a firm. Then, the observation "*It is not generally realized, that corporate liabilities other than warrants may be viewed as options*" transformed their stock price model into the first structural default model. Their idea was worked out in detail by Merton (1974), who slightly changed the underlying stochastic differential equation of the firm-value process to include dividends and interest payments. However, the solution to that equation is still a geometric Brownian motion. Moreover, as in Black and Scholes (1973), default within the original Merton model is only possible at maturity. This shortcoming was corrected by Black and Cox (1976), who criticize the original model as follows: "*Furthermore, it assumes that the fortunes of the firm may cause its value to rise to an arbitrary high level or dwindle to nearly nothing without any sort of reorganization occurring in the firm's financial arrangement. More generally, there may be both lower and upper boundaries at which the firm's securities must take on specific values.*" To correct this unrealistic assumption, they propose to continuously test for default and define the time of default as the first-passage time of the firm-value process below a given barrier. Further generalizations of the model address the economic framework, allowing the valuation of coupon bonds and bonds indenture provisions as in Geske (1977), or include stochastic interest rates as in Longstaff and Schwartz (1995).

Still, all these models suffer from the same defect. In pure diffusion models, the time of default, defined as a first-passage time of the firm-value process, is a predictable stopping time with respect to the filtration generated by the Brownian motion. This property turns out to imply vanishing credit spreads for bonds with short maturities, which contradicts the empirical observation that credit spreads have a positive limit at the short end of the term structure. This problem can be approached from two sides. First of all, it is possible to reduce or blur the filtration available to all investors. Alternatively, one can relax the assumption of a continuous firm-value process. In either case, the aim is to modify the model such that the time of default is no longer announced in advance.

Duffie and Lando (2001)'s model is still based on a geometric Brownian motion, but default probabilities are obtained conditional on noisy accounting data and survivorship of the firm. In contrast to Duffie and Lando, Zhou (2001a) does not change the filtration but suggests modelling the firm-value process as the superposition of a diffusion and a jump component instead, the latter with normally distributed jumps. He also presents a simple Monte Carlo algorithm to evaluate bond prices within his model. Moreover, he shows that the limit of credit spreads as implied by the model is positive.

Finally, Leland (1994), with Toft (1996) proposed a framework in which the default threshold is not exogenously given. Instead, shareholders are free to choose the default threshold such that the value of the firm's equity is maximized. Mathematically, this translates in an optimal stopping problem. Generalizations of this approach have been proposed by Hilberink and Rogers (2002), allowing downward jumps in the firm-value process, and recently by Chen and Kou (2005), Acar (2006) and Dao and Jeanblanc (2006) to jump-diffusion processes with two-sided exponentially distributed jumps.

1.1.2.2. *Univariate Reduced-form Models.* Unlike structural default models, reduced-form models do not intend to explain the default of a company by means of an economic construction. Instead, the time of default is exogenously given and assumed to agree with the jump time of some stochastic process. The distribution of this totally inaccessible random variable depends on its default-intensity process, for which models with different complexity exist. Recent models often allow this default-intensity process to depend on a vector of state variables. Another important issue is the amount of information based on which bond and derivative prices are derived. Typical examples are the filtration generated by the default indicator process, the filtration of the state variables, or some given filtration enlarged by the default indicator. Each investor then calculates default probabilities conditional on the available information.

1.1.2.3. *Incomplete Information Credit Models.* The incomplete information framework provides a common perspective on the structural and reduced form approaches to analyzing credit. This perspective enables us to see models of both types as members of a common family. This family contains previously unrecognized structural/reduced form hybrids, some of which incorporate the best features of both traditional approaches. Incomplete information credit models were introduced by Duffie & Lando (2001), Giesecke (2001) and Jarrow, Protter & Yildirim (2002). A non-technical discussion of incomplete information models is in Goldberg (2004).

1.2. Stochastic Processes in Credit Risk

1.2.1. General Framework. Let us recall the definition and basic properties of Brownian motion. Let $(\Omega, \mathcal{F}, P)$ be a probability space. Suppose that for each $\omega \in \Omega$, there is a continuous function

$W_t = W_t(\omega)$ of $t \geq 0$ that satisfies $W_0 = 0$. Then $(W_t)_{t\geq 0}$ is a standard Brownian motion $(w.r.t.((\mathcal{F}_t)_{t\geq 0}, P))$ if for every m partition ($m \geq 1$): $0 = t_0 < t_1 < \ldots < t_m$ the increments

$$W_{t_1} \equiv W_{t_1} - W_{t_0}, W_{t_2} - W_{t_1}, \ldots, W_{t_m} - W_{t_{m-1}}$$

are independent random variables, and each of these increments is normally distributed with

$$E[W(t_{i+1}) - W(t_i)] = 0 \text{ and } Var[W(t_{i+1}) - W(t_i)] = t_{i+1} - t_i.$$

The model for the evolution of asset prices $(V_t)_{t\geq 0}$ over time is a geometric Brownian motion (GBM) if the following stochastic differential equation is satisfied:

$$\frac{dV_t}{V_t} = (\mu + \sigma^2/2)dt + \sigma dW_t, \quad t > 0, \quad V_0 > 0, \tag{1.1}$$

where $\mu \in \mathbb{R}$ is a drift parameter, $\sigma > 0$ is a volatility parameter. Itô's lemma implies that the strong solution to (1.1) is given by

$$V_t = V_0 e^{\mu t + \sigma W_t}, \quad t \geq 0. \tag{1.2}$$

Under the risk-neutral measure we have that $\mu = r - \sigma^2/2$, where r is the risk-free interest rate.

If the value of the assets of a firm is smaller than the value of its liabilities, then the firm is in default. According to Black and Cox (1976), the default boundary, for which the time dependence takes an exponential form, is given by $C_t = e^{\lambda t}K$, $t \geq 0$. When the asset value V_t is larger than C_t, the firm continues to operate and meets its contractual obligations. However, if $V_t \leq C_t$ for some $t \geq 0$, then the firm immediately defaults on all of its obligations. In Figure 1.1, we see that default occurs at time τ since the asset value is not sufficient to cover all the firm's liabilities.

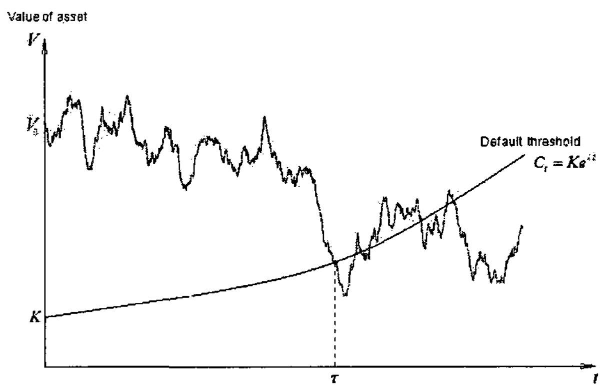

Figure 1.1: A sample path of the firm's asset process with default at time τ

A common assumption is that the value of a financial asset follows a lognormal distribution, i.e, that the logarithm of the asset value is normally distributed. How does one simulate the asset values under such an assumption such that the model predictions are correct? By the above assumption, $X_t := \ln V_t$ is a Gaussian process, which can be simulated exactly. One may use the *Euler* scheme to obtain sample paths of X_t. This approach will be discussed in Chapter 3.

1.2.2. The First Hitting Time Distribution for One Firm to Linear Boundaries. Let $t \to b(t), t \geq 0$, be a smooth function, and $W = \{W_s; s \geq 0\}$ be a standard Brownian motion. We are interested in the distribution of the first hitting time defined by

$$\tau_{x,b} := \inf\{t \geq 0 : x + W_t = b(t)\}, x \in \mathbb{R}, x \neq 0. \tag{1.3}$$

There are many papers devoted to this topic. We refer to one of them, Paavo Salminen (1988), for further references. For the case of linear boundaries, suppose $b(t) = \alpha t$, $\alpha \in \mathbb{R}$. Then we have

$$P(\tau_{x,b} \in (t, t + dt)) = \exp(\alpha x - \frac{1}{2}\alpha^2 t)\frac{|x|}{\sqrt{2\pi t^3}} \exp\left(-\frac{x^2}{2t}\right) dt. \tag{1.4}$$

The first hitting time for $b(t) = \alpha t$ also can be recast as

$$\begin{aligned}\tau_{x,b} = \inf\{t \geq 0 : x + W_t = \alpha t\} &= \inf\{t \geq 0 : x\sigma + \sigma W_t - \alpha\sigma t = 0\} \\ &= \inf\{t \geq 0 : W_t - \alpha t = -x\} \\ &= \inf\{t \geq 0 : x - \alpha t + W_t = 0\}.\end{aligned} \tag{1.5}$$

We hence consider the scaled Brownian motion with drift:

$$X_t = X_0 + \mu t + \sigma W_t. \tag{1.6}$$

X_t presents a log-price of an asset, defined as the logarithm of a geometric Brownian motion process, and $X_0 \neq 0$ is the initial point. Consider the first hitting time

$$\begin{aligned}\tau &= \inf\{t \geq 0 : X_t = 0\} = \inf\{t \geq 0 : X_0 + \mu t + \sigma W_t = 0\} \\ &= \inf\{t \geq 0 : X_0/\sigma - (-\mu/\sigma)t + W_t = 0\}.\end{aligned} \tag{1.7}$$

We have that $\tau = \tau_{x,b}$ when $x = X_0/\sigma$, $\alpha = -\mu/\sigma$. The following formula for the PDF of τ is obtained from (1.4)

$$\begin{aligned} P(\tau \in (t, t+dt)) &= \exp(\alpha X_0/\sigma)\frac{|X_0/\sigma|}{\sqrt{2\pi}}\frac{1}{\sqrt{t^3}}\exp\left(-\frac{1}{2}\frac{\mu^2}{\sigma^2}t - \frac{1}{2}\frac{X_0^2}{\sigma^2 t}\right)dt \\ &= \frac{|X_0|}{\sigma\sqrt{2\pi t^3}}e^{-(X_0+\mu t)^2/2\sigma^2 t}dt \\ &= \frac{e^{-\frac{\mu}{\sigma^2}X_0 - \frac{1}{2}\frac{\mu^2}{\sigma^2}t}}{\sigma\sqrt{2\pi t^3}}|X_0|e^{-\frac{X_0^2}{2\sigma^2 t}}dt. \end{aligned} \tag{1.8}$$

The first hitting time given here will be discussed in detail in Section 4 of this chapter.

1.3. Multivariate Extension

1.3.1. Notation and Approach for the GBM Model. Our model has the following parameters:

σ_i = the volatility of a diffusion component ($i = 1, 2$);

μ_i = the drift of a diffusion component ($i = 1, 2$);

$V_0^{(i)}$ = the initial value of the ith company ($i = 1, 2$);

K_i = the initial value of the default threshold ($i = 1, 2$);

λ_i = the constant of the exponential form $C_t = Ke^{\lambda_i t}$ ($i = 1, 2$);

T = the maturity time.

1.3.2. Multiple Default Probability. Typically, several factors can affect borrower's default probability. In the retail segment, one would consider the salary, occupation, age and other characteristics of the loan applicant; when dealing with corporate clients, one would examine the firm's leverage, profitability, and cash flows.

The first passage time approach allows a greater flexibility in comparison with the Merton model, since in Merton's model (1974), default can only occur at the maturity of a bond, that is restrictive and unrealistic. We extend the original Merton model by accounting for the observed feature that the default may occur not only at the debt's maturity, but also prior to this date, and evaluation using the first passage time model provides more practical results.

Let us introduce the two random variables $D_1(t)$ and $D_2(t)$, called the default indicators that describe the default status of two firms. For each firm we denote:

$$D_i(t) = \begin{cases} 1, & \text{if firm } i \text{ defaults by time } t \text{ ;} \\ 0, & \text{otherwise.} \end{cases}$$

In our model with two firms, we assume the default barriers are given by

$$C_t^{(i)} = K_i e^{\lambda_i t}, \quad t \geq 0, \quad i \in \{1, 2\},$$

where $K_i > 0$ is the initial value of the default threshold, and λ_i is a constant, with $i \in \{1, 2\}$. Let's define the first passage times:

$$\tau_i = \inf \left\{ t \geq 0 : V_t^{(i)} \leq C_t^{(i)} \right\}, \quad i \in \{1, 2\}; \tag{1.9}$$

where $V_t^{(i)}$ is the value of firm i at time t. Then, the default indicator can be represented as follows:

Firm i defaults by time $t \Leftrightarrow D_i(t) = 1 \Leftrightarrow \tau_i \leq t$;

Firm i has no default by time $t \Leftrightarrow D_i(t) = 0 \Leftrightarrow \tau_i > t$.

Based on the above definitions, we have the following formula for the default probability

$$P(D_i(t) = 1) = P(\tau_i \leq t).$$

Assumption 1. Let $V_t^{(1)}$ and $V_t^{(2)}$ denote the asset values of firms 1 and 2 at time $t \geq 0$, respectively. Assume that $\left(V_t^{(1)}\right)_{t\geq 0}$ and $\left(V_t^{(2)}\right)_{t\geq 0}$ satisfy the following vector stochastic differential equation

$$\begin{bmatrix} d\ln(V_t^{(1)}) \\ d\ln(V_t^{(2)}) \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \Omega \begin{bmatrix} dW_t^{(1)} \\ dW_t^{(2)} \end{bmatrix}, \tag{1.10}$$

where μ_1 and μ_2 are constant drift coefficients, $\left(W_t^{(1)}\right)_{t\geq 0}$ and $\left(W_t^{(2)}\right)_{t\geq 0}$ are two independent standard Brownian motions, and Ω is a constant 2×2 matrix such that

$$\Omega \cdot \Omega^T = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \tag{1.11}$$

where $\sigma_1, \sigma_2 > 0$ and $\rho \in [-1, 1]$. At time $t = 0$, we assume $V_0^{(i)} > 0, i \in \{1, 2\}$.

The coefficient $\rho \equiv Corr(d\ln(V_t^{(1)}), d\ln(V_t^{(2)}))$, reflects the correlation between the movements in the asset values of the two firms. This correlation coefficient plays an important role in determining the default correlation between the firms.

Now, we can move to the next step. Assuming the independence of default events, the joint default probability of the two firms is

$$P(D_1(t) = 1 \text{ and } D_2(t) = 1) = P(D_1(t) = 1) \cdot P(D_2(t) = 1). \tag{1.12}$$

When we exam the joint probability, however, it is reasonable to assume that if one entity is in default, the other entity may have a higher likelihood of defaulting. Thus, it is possible that the two entities may have a positive default correlation.

We define the default correlation $Corr(D_1(t), D_2(t))$ as follows:

$$Corr[D_1(t), D_2(t)] \equiv \frac{E[D_1(t) \cdot D_2(t)] - E[D_1(t)] \cdot E[D_2(t)]}{\sqrt{Var[D_1(t)] \cdot Var[D_2(t)]}}. \quad (1.13)$$

Since $D_1(t)$ and $D_2(t)$ are Bernoulli random variables, we have that

$$E[D_i(t)] = P(D_i(t) = 1),$$

$$Var[D_i(t)] = P(D_i(t) = 1) \cdot [1 - P(D_i(t) = 1)].$$

From equation (1.13), we can derive a probability that both firms are in default at time t as

$$P(D_1(t) = 1 \text{ and } D_2(t) = 1)$$

$$= E[D_1(t) \cdot D_2(t)]$$

$$= E[D_1(t)] \cdot E[D_2(t)] + Corr[D_1(t), D_2(t)] \cdot \sqrt{Var[D_1(t)] \cdot Var[D_2(t)]}.$$

The default correlation is also taken in evaluating the probability that either firm defaults:

$$P(D_1(t) = 1 \text{ or } D_2(t) = 1)$$

$$= P(D_1(t) = 1) + P(D_2(t) = 1) - P(D_1(t) = 1 \text{ and } D_2(t) = 1)$$

$$= E[D_1(t)] + E[D_2(t)] - E[D_1(t) \cdot D_2(t)].$$

1.4. Models with a Stochastic Time Change

1.4.1. Time-changed Brownian Motion. Let $(\Omega, \mathcal{F}, P)$ be a probability space that supports a Brownian motion $(W_t)_{t\geq 0}$ and a non-decreasing process $(G_t)_{t\geq 0}$ with $G_0 = 0$ called a stochastic *time-change* process. The measure P may be thought of as either the physical or risk-neutral measure. Let $B_t = x + \mu t + \sigma W_t$ be a scaled Brownian motion starting at x and having constant drift parameter μ. We henceforth restrict our scope by assuming the following.

Assumption 2. B and G are independent stochastic processes under the probability measure P.

This assumption is introduced mostly for simplicity. The more general case where B and G are dependent stochastic processes is also of interest in finance. A *time-changed* Brownian motion (TCBM) is defined to be a process of the form

$$X_t := B_{G_t}, \qquad t \geq 0. \tag{1.14}$$

Identification of the components of such a TCBM leads to two subfiltrations of the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ (which we assume satisfies the "usual conditions"):

$$\mathcal{X}_t = \sigma\{X_s : s \leq t\}, \tag{1.15}$$

$$\mathcal{G}_t = \sigma\{G_s : s \leq t\}. \tag{1.16}$$

We also consider the Brownian filtration $\mathcal{W}_t = \sigma\{W_s : s \leq t\}$.

1.4.2. The First Passage Time for TCBM. In this section, we define two distinct notions of the first passage time for a TCBM starting at a point $x \geq 0$ to hit zero.

Definition. For any TCBM $X_t = B_{G_t}$, $t \geq 0$, we define

1. The *first passage time* $t_X^{(1)}$ *of the first kind* is a $\mathcal{X}$-stopping time defined by

$$t_X^{(1)} = \inf\{t \geq 0 : X_t \leq 0\}. \tag{1.17}$$

The corresponding stopped TCBM process is $X_t^{(1)} = X_{t \wedge t_X^{(1)}}$. Note that in general $X_{t_X^{(1)}} \leq 0$, with strict inequality possible at a time when G makes a jump.

2. The *first passage time* $t_X^{(2)}$ *of second kind* is a $\mathcal{F}$-stopping time defined by

$$t_X^{(2)} = \inf\{t \geq 0 : G_t \geq t_B^{(1)}\}. \tag{1.18}$$

where $t_B^{(1)} = \inf\{t \geq 0 : B_t \leq 0\}$. The corresponding stopped TCBM process is $X_t^{(2)} = B_{G_{t \wedge t_B^{(1)}}}$. Note that $X_{t_X^{(2)}} = 0$.

For ease of notation, we let τ^* represents $t_X^{(2)}$ and τ represent $t_B^{(1)}$.

1.4.3. Notation and Approach for the VG Model. Our model consists of the same parameters as the GBM model: μ_i, σ_i, $V_0^{(i)}$, K_i, λ_i, $i = 1, 2$, and T.

This section extends the GBM process to another process obtained by evaluating the GBM process at a random time given by a gamma process. Each unit of calendar time is viewed as having an economically relevant time length given by an independent random variable which has a gamma density with unit mean. Also, we focus on the case with two dependent firms.

First, we consider the probability of default for the special case ($\lambda_i = \mu_i, i = 1, 2$) and then consider the general case. Based on the result of Harrison (1990) we derive the formulas for the two above cases. Moreover, we obtain the analytical formula for the special case of two firms with the use of modified Bessel functions. Also the Monte Carlo simulation is used to compute the probability of default for the general case of two firms. Specifically, the default indicator can be represented as

$$D_i^*(t) = 1 \Leftrightarrow \text{firm } i \text{ default} \Leftrightarrow \tau_i^* \leq t,$$

where $\tau_i^* = \inf\{t \geq 0 : G_t \geq \tau_i\}$, τ_i is defined by (1.9), $i = 1, 2$. Notice that the probability of default in the VG model refers to the corresponding probability in the GBM model as follows

$$P(D_i^*(t) = 1) = \int_0^\infty P(D_i(s) = 1) \cdot f(s; t) ds, \tag{1.19}$$

where $f(s;t)$ is a gamma density function, which will be defined in Section 2 of Chapter 2.

CHAPTER 2

Evaluating the Default Probability for One Firm

2.1. The GBM Model

Following the result of Harrison (1990), we obtain the default probability of a single firm

$$\begin{aligned} P(D(t)=1) = N\left(-\frac{Z}{\sqrt{t}} - \frac{\mu-\lambda}{\sigma}\sqrt{t}\right) \\ + e^{\frac{2(\lambda-\mu)Z}{\sigma}} N\left(-\frac{Z}{\sqrt{t}} + \frac{\mu-\lambda}{\sigma}\sqrt{t}\right), \end{aligned} \tag{2.1}$$

where $(V_0 \geq K)$

$$Z \equiv \frac{\ln(V_0/K)}{\sigma} \tag{2.2}$$

is the standardized distance of firm to its default point and $N(\cdot)$ denotes the cumulative probability distribution function for a standard normal variable with $N(\cdot)$ defined in equation (2.14)

Also, there exists a simplified version of (2.1) when $\lambda = \mu$, given by

$$P(D(t)=1) = N\left(-\frac{\ln(V_0/K)}{\sigma\sqrt{t}}\right) + \frac{V_0}{K} N\left(-\frac{\ln(V_0/K)}{\sigma\sqrt{t}}\right). \tag{2.3}$$

2.2. The VG Model

2.2.1. Defining the VG Model. Here we consider a time-changed Brownian motion as the model of log- stock returns. The new process, termed the variance gamma (VG) process, is obtained by evaluating

Brownian motion with drift at a random time given by a gamma process.

The first important class of TCBMs arises by taking G to be a *Lévy* time change, in other words, a *Lévy subordinator. Lévy* processes form a general class of continuous time stochastic processes with stationary and independent increments. In addition to their interest in the theory of stochastic processes, they have found important uses in mathematical finance, where they are applied as models for log-stock price processes.

Let us begin by introducing the gamma process—a continuous-time process with stationary, independent gamma increments. Madan and Seneta (1990), Madan, Carr, and Chang (1998) introduced in the context of financial option pricing a continuous-time stochastic process termed *variance gamma* that is a Brownian motion with random time change, where the random time change is a gamma process. The authors argued that the variance gamma model permits more flexibility in modelling skewness and kurtosis relative to Brownian motion. They developed closed-form solutions for European option prices under the VG model and provided empirical evidence that the VG option pricing model gives a better fit to market option prices than the classical Black-Scholes model.

Many papers, for example Geman and Ané (1996), show that Brownian motion time-changed by a gamma process provides a significantly better description of historical asset returns and the risk-neutral return distribution embedded in option prices. In fact, the time-changed process successfully combats the well-known smile effects of Black-Scholes pricing for short maturity $S\&P$ 500 index options. Bakshi and Madan (1999), on the other hand, use this process to learn about the probabilities of large market moves from small ones.

In what follows, we describe the statistical and risk neutral dynamics of the stock price in term of the VG process, and derive closed form expressions for the return density.

Let us define

$$B(t;\mu,\sigma) = \mu t + \sigma W_t,$$

where $(W_t)_{t\geq 0}$ is a standard Brownian motion. The process $B(t;\mu,\sigma)$ is a scaled Brownian motion with drift μ and volatility σ. We define the gamma process $\gamma(t) := \gamma(t;\theta,\nu)$ with mean rate θ and variance rate ν as a process with independent gamma increments over non-overlapping intervals of time. The probability density $f(s;\Delta t)$ of the increment $s = \gamma(t+\Delta t) - \gamma(t)$, $t, \Delta t \geq 0$ over the time interval $(t, t+\Delta t)$ is a gamma probability density function with mean $\theta \Delta t$ and variance $\nu \Delta t$. Specifically,

$$f(s;\Delta t) = \left(\frac{\theta}{\nu}\right)^{\frac{\theta^2 \Delta t}{\nu}} \frac{s^{\frac{\theta^2\Delta t}{\nu}-1}\exp\left(-\frac{\theta}{\nu}s\right)}{\Gamma(\frac{\theta^2\Delta t}{\nu})}, \quad s > 0, \tag{2.4}$$

where $\Gamma(x)$ is the gamma function. The VG process $X(t;\sigma,\nu,\mu)$, is defined in terms of the scaled Brownian motion with drift $B(t;\mu,\sigma)$ and the gamma process $\gamma(t;1,\nu)$ with unit mean rate. We assume that $\gamma(0) = 0$, hence $f(s;t)$ is the transition PDF for $\gamma(t)$. Therefore, we define the VG (gamma time-changed drift BM) process by

$$X(t;\sigma,\nu,\mu) = B(\gamma(t;1,\nu);\mu,\sigma), \quad t \geq 0.$$

The VG process has three parameters: σ—the volatility of the Brownian motion, ν—the variance rate of the gamma time change, and μ—the drift of the Brownian motion.

The transition probability density function of the VG process $X(t)$ is a mixture density

$$\mathsf{p}_X(x, x_0, t) = \int_0^\infty \mathsf{p}_B(x, x_0, s) \cdot f(s;t)ds, \tag{2.5}$$

where $\mathsf{p}_B(x, x_0, s)$ is the transition PDF of the process $B(s;\mu,\sigma)$. Since for any $s > 0$, we have $B(s) \sim N(\mu s, \sigma^2 s)$, p_B is a normal PDF.

The new specification for the statistical stock price dynamics is obtained by replacing the role of Brownian motion in the original Black-Scholes geometric Brownian motion model by the VG process.

Under the risk-neutral measure, money market account discounted stock prices are martingales and it follows that the mean rate of return on the stock under this probability measure is the continuously compounded interest rate r. Let the risk neutral process be given by

$$V_t = V_0 \exp(rt + X(t;\sigma_{RN},\nu_{RN},\mu_{RN}) + \omega_{RN}t), \tag{2.6}$$

where the subscript RN on the VG parameters indicates that these are the risk neutral parameters, and

$$\omega_{RN} = \frac{1}{\nu_{RN}} \ln(1 - \mu_{RN}\nu_{RN} - \sigma^2\nu_{RN}/2). \tag{2.7}$$

The density of the log-stock price over an interval of length t is, conditional on the realization of the gamma time change, a normal density function. The unconditional density is obtained by integrating out the gamma variate and the result is expressed in terms of the modified Bessel functions of the second kind.

2.2.2. The VG Model and First Passage Time. Let the firm value process $(V_t)_{t\geq 0}$ follows the VG model as given by (2.6). The default time is $\tau_d = \inf\{t : V_t \leq C_t\}$, where $V_t = V_0 e^{(r+\omega)t + X(t;\sigma,\nu,\mu)}$ and $C_t = Ke^{\lambda t}$. The default event is represented as follows:

$$V_t \leq C_t \Leftrightarrow \ln(V_t) \leq \ln(C_t) \Leftrightarrow \ln(V_0/K) + (r+\omega-\lambda)t + X(t;\sigma,\nu,\mu) \leq 0. \tag{2.8}$$

Assumption 3. For all VG models considered here we assume $r+\omega-\lambda = 0$, where ω is given in (2.7).

Under this assumption the default time τ_d is just the first passage time $t_X^{(1)}$ of the first kind for the subordinate process

$$X_t = B_{G_t} = \ln(V_0/K) + \mu G_t + \sigma W_{G_t}, \tag{2.9}$$

where $G_t = \gamma(t; 1, \nu)$ is a gamma process and $B_t = \ln(V_0/K) + \mu t + \sigma W_t$ is a scaled Brownian motion with drift.

In contrast to traditional Brownian motion, the VG process is a pure jump process with an infinite arrival rate of jumps. Therefore, the paths of the VG process are discontinuous functions of time t. It is reasonable to use the first passage time of the second kind for modelling the time of default. Normally, the usual structural approach for credit risk is based on the first passage time of the first kind, but it leads to technical difficulties. General properties have been presented via fluctuation methods (Bingham, 1975) and Wiener-Hopf factorization (Bertoin, 1996). For this reason, we focus our efforts on $t_X^{(2)}$.

Remark 1. $t_X^{(2)}$ can be viewed as an approximation of the usual first passage time $t_X^{(1)}$ with $t_X^{(1)} \geq t_X^{(2)}$. When the stochastic time change G is a process with continuous paths, the two definitions coincide.

Remark 2. In general, $t_X^{(2)}$ is not an $\mathcal{X}$-stopping time. For more details, see Geman (2001) who discuss the problem of inferring the time change G from observing the history of X.

As stated earlier in (1.17) and (1.18), we will use τ^* to represent $t_X^{(2)}$, and τ for $t_B^{(1)}$ for the ease of notation. In our model for the case with one default probability, we define $\tau^* = \inf\{t|G_t \geq \tau\}$, where $\tau = \inf\{t|B_t \leq 0\} = \inf\{t|V_t \leq C_t\}$. Here, $B_t = \ln(V_0/K) + \mu t + \sigma W_t$, $t \geq 0$, $(V_t)_{t\geq 0}$ defined by equation (1.2) is the asset value process of one firm and $C_t = K$, $t > 0$, where K is the initial value of the default threshold (*i.e.*, $\lambda = 0$). Then the default indicator $D^*(t)$ can be defined

$$D^*(t) = 1 \Leftrightarrow \text{firm defaults} \Leftrightarrow \tau^* \leq t.$$

Thus, we have the following expression for the probability of default in our VG model:

$$P(D^*(t) = 1) = P(\tau^* \leq t) = P(\tau \leq G_t) = \int_0^\infty P(\tau \leq s) \cdot f(s;t)ds, \tag{2.10}$$

where t is the maturity time, and f is the gamma density in (2.4). In this formula, the time of default is a first passage time of the second kind.

2.3. Evaluation of the Default Probability for the VG Model

Before we estimate the default probability using equation (2.10), we have to mention another formula which can also be used to evaluate the default probability for one firm. That is,

$$P(\tau^* < t) = \int_0^t p_{\tau^*}(s)ds, \tag{2.11}$$

where p_{τ^*} is the PDF for τ^* given by

$$p_{\tau^*}(t) = \frac{\partial}{\partial t}P(\tau \leq G_t) = \int_0^\infty P(\tau \leq s)\frac{\partial}{\partial t}f(s;t)ds. \tag{2.12}$$

In this case, we prefer to use the first method to compute the default probability than the latter, since it is not easy to evaluate the integral of the default probability using equation (2.11).

Let us recall the error function in advance. The error function (also called the Gauss error function) is a special function (non elementary) which occurs in probability, statistics, materials science, and partial differential equations. It is defined as

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt. \tag{2.13}$$

Also, the cumulative probability distribution function (CDF) of a standard normal variable is expressed as

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-t^2/2} dt. \tag{2.14}$$

Notice that

$$N(x) = 1 - N(-x). \tag{2.15}$$

From (2.13) and (2.14) we have

$$N(x) = \frac{1}{2} + \frac{\mathrm{erf}(\frac{x}{\sqrt{2}})}{2}. \tag{2.16}$$

Now, we can get useful results using the error function instead of the normal CDF and the gamma density function. Using the formulae (2.3) and (2.10), we obtain the computational formula for the special case when $\mu = 0$:

$$\begin{aligned} P(\tau^* \leq T) = \int_0^{\infty} & \left[N\left(-\frac{\ln(V_0/K)}{\sigma\sqrt{s}}\right) + \frac{V_0}{K} N\left(-\frac{\ln(V_0/K)}{\sigma\sqrt{s}}\right)\right] \\ & \cdot \frac{1}{\nu^{\frac{T}{\nu}}\Gamma(\frac{T}{\nu})} s^{(\frac{T}{\nu}-1)} e^{-\frac{s}{\nu}} ds. \end{aligned} \tag{2.17}$$

To solve the above the (2.17), we simplify the following two terms. One is written as

$$\begin{aligned} & \int_0^{\infty} N\left(-\frac{\ln(V_0/K)}{\sigma\sqrt{s}}\right) \cdot \frac{1}{\nu^{\frac{T}{\nu}}\Gamma(\frac{T}{\nu})} s^{(\frac{T}{\nu}-1)} e^{-\frac{s}{\nu}} ds \\ & = \int_0^{\infty} \left[\frac{1}{2} + \frac{1}{2}\mathrm{erf}\left(-\frac{\ln(V_0/K)}{\sigma\sqrt{s}\sqrt{2}}\right)\right] \cdot \frac{1}{\nu^{\frac{T}{\nu}}\Gamma(\frac{T}{\nu})} s^{(\frac{T}{\nu}-1)} e^{-\frac{s}{\nu}} ds \\ & = \frac{1}{2}\left(1 + \int_0^{\infty} \mathrm{erf}\left(-\frac{\ln(V_0/K)}{\sigma\sqrt{s}\sqrt{2}}\right) \cdot \frac{1}{\nu^{\frac{T}{\nu}}\Gamma(\frac{T}{\nu})} s^{(\frac{T}{\nu}-1)} e^{-\frac{s}{\nu}} ds\right). \end{aligned}$$

Let $e^{-\frac{s}{\nu}} = y$, so we have $s = -\nu \ln y$, hence $ds = -\nu \cdot \frac{1}{y} dy$. Also we have $s = 0 \Rightarrow y = 1$ and $s = \infty \Rightarrow y = 0$. So, by completing the integration and simplifying, we obtain the following result for the first term

$$\frac{1}{2}\left(1 + \int_0^1 \operatorname{erf}(-\frac{\ln(\frac{V_0}{K})}{\sigma\sqrt{-2\nu \ln y}}) \cdot \frac{(-\ln y)^{(\frac{T}{\nu}-1)}}{\Gamma(\frac{T}{\nu})} dy\right). \tag{2.18}$$

Similarly, the second term can be rewritten as follows:

$$\frac{V_0}{2K}\left(1 + \int_0^1 \operatorname{erf}(-\frac{\ln(\frac{V_0}{K})}{\sigma\sqrt{-2\nu \ln y}}) \cdot \frac{(-\ln y)^{(\frac{T}{\nu}-1)}}{\Gamma(\frac{T}{\nu})} dy\right). \tag{2.19}$$

The final result for the special case is obtained by combining equations (2.18) and (2.19).

Next, we present the formula for the general case ($\mu \neq 0$). Similarly, using (2.1) and (2.10), the default probability can be calculated as follows:

$$P(\tau^* \leq T) = \int_0^\infty P(\tau \leq s) \cdot \frac{1}{\nu^{\frac{T}{\nu}} \Gamma(\frac{T}{\nu})} s^{(\frac{T}{\nu}-1)} e^{-\frac{s}{\nu}} ds. \tag{2.20}$$

Following the formula (2.1), we need to simplify the following two parts. One is written as

$$\begin{aligned}
&\int_0^\infty N\left(-\frac{Z}{\sqrt{t}} - \frac{\mu}{\sigma}\sqrt{t}\right) \cdot \frac{1}{\nu^{\frac{T}{\nu}} \Gamma(\frac{T}{\nu})} s^{(\frac{T}{\nu}-1)} e^{-\frac{s}{\nu}} ds \\
&= \int_0^\infty [\frac{1}{2} + \frac{1}{2}\operatorname{erf}(-\frac{\ln(\frac{V_0}{K})}{\sigma\sqrt{s}\sqrt{2}} - \frac{\mu}{\sigma\sqrt{2}}\sqrt{s})] \cdot \frac{1}{\nu^{\frac{T}{\nu}} \Gamma(\frac{T}{\nu})} s^{(\frac{T}{\nu}-1)} e^{-\frac{s}{\nu}} ds \\
&= \int_0^\infty \frac{1}{2}[1 + \operatorname{erf}(-\frac{\ln(\frac{V_0}{K})}{\sigma\sqrt{s}\sqrt{2}} - \frac{\mu}{\sigma\sqrt{2}}\sqrt{s})] \cdot \frac{1}{\nu^{\frac{T}{\nu}} \Gamma(\frac{T}{\nu})} s^{(\frac{T}{\nu}-1)} e^{-\frac{s}{\nu}} ds.
\end{aligned}$$

Let $e^{-\frac{s}{\nu}} = y$, then $s = -\nu \ln y \Rightarrow ds = -\nu \cdot \frac{1}{y} dy$. That is, $s = 0 \Rightarrow y = 1$ and $s = \infty \Rightarrow y = 0$.

Thus, the following result is obtained for the first part

$$\frac{1}{2}\int_0^1 [1 + \mathrm{erf}(-\frac{\ln(\frac{V_0}{K})}{\sigma_i\sqrt{-2\nu \ln y}} - \frac{\mu}{\sigma\sqrt{2}}\sqrt{-\nu \ln y})] \cdot \frac{(-\ln y)^{\frac{T}{\nu}-1}}{\Gamma(\frac{T}{\nu})} dy. \tag{2.21}$$

The other part is expressed as follows:

$$\int_0^\infty e^{\frac{2(-\mu)Z}{\sigma}} N\left(-\frac{Z}{\sqrt{s}} + \frac{\mu}{\sigma}\sqrt{s}\right) \cdot \frac{1}{\nu^{\frac{T}{\nu}}\Gamma(\frac{T}{\nu})} s^{(\frac{T}{\nu}-1)} e^{-\frac{s}{\nu}} ds$$

$$= \frac{1}{2} e^{\frac{2(-\mu)\ln(\frac{V_0}{K})}{\sigma^2}} \int_0^1 [1 + \mathrm{erf}(-\frac{\ln(\frac{V_0}{K})}{\sigma\sqrt{-2\nu \ln y}} + \frac{\mu}{\sigma\sqrt{2}}\sqrt{-\nu \ln y})] \tag{2.22}$$

$$\cdot \frac{(-\ln y)^{(\frac{T}{\nu}-1)}}{\Gamma(\frac{T}{\nu})} dy.$$

The final result for the general case is given by combining equations (2.21) and (2.22).

CHAPTER 3

Evaluating the Default Probability for Two Firms

So far, we were primarily concerned with the default probability for a single firm. More generally, we consider the mulitple default probability, that is, the probability that at least one default has occurred by time t. In this chapter, we generalize the VG model presented in section 2.2 to the case with two firms. This model is able to explain the multiple default probabilities of several firms and their default correlation. Also, our intention is to present analytical formulas, which allow us to compute the multiple default probability.

3.1. The Bivariate GBM Model

Based on the result of Harrison (1990) and Rebholz (1994), we derive the formulas of probability of default for the special case when $\lambda_i = \mu_i, i = 1, 2$, and the general case, when $\lambda_i \neq \mu_i$ for at least one $i \in \{1, 2\}$.

3.1.1. The Special Case. Assume that $\lambda_i = \mu_i$, for $i = 1,2$. We have the following formula based on the result of Rebholz (1994):

$$
\begin{aligned}
P(D_1(t) &= 1 \text{ or } D_2(t) = 1) \\
&= 1 - \frac{2r_0}{\sqrt{2\pi t}} \cdot e^{-\frac{r_0^2}{4t}} \cdot \sum_{n=1,3,\ldots} \frac{1}{n} \cdot \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \\
&\cdot \left[I_{\frac{1}{2}((\frac{n\pi}{\alpha}+1)}\left(\frac{r_0^2}{4t}\right) + I_{\frac{1}{2}((\frac{n\pi}{\alpha}-1)}\left(\frac{r_0^2}{4t}\right) \right],
\end{aligned} \tag{3.1}
$$

where $I_\nu(z)$ is the modified Bessel function I of the first kind with order ν and

$$
\begin{aligned}
\alpha &= \begin{cases} \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \text{if } \rho < 0, \\ \pi + \tan^{-1}\left(-\frac{\sqrt{1-\rho^2}}{\rho}\right) & \text{otherwise;} \end{cases} \\
\theta_0 &= \begin{cases} \tan^{-1}\left(\frac{Z_2\sqrt{1-\rho^2}}{Z_1-\rho Z_2}\right) & \text{if } (\cdot) > 0, \\ \pi + \tan^{-1}\left(\frac{Z_2\sqrt{1-\rho^2}}{Z_1-\rho Z_2}\right) & \text{otherwise;} \end{cases} \\
r_0 &= Z_2/\sin(\theta_0).
\end{aligned} \tag{3.2}
$$

Here we denote

$$
Z_i = \frac{\ln(\frac{V_0^{(i)}}{K_i})}{\sigma_i}, \quad i = 1,2.
$$

3.1.2. The General Case. Let λ_i and μ_i, $i = 1,2$, be any given constants. We have

$$
\begin{aligned}
P(D_1(t) &= 1 \text{ or } D_2(t) = 1) \\
&= 1 - \frac{2}{\alpha t} e^{a_1 x_1 + a_2 x_2 + a_t t} \cdot \sum_{n=1}^{\infty} \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \cdot \\
& e^{-\frac{r_0^2}{2t}} \int_0^\alpha \sin\left(\frac{n\pi\theta}{\alpha}\right) g_n(\theta)\, d\theta,
\end{aligned} \tag{3.3}
$$

where θ_0, r_0, and α are defined as in (3.2) and

$$g_n(\theta) = \int_0^\infty r \cdot e^{-\frac{r^2}{2t}} \cdot e^{d_1 r \sin(\theta-\alpha) - d_2 r \cos(\theta-\alpha)} \cdot I_{\frac{n\pi}{\alpha}}\left(\frac{r r_0}{t}\right) dr,$$

$$a_1 = \frac{(\lambda_1 - \mu_1)\sigma_2 - (\lambda_2 - \mu_2)\rho\sigma_1}{(1-\rho^2)\sigma_1^2\sigma_2},$$

$$a_2 = \frac{(\lambda_2 - \mu_2)\sigma_1 - (\lambda_1 - \mu_1)\rho\sigma_2}{(1-\rho^2)\sigma_2^2\sigma_1},$$

$$a_t = \frac{a_1^2\sigma_1^2}{2} + \rho a_1 a_2 \sigma_1 \sigma_2 + \frac{a_2^2\sigma_2^2}{2} - a_1(\lambda_1 - \mu_1) - a_2(\lambda_2 - \mu_2),$$

$$d_1 = a_1\sigma_1 + \rho a_2 \sigma_2,$$

$$d_2 = a_2\sigma_2\sqrt{1-\rho^2},$$

$$x_1 = b_1 - \sigma_1[(\sqrt{1-\rho^2} r \cos(\theta) + \rho r \sin(\theta)],$$

$$x_2 = b_2 - \sigma_2 r \sin(\theta),$$

$$b_i = -\ln[K_i/V_0^{(i)}].$$

Since equation (3.3) involves a modified Bessel function and a double integral, it makes the calculation of the probability difficult. To overcome this difficulty, we use the Monte Carlo method to compute the default probability.

3.2. The Bivariate VG model

A bivariate VG model is obtained by evaluating the bivariate GBM process at a random time given by a gamma process. Each unit of calendar time is viewed as having an economically relevant time length given by an independent random variable that has a gamma density with unit mean and positive variance.

3.2.1. The Special Case. When $\mu_i = 0, i \in \{1,2\}$, based on the formulas (2.4) and (2.10), the probability of default is given as

$$P(D_1^*(t) = 1 \text{ or } D_2^*(t) = 1)$$
$$= \int_0^\infty P(D_1(s) = 1 \text{ or } D_2(s) = 1) f(s;t) ds$$
$$= \int_0^\infty \left\{ 1 - \frac{2r_0}{\sqrt{2\pi s}} \cdot e^{-\frac{r_0^2}{4s}} \cdot \sum_{n=1,3,\ldots} \frac{1}{n} \cdot \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \right.$$
$$\left. \cdot \left[I_{\frac{1}{2}((\frac{n\pi}{\alpha}+1)}\left(\frac{r_0^2}{4s}\right) + I_{\frac{1}{2}((\frac{n\pi}{\alpha}-1)}\left(\frac{r_0^2}{4s}\right) \right] \right\} \cdot \left(\frac{\theta}{\nu}\right)^{\frac{\theta^2 t}{\nu}} \frac{s^{\frac{\theta^2 t}{\nu}-1} \exp\left(-\frac{\theta}{\nu}s\right)}{\Gamma(\frac{\theta^2 t}{\nu})} ds. \tag{3.4}$$

There are two approaches to the evaluation of the probability of default. One approach is to change the order of summation and integration in equation (3.4) and calculate the truncated series. The other approach, presented in the Appendix, is to use the series representation of the function I and to simply the integrals.

3.2.2. The General Case. Here we consider the general case that allows us $\mu_i \neq 0,\ i \in \{1,2\}$. We have the following formula for the probability of default of two firms.

$$P(D_1^*(t) = 1 \text{ or } D_2^*(t) = 1)$$
$$= \int_0^\infty P(D_1(s) = 1 \text{ or } D_2(s) = 1) f(s;t) ds$$
$$= \int_0^\infty \left\{ 1 - \frac{2}{\alpha s} e^{a_1 x_1 + a_2 x_2 + a_t s} \cdot \sum_{n=1}^\infty \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \right. \tag{3.5}$$
$$\left. \cdot e^{-\frac{r_0^2}{2s}} \int_0^\alpha \sin\left(\frac{n\pi\theta}{\alpha}\right) g_n(\theta)\, d\theta \right\} \cdot \left(\frac{\theta}{\nu}\right)^{\frac{\theta^2 t}{\nu}} \frac{s^{\frac{\theta^2 t}{\nu}-1} \exp\left(-\frac{\theta}{\nu}s\right)}{\Gamma(\frac{\theta^2 t}{\nu})} ds,$$

where all parameters are obtained by applying formulas of Subsection 3.1.2 with $\lambda_1 = \lambda_2 = 0$.

Computation of the multiple default probability given by equation (3.5) is technically difficult. How to deal with this problem? The Monte Carlo method can be used to estimate the probability of default for two firms instead of the direct application of equation (3.5).

3.3. The Monte Carlo Method

The Monte Carlo method is proposed in order to efficiently compute the multiple default probability. As we see, calculations of the default probability for two firms involve the evaluation of high-dimensional definite integrals, so the Monte Carlo method is now used here.

We represent the default event as an indicator function. As we discuss as before, D_i $(i = 1, 2)$, takes the value 1 if the obligor defaults and 0 otherwise. Our goal now is to show how the indicator function can be computed to estimate default probabilities.

Let A be an event of the sample space Ω, and $\omega \in \Omega$ be an outcome. The indicator $\mathbb{1}_A$ of the event is a random variable defined by

$$\mathbb{1}_A(\omega) = \begin{cases} 1, & \omega \in A; \\ 0, & \text{otherwise}, \end{cases}$$

that is, $\mathbb{1}_A = \begin{cases} 1, & \text{with probability } P(A); \\ 0, & \text{with probability } 1 - P(A). \end{cases}$ Therefore,

$$E[\mathbb{1}_A] = 1 \cdot P(A) + 0 \cdot (1 - P(A)) = P(A).$$

If we obtain N independent realizations $\omega_1, \omega_2, \ldots, \omega_N$, by the law of large numbers, we have

$$P(A) \simeq \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}_A(\omega_i).$$

3.3.1. The Brownian Bridge. A standardized Brownian bridge is a continuous-time stochastic process, whose probability distribution is the *conditional* probability distribution of a Wiener process W_t given that $W_0 = W_T = 0$ for some $T > 0$.

The Brownian bridge can also be used in estimating the first passage time. Suppose we have a single Brownian motion (with / without drift) $(B_t)_t \geq 0$ and a lower barrier b. Let the process be sampled at a finite number of times $0 < t_1 < t_2 < \ldots < t_n$ with time step $\Delta t = t_k - t_{k-1}$. One may estimate the first passage time $\tau = \inf\{t \geq 0 : B_t \leq b\}$ by using the approximation $\tau^{\Delta t} := \min\{t_i \geq 0 : B_{t_i} \leq b\}$. This estimate of the first passage time is biased, since during any time interval $[t, t+\Delta t]$ the process may cross the barrier and return back. To catch these events, we use the construction of the Brownian bridge. We calculate the probability *prob* that the process crosses the barrier conditional on the event that it doesn't cross the barrier at t and $t + \Delta t$. Then we simulate the event of crossing the barrier by sampling random variable α from $U(0, 1)$ and checking if $\alpha <$ *prob*. The same method can be used in several dimensions if the multivariate Brownian bridge distribution is available. In practice, we may approximate such a probability distribution by a product of one dimensional distributions assuming uncorrelation of the coordinates of the Brownian bridge process on a short time interval.

The Distribution of the Minimum of the Brownian Bridge. L.Beghin and E.Orsingher (1999) present some extensions of the distributions of the maximum of the Brownian bridge in $[0, t]$ when the conditioning event is placed at a future time or at an intermediate time. The standard distributions for Brownian motion and Brownian bridge are obtained as limiting cases. These results permit us to derive also the distribution of the first-passage time of the Brownian bridge. Let W be the standard Brownian motion, then for any positive β, $\eta \in R$, the following distribution of the maximum of the one-dimensional Brownian bridge holds:

$$P\left\{\max_{0\leq s\leq t} W_s \geq \beta | W_t = \eta\right\} = \begin{cases} \exp\left\{-\frac{2\beta(\beta-\eta)}{t}\right\}, & \beta > \eta, \\ 1, & \beta \leq \eta. \end{cases} \tag{3.6}$$

In our model, we need to use a distribution of the minimum of the bridge process for a Brownian motion with drift. We obtain it through the following steps. Define $B_t = \mu t + \sigma W_t$, $t \geq 0$, $\sigma > 0$. Let $0 \leq t_1 < t_2$, and $b < a_i$ $(i = 1, 2)$. Consider the distribution of B_s, $t_1 \leq s \leq t_2$, conditional on $B_{t_1} = a_1$ and $B_{t_2} = a_2$. Of interest is the probability

$$P\left(\min_{t_1<s<t_2} B_s < b \middle| B_{t_1} = a_1, B_{t_2} = a_2\right).$$

For $s \in [t_1, t_2]$, we have

$$B_s = B_{t_1} + \mu(s - t_1) + \sigma(W_s - W_{t_1}),$$

That is we can write $B_s = B_{t_1} + \widetilde{B}_\tau$, $\tau = s - t_1$, where we define $\widetilde{B}_\tau = \mu\tau + \sigma\widetilde{W}_\tau$, $\tau \geq 0$, with $\widetilde{W}_\tau$ a standard Brownian motion independent of the process $(W_t)_{0\leq t\leq t_1}$. Next, the distribution of the minimum of the Brownian bridge is obtained by shifting in time and in space:

$$\begin{aligned}
&P\left(\min_{0<\tau<t_2-t_1} \widetilde{B}_\tau + a_1 \leq b \,\middle|\, \widetilde{B}_0 = 0, \widetilde{B}_{t_2-t_1} = a_2 - a_1\right)\\
&= P\left(\max_{0<\tau<t_2-t_1} (-\widetilde{B}_\tau) - a_1 \geq -b \,\middle|\, -\widetilde{B}_0 = 0, -\widetilde{B}_{t_2-t_1} = a_1 - a_2\right)\\
&= P\left(\max_{0<\tau<t_2-t_1} \widehat{B}_\tau \geq a_1 - b \,\middle|\, \widehat{B}_0 = 0, \widehat{B}_{t_2-t_1} = a_1 - a_2\right) \qquad (3.7)\\
&= P\left(\max_{0<\tau<t_2-t_1} \frac{1}{\sigma}\widehat{B}_\tau \geq \frac{a_1 - b}{\sigma} \,\middle|\, \tfrac{1}{\sigma}\widehat{B}_0 = 0, \tfrac{1}{\sigma}\widehat{B}_{t_2-t_1} = \tfrac{a_1-a_2}{\sigma}\right)\\
&= P\left(\max_{0<\tau<t_2-t_1} \widehat{W}_\tau \geq \frac{a_1 - b}{\sigma} \,\middle|\, \widehat{W}_0 = 0, \widehat{W}_{t_2-t_1} = \tfrac{a_1-a_2}{\sigma}\right),
\end{aligned}$$

Since $a_2 > b$, then $-b > -a_2$, and $a_1 - b > a_1 - a_2$, hence $\frac{a_1-b}{\sigma} > \frac{a_1-a_2}{\sigma}$. Using the formula (3.6), we continue formula (3.7). Here we denote $\widehat{B}_t := -\widetilde{B}_t$, $\widehat{W}_t := -\widetilde{W}_t$. We have also used the fact that the distribution of the max / min of the Brownian bridge does not depend on the drift. As a result, we have

$$\begin{aligned}
P\left(\max_{0<\tau<t_2-t_1} \widehat{W}_\tau \geq \frac{a_1 - b}{\sigma} \,\middle|\, \widetilde{W}_{t_2-t_1} = \tfrac{a_1-a_2}{\sigma}\right) &= e^{-\frac{2(\frac{a_1-b}{\sigma})(\frac{a_1-b}{\sigma} - \frac{a_1-a_2}{\sigma})}{t_2-t_1}}\\
&= e^{-\frac{2(\frac{a_1-b}{\sigma})(\frac{a_2-b}{\sigma})}{t_2-t_1}}\\
&= e^{-\frac{2(a_1-b)(a_2-b)}{\sigma^2(t_2-t_1)}}.
\end{aligned} \qquad (3.8)$$

using (3.6) for $\beta > \eta$.

3.3.2. The Control Variate Method. The method of control variates is among the most effective and broadly applicable techniques for improving the efficiency of the Monte Carlo integration method. It exploits information about the errors in estimates of known quantities to reduce the error in an estimate of an unknown quantity.

To describe the method, we let $Y_1, \ldots, Y_n$ be outputs from n replications of a MC simulation. For example, Y_i could be the discounted payoff of a derivative security on the ith simulated path. Suppose that all Y_i are independent and identically distributed and that our objective is to estimate $\mathsf{E}[Y]$. Since the usual estimator is the sample mean $\bar{Y} = (Y_1 + \ldots + Y_n)/n$. This estimator is unbiased and converges to $\mathsf{E}[Y]$ with probability 1 as $n \to \infty$.

Suppose, now, that on each replication we calculate another output X_i along with Y_i. Suppose that the pairs (X_i, Y_i), $i = 1, \ldots, n$, are i.i.d. and that the expectation $\mathsf{E}[X]$ of each X_i is known. We use (X, Y) to denote a generic pair of random variables with the same distribution as each $(X_i, , Y_i)$. Then for any fixed b we can calculate

$$Y_i(b) = Y_i - b(X_i - \mathsf{E}[X])$$

from the ith replication and then compute the sample mean

$$\bar{Y}(b) = \bar{Y} - b(\bar{X} - \mathsf{E}[X]) = \frac{1}{n}\sum_{i=1}^{n}(Y_i - b(X_i - \mathsf{E}[X])). \qquad (3.9)$$

This is a control variate estimator. The observed error $\bar{X} - \mathsf{E}[X]$ serves as a control in estimating $\mathsf{E}[\mathrm{Y}]$.

As an estimator of E[Y], the control variate estimator (3.9) is unbiased because

$$\mathsf{E}[\bar{Y}(b)] = \mathsf{E}[\bar{Y} - b(\bar{X} - \mathsf{E}[X])] = \mathsf{E}[\bar{Y}] = \mathsf{E}[Y], \tag{3.10}$$

and it is consistent because, with probability 1,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} Y_i(b) &= \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n}(Y_i - b(X_i - \mathsf{E}[X])) \\ &= \mathsf{E}[Y - b(X - \mathsf{E}[X])] \\ &= \mathsf{E}[Y].\end{aligned}$$

Each $Y_i(b)$ has variance

$$\begin{aligned}Var[Y_i(b)] &= Var[Y_i - b(X_i - \mathsf{E}[X])] \\ &= \sigma_Y^2 - 2b\sigma_X\sigma_Y\rho_{XY} + b^2\sigma_X^2 \equiv \sigma^2(b),\end{aligned} \tag{3.11}$$

where $\sigma_X^2 = Var[X], \sigma_Y^2 = Var[Y]$, and ρ_{XY} is the correlation between X and Y. The control variate estimator $\bar{Y}(b)$ has variance $\sigma^2(b)/n$ and the ordinary sample mean $\bar{Y}$ (which corresponds to $b = 0$) has variance σ_Y^2/n. Hence, the control variate estimator has smaller variance than the standard estimator if $b^2\sigma_X < 2b\sigma_Y\rho_{XY}$.

The optimal coefficient b^* that minimizes the variance in (3.11) is given by

$$b^* = \frac{\sigma_Y}{\sigma_X}\rho_{XY} = \frac{Cov[X,Y]}{Var[X]}. \tag{3.12}$$

Substituting this value in (3.11) and simplifying, we find that the ratio of the variances of the optimally controlled estimator and of the uncontrolled estimator is

$$\frac{Var[\bar{Y} - b^*(\bar{X} - \mathsf{E}[X])]}{Var[\bar{Y}]} = 1 - \rho_{XY}^2. \tag{3.13}$$

3.3.3. Computation of the Default Probability and Algorithm. Let us describe how to compute the default probability using the Monte Carlo method. First, we focus on the univariate case with one firm. In Section 1.2, we already have defined the firm value process $V_t = V_0 e^{\mu t + \sigma W_t}$ and the default event given by $V_t \leq Ke^{\lambda t}$. Also we have defined $X_t = \ln(V_t) = X_0 + \mu t + \sigma W_t$, where $X_0 := \ln(V_0)$, thus we have default if

$$X_t \leq \ln(K) + \lambda t. \tag{3.14}$$

Let's introduce a new process Y_t with drift coefficient $\mu - \lambda$ given by

$$Y_t := \ln(V_0/K) + (\mu - \lambda)t + \sigma W_t. \tag{3.15}$$

Let $0 = t_0 < t_1 < t_2 < \ldots < t_n$ be a time partition. Using the Euler scheme, we obtain

$$Y_{t_{i+1}} = Y_{t_i} + (\mu - \lambda)\Delta t_i + \sigma Z_i \sqrt{\Delta t_i}, \quad i \geq 0, \quad Y_0 = \ln(V_0/K), \tag{3.16}$$

where $(Z_i)_{i\geq 0}$ are *i.i.d* random variables generated from the standard normal distribution, and $\Delta t_i = t_{i+1} - t_i$ are time increments.

The first hitting time is given by the following formula for this new process Y_t:

$$\tau = \inf\{t \geq 0 : V_t \leq Ke^{\lambda t}\} = \inf\{t \geq 0 : Y_t \leq 0\}. \tag{3.17}$$

Denote $h = \max_i\{t_{i+1} - t_i\}$, $i = 0, \ldots, N-1$. The first hitting time τ in formula (3.17) is approximated by τ^h, which is defined by

$$\tau^h = \min_i\{t_i \geq 0 : Y_{t_i} \leq 0\}, \qquad i = 0, \ldots, N.$$

To estimate the default probability for either one firm or several firms, we sample M paths, $k = 1, \ldots, M$, and then calculate

$$P(D(t) = 1) = P(\tau \leq t) \approx P(\tau^h \leq t) \approx \frac{1}{M}\sum_{k=1}^{M} \mathbb{1}_{\{\tau^h \leq t\}}.$$

Now we consider the two firm case. The asset values of firms still follow a geometric Brownian motion, but the asset value processes are coupled. Thus we have two equations to simulate two new firm value processes $Y_t^{(j)}$ $(j = 1, 2)$ using the *Euler* scheme:

$$Y_{t_{i+1}}^{(1)} = Y_{t_i}^{(1)} + (\mu_1 - \lambda_1)\Delta t_i + \sigma_1 Z_i^1 \sqrt{\Delta t_i}, \tag{3.18}$$

$$Y_{t_{i+1}}^{(2)} = Y_{t_i}^{(2)} + (\mu_2 - \lambda_2)\Delta t_i + \sigma_2 Z_i^2 \sqrt{\Delta t_i}, \tag{3.19}$$

where Z_i^1 and Z_i^2 are coordinates of Z_i, which is a random vector chosen from the bivariate normal distribution with mean vector zero and covariance matrix $\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$.

To evaluate the default probability with multiple firms, we define $\tau_j^h = \min_i\{t_i : Y_{t_i}^{(j)} \leq 0\}$ ($j = 1, 2$). Thus we have

$$P(D_1(t) = 1 \text{ or } D_2(t) = 1) = P(\tau_1 \leq t \text{ or } \tau_2 \leq t)$$

$$\approx P(\tau_1^h \leq t \text{ or } \tau_2^h \leq t) \approx \frac{1}{M}\sum_{k=1}^{M} \mathbb{1}_{\{\tau_1^h \leq t \text{ or } \tau_2^h \leq t \text{ for the } k\text{th path}\}}.$$

Let $I(t)$ denote the indicator function of the event $D_1(t) = 1$ or $D_2(t) = 1$. We now have the default indicator estimator of the target probability of default using the control variate method:

$$I_b(t) = I(t) - b \cdot (I_c(t) - E[I_c(t)]), \tag{3.20}$$

where I_b is the controlled estimator of the default indicator, I is the default indicator without using the control variate, I_c is the control variate default indicator, $E[I_c] = P(I_c = 1)$ is the expectation of I_c. We use the special case estimator $(\mu_i = \lambda_i)$ as a control variate.

To reduce error in the computation of the default probability, we do the following. Each time step is taken to be small, thus reducing the deterministic bias. In doing so we apply the Brownian bridge construction. For the control variate method, we choose a large number of paths, hence reducing the stochastic error. The drawback of these changes is that it takes a longer time to run the code.

Algorithm 1 . Estimation of the default probability for two firms using the control variate method in GBM model (or VG model)

Input: M is the number of sample paths, N is the number of time steps. Model parameters are $V_0^{(j)}, K_j, \lambda_j, \mu_j, \nu_j, \sigma_j, j = 1, 2$
Output: The sample value of the indicator function $I(t)$
Compute the expectation of I_c using the formula (3.1) (or (3.8) for the VG model). For the VG model, generate random numbers from the gamma distribution

```
if V_0^(1) ≤ K_1 or V_0^(2) ≤ K_2 then
  I = 1
  Return
end if
for k = 1, ..., M do
  for i = 1, ..., N do
    Compute Y_{t_i}^(j), j = 1, 2, using (3.18) and (3.19)
    if Y_{t_i}^(1) ≤ 0 or Y_{t_i}^(2) ≤ 0 then
      I = 1
      Return
    else
      For j = 1, 2, compute prob_j using (3.8)
      with a_1 = Y_{t_{i-1}}^(j), a_1 = Y_{t_i}^(j), b = 0, t_2 − t_1 = Δt_i, σ = σ_j;
      Generate α_j ~ U(0, 1);
      if α_1 ≤ prob_1 or α_2 ≤ prob_2 then
        I = 1
        Return
      end if
    end if
  end for
end for
```

CHAPTER 4

Numerical Results

Let us recall the notation used in our model:

T = the maturity time(s);

N = the number of simulated paths in the Monte Carlo method;

ν = the parameter for the gamma distribution;

ρ = the correlation coefficient;

r = the risk neutral interest rate (drift parameter);

μ_1, μ_2 = the drift coefficients for two firms;

σ_1, σ_2 = the volatility coefficients for two firms;

λ_1, λ_2 = the constants of the barrier function $C(t) = Ke^{\lambda t}$;

$V_0^{(1)} \equiv V_1$, $V_0^{(2)} \equiv V_2$ = the initial asset values of two firms;

K_1, K_2 = the respective constant values for K of the barrier function $C(t) = Ke^{\lambda t}$;

4.1. Comparison of the PDFs of the FHT τ and τ^*

In Figure 4.1, we notice that the first hitting time (FHT) density approaches zero as the maturity time T increases. The PDF becomes more heavy-tailed as ν increases.

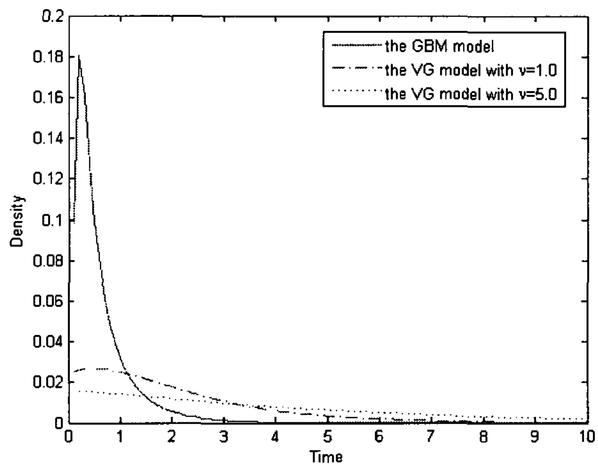

Figure 4.1: The PDFs of the First Hitting Times τ^* and τ, $X_0 = 0.5, \mu = 0.8, \sigma = 0.6$, $\lambda = 0.03$, $V_1/K_1 = 2$.

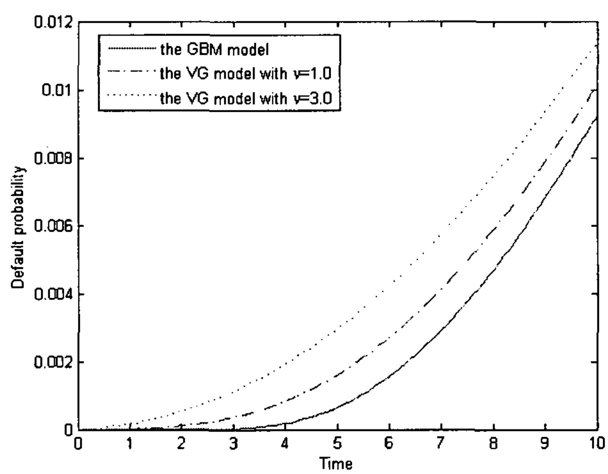

Figure 4.2: The GBM model vs. the VG model for one firm with $\sigma = 0.1$, $V_1/K_1 = 2$, $r = 0.05$, $\lambda = 0.03$.

4.2. Comparison of the Probabilities $P(\tau > t)$ and $P(\tau^* > t)$

In Figure 4.2, we note that, for fix $\sigma = 0.1$, the default probability plot for the VG model with $\nu = 1.0$ is closer to that of the GBM than for the case when $\nu = 3.0$.

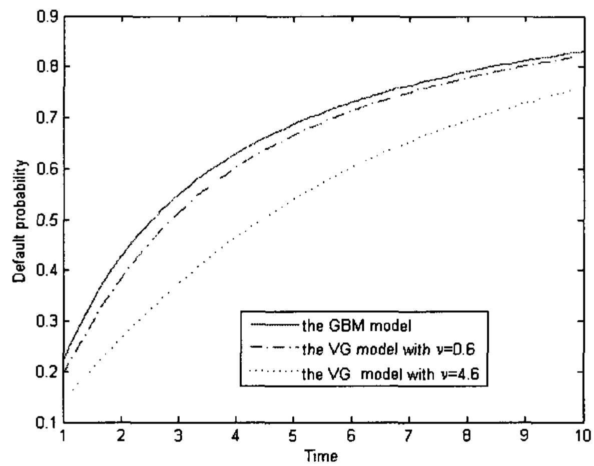

Figure 4.3: The GBM model vs. the VG model for one firm with $\sigma = 0.5$, $V_1/K_1 = 2$, $r = 0.05$, $\lambda = 0.03$.

In Figure 4.3, we observe that, for fix $\sigma = 0.5$, the default probability plot for $\nu = 0.6$ is closer to that of the GBM than when $\nu = 0.4$.

Both Figures indicate that the default probability plot for the VG model is closer to the GBM model as ν gets smaller and time gets larger. This conclusion is also obtained from formulas (2.4) and (2.5) since the no time change happens when ν becomes smaller enough.

4.3. The Default Correlations in the GBM and VG Models

4.3.1. The Special Case of the GBM Model. Figure 4.4 illustrates the following results.

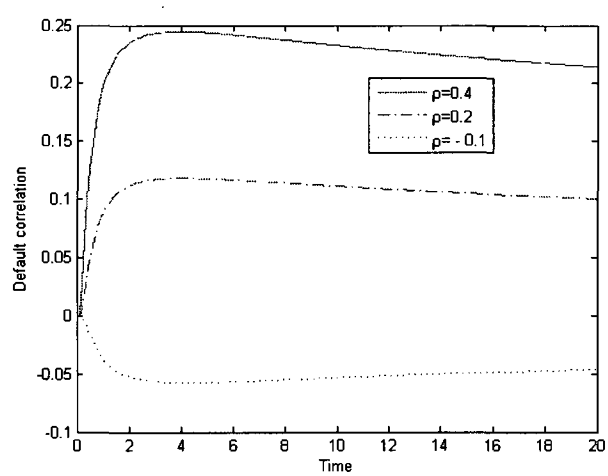

Figure 4.4: Default correlations for the GBM model for different value of ρ; $V_1/K_1 = V_2/K_2 = 1.8$, $\sigma_1 = \sigma_2 = 0.4$, and $N = 20000$.

a. *The default correlation and the asset correlation ρ have the same sign. The higher ρ, the higher the default correlation.* This result is intuitive. For instance, if the asset level correlation ρ is positive, when one firm defaults, it is likely that the value of the other firm has also declined and moved closer to its default boundary. The results explains why firms in the same industry often have higher default correlations than the firms in different industries.

b. *Default correlations are generally very small over short horizons. They first increase and slowly decrease with time.* Over a short horizon, default correlations are low because quick defaults are rare and may converge to a stable value. Default correlations eventually decrease with time because over a sufficiently long time horizon, the default of a firm is virtually inevitable in the model, and nondefault events become rare.

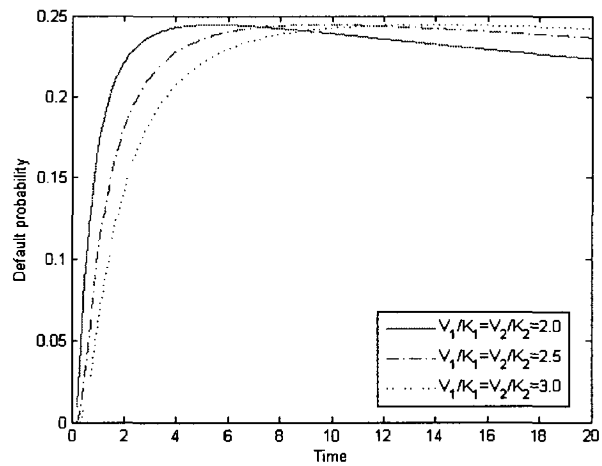

Figure 4.5: Default Correlations for the GBM model for different values of $V_1/K_1 = V_2/K_2$; $\rho = 0.4$, $\sigma_1 = \sigma_2 = 0.4$, and $N = 20000$.

Figure 4.5 illustrates the relation between the default correlations and time for various levels of the credit quality of the firms, which is effected by the initial value of V/K. This figure has some interesting results.

a. *High credit quality implies a low default correlation over typical horizons.* For the higher credit quality firms, the conditional default probability $P(D_2(t) = 1|D_1(t) = 1)$ is small. Although the default of firm 1 signals that the value of firm 2, V_2, may have declined, because the original ration V_2/K_2 is high, the probability that V_2 falls below K_2 is still very small.

b. *The time of peak default correlation depends on the credit quality of the underlying firms.* The higher quality firms take a longer time to peak.

c. *Because the credit quality of firms is time-varying, the default correlation is dynamic.* A rise in the credit quality leads to a substantial drop in the default correlation, and a decline in the credit quality leads to a rise in the default correlation. This dynamic behavior arises even when the underlying assets and liabilities of the firm have constant expected returns and risks.

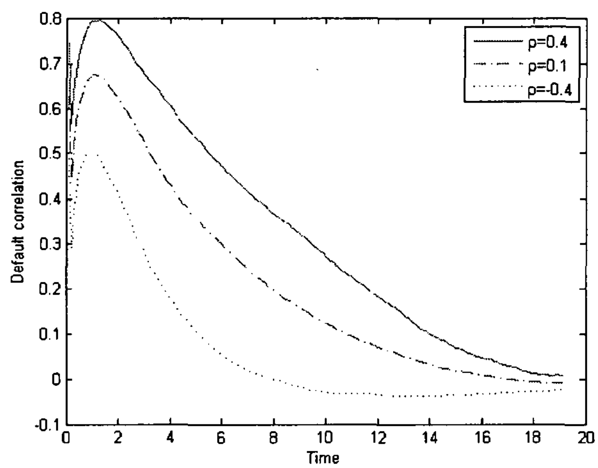

Figure 4.6: Default Correlations in the VG model for different values of ρ; $\mu_1 = 0.55$, $\lambda_1 = 0.65$, $\mu_2 = 0.35$, $\lambda_2 = 0.65$, $V_1/K_1 = 2$, $V_2/K_2 = 3$, $\sigma_1 = 0.4$, $\sigma_2 = 0.6$, $\nu = 0.1$, and $N = 100000$.

4.3.2. The General Case of the VG Model. In Figure 4.6, we notice that the peak of the default correlation becomes higher and the correlation function tends to zero faster with the increase of time as ρ becomes larger.

4.4. The Control Variate Method

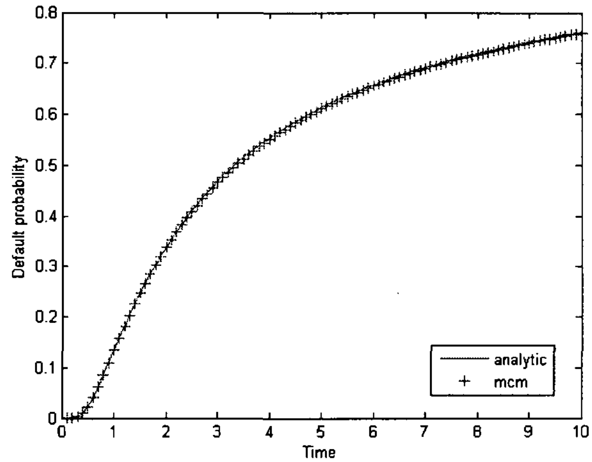

Figure 4.7: The default probability computed with the analytic formula (5.1) vs. the Monte Carlo method for the GBM model (for the special case), $\rho = 0.4, V_1/K_1 = 2, V_2/K_2 = 3, \sigma_1 = 0.4, \sigma_2 = 0.6$, $N = 100000$.

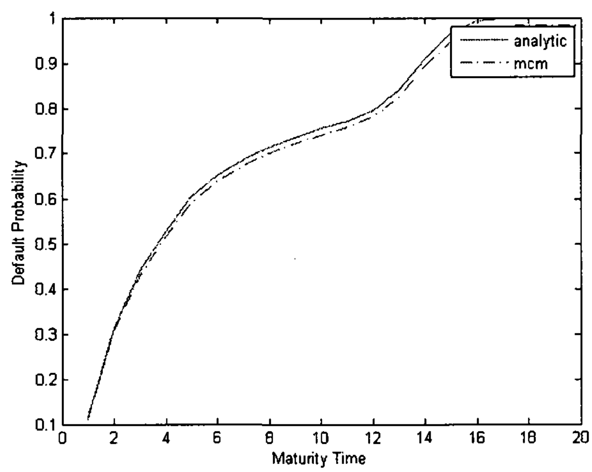

Figure 4.8: The default probability computed with the analytic formula (5.9) vs. the Monte Carlo method for the VG model (for the special case), $\rho = 0.4, \mu_1 = \mu_2 = 0, \lambda_1 = 0.55, \lambda_2 = 0.65, V_1/K_1 = 2, V_2/K_2 = 1.8, \sigma_1 = 0.4, \sigma_2 = 0.3, \nu = 0.1$, $N = 1000000$.

Table 1: Comparison of analytic values and Monte Carlo estimates in the GBM model

Time	Analytical Result	MCM Estimate	Standard Deviation
5	0.6898	0.6898	0.0015
10	0.7851	0.7851	0.0014
15	0.8344	0.8344	0.0012

Table 2: Comparison of analytic values and Monte Carlo estimates in the VG model

Time	Analytical Result	MCM	Standard Deviation
5	0.5319	0.5320	0.0016
10	0.7547	0.7545	0.0014
15	0.8835	0.8829	0.0013

4.4.1. Comparison of the Analytic Results with the Monte Carlo Result. In table 1 and Figure 4.7, we we can conclude that the default probability obtained by the analytic formula (5.1) and the result from the Monte Carlo method for the special case are the same.

In table 2 and Figure 4.8, we we can also conclude that the default probability obtained by the analytic formula (5.9) and the result from the Monte Carlo method for the special case have a little difference. Indeed, the difference between the analytic values and Monte Carlo estimates has a trend to increase as the maturity time gets larger.

4.4.2. Efficiency of the Control Variate Method. In Figure 4.9 and 4.10, we note that as the maturity time gets larger, the ratio of variances $1/(1-\rho^2)$ approaches 1. As for two firms, their asset level correlations ρ have decreased as the maturity time gets larger, so their ratio of variances decrease to approach 1 along ρ close to 0.

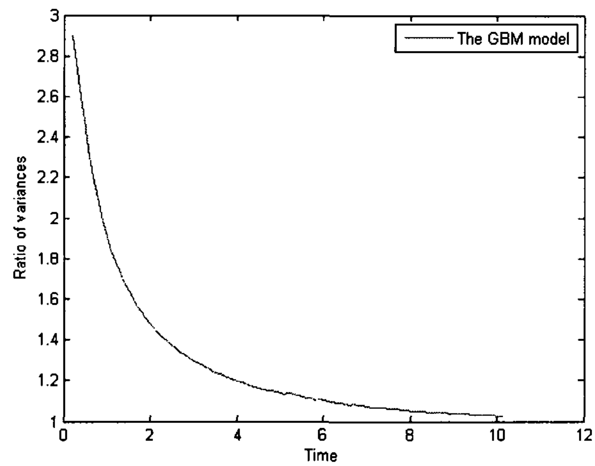

Figure 4.9: The ratio of variances in the control variate MCM for the bivariate GBM model; $\rho = 0.4, r = 0.55, \lambda_1 = 0.75, \sigma_1 = 0.4, V_1/K_1 = 1.8, \lambda_2 = 0.75, \sigma_2 = 0.4, V_2/K_2 = 2$, and $N = 10000000$.

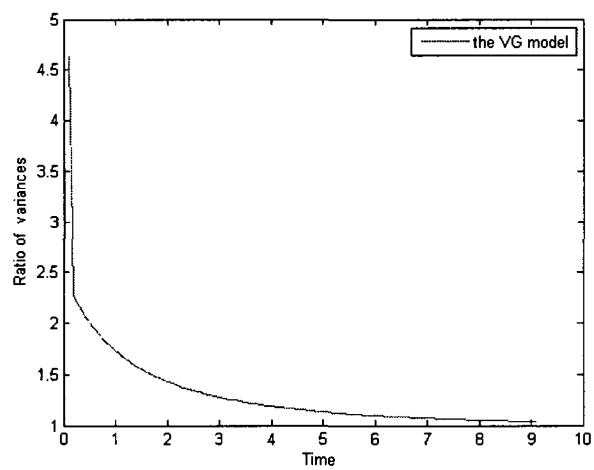

Figure 4.10: The ratio of variances in the control variate MCM for the bivariate VG model; $\rho = 0.4, \nu = 0.5, \mu_1 = 0.55, \lambda_1 = 0.75, \sigma_1 = 0.4, V_1/K_1 = 1.8, \mu_2 = 0.55, \lambda_2 = 0.75, \sigma_2 = 0.4, V_2/K_2 = 2$, and $N = 1000000$.

CHAPTER 5

Conclusion

This thesis has studied the variance gamma process as a model of credit risk. The first passage time of the second kind presents some key advantages over the classic definition of first passage time for a diffusion process. We provide analytical formulas for calculating probabilities of single and joint probabilities of defaults that are easily implemented. The formulas obtained provide a convenient tool for credit evaluation we demonstrated and risk management.

Moreover, the application of the Monte Carlo method to the solution of the GBM and VG model. This was achieved by the construction of the Brownian bridge, which is used in estimating first passage times. For improving the efficiency of the Monte Carlo method, we have implemented the control variate technique.

Appendix

By using the series representation for the modified Bessel function $I_\mu(Z)$, equation (3.1) has the following form.

$$
\begin{aligned}
&P(D_1(t) = 1 \text{ or } D_2(t) = 1) \\
&\quad = 1 - \frac{2r_0}{\sqrt{2\pi t}} \cdot e^{-\frac{r_0^2}{4t}} \cdot \sum_{n=1,3,\ldots} \frac{1}{n} \cdot \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \\
&\quad \cdot \left[I_{\frac{1}{2}((\frac{n\pi}{\alpha}+1)}\left(\frac{r_0^2}{4t}\right) + I_{\frac{1}{2}((\frac{n\pi}{\alpha}-1)}\left(\frac{r_0^2}{4t}\right) \right] \\
&\quad = 1 - \frac{2r_0}{\sqrt{2\pi t}} \cdot e^{-\frac{r_0^2}{4t}} \cdot \sum_{n=1,3,\ldots} \frac{1}{n} \cdot \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \\
&\quad \cdot \left(\sum_{k=0}^{\infty} \frac{(\frac{r_0{}^2}{8t})^{2k+\frac{1}{2}(\frac{n\pi}{\alpha}+1)}}{k!\Gamma[\frac{1}{2}(\frac{n\pi}{\alpha}+1)+k+1]} + \sum_{k=0}^{\infty} \frac{(\frac{r_0{}^2}{8t})^{2k+\frac{1}{2}(\frac{n\pi}{\alpha}-1)}}{k!\Gamma[\frac{1}{2}(\frac{n\pi}{\alpha}-1)+k+1]} \right) .
\end{aligned}
\tag{5.1}
$$

In our VG model, the default probability is an integral of a product between the expression in (5.1) and the gamma density. That is,

$$P(D_1^*(t)=1 \text{ or } D_2^*(t)=1)$$
$$=\int_0^\infty \left(1-\frac{2r_0}{\sqrt{2\pi s}}\cdot e^{-\frac{r_0^2}{4s}}\cdot \sum_{n=1,3,\ldots}\frac{1}{n}\cdot\sin\left(\frac{n\pi\theta_0}{\alpha}\right)\right.$$
$$\cdot\left.\left(\sum_{k=0}^{\infty}\frac{(\frac{r_0{}^2}{8t})^{2k+\frac{1}{2}(\frac{n\pi}{\alpha}+1)}}{k!\Gamma[\frac{1}{2}(\frac{n\pi}{\alpha}+1)+k+1]}+\sum_{k=0}^{\infty}\frac{(\frac{r_0{}^2}{8t})^{2k+\frac{1}{2}(\frac{n\pi}{\alpha}-1)}}{k!\Gamma[\frac{1}{2}(\frac{n\pi}{\alpha}-1)+k+1]}\right)\right)$$
$$\cdot\frac{1}{\nu^{\frac{T}{\nu}}\Gamma(\frac{T}{\nu})}s^{(\frac{T}{\nu}-1)}e^{-\frac{s}{\nu}}ds. \tag{5.2}$$

Now, we need to simplify the following formulas

$$\int_0^\infty\frac{2r_0}{\sqrt{2\pi s}}e^{-\frac{r_0{}^2}{4s}}(\frac{r_0{}^2}{8s})^{2k+\frac{1}{2}(\frac{n\pi}{\alpha}-1)}\frac{1}{\nu^{\frac{T}{\nu}}\Gamma(\frac{T}{\nu})}s^{(\frac{T}{\nu}-1)}e^{-\frac{s}{\nu}}ds$$
$$=\int_0^\infty\frac{\sqrt{2}r_0}{\sqrt{\pi s}}e^{-\frac{r_0{}^2}{4s}}\left(\frac{r_0{}^2}{8s}\right)^{2k+\frac{n\pi}{2\alpha}}\left(\frac{r_0{}^2}{8s}\right)^{-\frac{1}{2}}e^{-\frac{s}{\nu}}(\frac{s}{\nu})^{(\frac{T}{\nu}-1)}\frac{1}{\nu\Gamma\frac{T}{\nu}}ds$$
$$=\int_0^\infty\frac{4}{\sqrt{\pi}\Gamma(\frac{T}{\nu})}\left(\frac{r_0{}^2}{8}\right)^{2k+\frac{n\pi}{2\alpha}}e^{(-\frac{r_0{}^2}{4s}-\frac{s}{\nu})}s^{-(2k+\frac{n\pi}{2\alpha})+\frac{T}{\nu}-1}ds \tag{5.3}$$

and

$$\int_0^\infty \frac{2r_0}{\sqrt{2\pi s}} e^{-r_0^2/4s} \left(\frac{r_0{}^2}{8s}\right)^{2k+\frac{1}{2}(\frac{n\pi}{\alpha}+1)} \frac{1}{\nu^{\frac{T}{\nu}}\Gamma(\frac{T}{\nu})} s^{(\frac{T}{\nu}-1)} e^{-\frac{s}{\nu}} ds$$

$$= \int_0^\infty \frac{2r_0}{\sqrt{2\pi s}} e^{-r_0^2/4s} \left(\frac{r_0{}^2}{8s}\right)^{2k+\frac{n\pi}{2\alpha}} \left(\frac{r_0{}^2}{8s}\right)^{\frac{1}{2}} \frac{1}{\nu^{\frac{T}{\nu}}\Gamma(\frac{T}{\nu})} s^{(\frac{T}{\nu}-1)} e^{-\frac{s}{\nu}} ds$$

$$= \int_0^\infty \frac{4r_0{}^2}{8\sqrt{\pi}} \left(\frac{r_0{}^2}{8}\right)^{2k+\frac{n\pi}{2\alpha}} s^{-(2k+\frac{n\pi}{2\alpha}+1)} e^{-\frac{r_0^2}{4s}} \frac{1}{\nu^{\frac{T}{\nu}}\Gamma(\frac{T}{\nu})} s^{(\frac{T}{\nu}-1)} e^{-\frac{s}{\nu}} ds$$

$$= \int_0^\infty \frac{4}{\sqrt{\pi}} \left(\frac{r_0{}^2}{8}\right)^{2k+\frac{n\pi}{2\alpha}+1} \frac{1}{\nu^{\frac{T}{\nu}}\Gamma(\frac{T}{\nu})} e^{(-(s/\nu+r_0{}^2/4s))} s^{(\frac{T}{\nu}-2k-\frac{n\pi}{2\alpha}-2)} ds. \tag{5.4}$$

The above integral can be evaluated as follows. Let us consider the Generalized Inverse Gaussian, with density function.

$$f(x) = \frac{1}{2\mu^\gamma K_\gamma\left(\frac{\lambda}{\mu}\right)} x^{\gamma-1} e^{-\frac{\lambda/x+(\lambda/\mu^2)x}{2}}, \tag{5.5}$$

where $\mu > 0$ and $\lambda \geq 0, \gamma \in \mathbb{R}$ and

$$K_\gamma(\lambda/\mu) = \int_0^\infty \frac{1}{2\mu^\gamma} x^{\gamma-1} e^{-\frac{\lambda/x+(\lambda/\mu^2)x}{2}} dx. \tag{5.6}$$

Now, the expressions (5.3) and (5.4) can be written more compactly as the expression (5.6):

$$(5.3) = \frac{4}{\sqrt{\pi}\Gamma(\frac{T}{\nu})} \left(\frac{r_0{}^2}{8}\right)^{2k+\frac{n\pi}{2\alpha}} 2\mu^\gamma K_\gamma(\lambda/\mu) \tag{5.7}$$

with $\gamma = -(2k+\frac{n\pi}{2\alpha}) + T/\nu$, $\lambda = r_0{}^2/2$ and $\mu = \frac{r_0}{2}\sqrt{\nu}$,

and

$$(5.4) = \frac{4}{\sqrt{\pi}} \left(\frac{r_0{}^2}{8}\right)^{2k+\frac{n\pi}{2\alpha}+1} \frac{1}{\nu^{\frac{T}{\nu}}\Gamma(\frac{T}{\nu})} 2\mu^\gamma K_\gamma(\lambda/\mu) \tag{5.8}$$

with $\gamma = -(2k + \frac{n\pi}{2\alpha}) + T/\nu - 1$, $\lambda = {r_0}^2/2$ and $\mu = \frac{r_0}{2}\sqrt{\nu}$.

Thus, the default probability (5.2) for two firms in the VG model can be written as:

$$
\begin{aligned}
&P(D_1^*(t) = 1 \text{ or } D_2^*(t) = 1) \\
&= 1 - \sum_{n=1,3,\dots} \frac{1}{n} \cdot \sin\left(\frac{n\pi\theta_0}{\alpha}\right) \\
&\cdot \left(\sum_{k=0}^{\infty} \frac{1}{k!\Gamma[\frac{1}{2}(\frac{n\pi}{\alpha}+1)+k+1]} \frac{4}{\sqrt{\pi}\Gamma(\frac{T}{\nu})} \left(\frac{{r_0}^2}{8}\right)^{2k+\frac{n\pi}{2\alpha}} 2\mu^{\gamma} K_{\gamma}(\lambda/\mu) \right. \\
&\left. + \sum_{k=0}^{\infty} \frac{1}{k!\Gamma[\frac{1}{2}(\frac{n\pi}{\alpha}-1)+k+1]} \frac{4}{\sqrt{\pi}} \left(\frac{{r_0}^2}{8}\right)^{2k+\frac{n\pi}{2\alpha}+1} \frac{1}{\nu^{\frac{T}{\nu}}\Gamma(\frac{T}{\nu})} 2\mu^{\gamma} K_{\gamma}(\lambda/\mu) \right).
\end{aligned}
\tag{5.9}
$$

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