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STRICT-DOMINANCE SOLVABILITY OF GAMES ON CONTINUOUS STRATEGY SPACES

by

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ABSTRACT. The concept of strict dominance provides a technique that can be used normatively to predict the play of games based only on the assumption of individual rationality. Such predictions, unlike those based on Nash equilibria, do not depend on players' beliefs about the behaviour of others. One strategy strictly dominates another if and only if the payoff from the first strategy is strictly greater than the payoff from the second, no matter how the opponent(s) plays. It is possible for iterated elimination of strictly dominated strategies to remove all but a single choice for each player, in which case we say that the game is strict-dominance solvable. Analysis of games with continuous strategy spaces reveals necessary and sufficient conditions on the payoff functions for strict-dominance solvability. These conditions will be identified first for symmetric two-player games with quadratic payoff functions, and then extended to higher-order payoff functions and asymmetric games. The conditions discovered can be applied to specific games, producing conclusions that can be compared to other solutions of those games.

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Contents

1. Introduction

The mathematical game is "the simplest and most powerful model to analyze situations involving mixed motives"[8]. Since its origination by von Neumann and Morgenstern in their work *Theory of Games and Economic Behavior* (1944), game theory has been applied to the social sciences to provide a model for the analysis of any situation where multiple agents influence the outcome and have differing interests. This tool has been used to glean information about a wide class of interactions, with the games falling under a myriad of classifications.

A non-cooperative game $\Gamma = \langle N; S_1, \ldots, S_n; \Pi_1, \ldots, \Pi_n \rangle$ consists of a set of players, $N = \{1, \ldots, n\}$, each of which has a set of possible strategies, $S_1, \ldots, S_n$, sometimes called pure strategies. The choice of a particular strategy $x_i \in S_i$ by every player i will determine an outcome of the game. Players' values for outcomes are measured by utilities [11]; player i's utility for the outcome that results from strategies $(x_1, x_2, \ldots, x_n)$ is given by i's payoff function $\Pi_i(x_1, \ldots, x_n)$. "Rational" players do the best they can for themselves through their choice of strategy (see below).

Depending on the interaction that is being modeled, other considerations may be taken into account. The players may or may not have complete information about the identity and number of their opponents, and the preferences of those opponents. Communication among players might or might not be allowed. Choices might be made in a specific order, or simultaneously. The game might only be played once, a certain number of times, or indefinitely. Random events might be involved at some stage.

Game theory then seeks to predict how the game will unfold given that the players are "rational", where "rational" is used to describe a course of action which is best for the player, and to describe a player who chooses such a course of action [7]. Will a specific outcome be determined? Can some outcomes be determined to be impossible under rational play? In some cases, equilibrium choices can be found such that players who anticipate the equilibrium are disinclined to depart from it. Such an equilibrium can be taken as a possible solution to the game.

1.1. Non-Cooperative Game Theory and Nash Equilibria.

Non-Cooperative game theory is the study of games in which players act on their own to achieve the most preferable (greatest utility) outcome. Communication between players is usually assumed to be impossible.

A common solution concept for non-cooperative games is Nash equilibrium. A Nash equilibrium is a strategy selection for every player, or strategy profile, such that no player has a rational inclination to change his or her strategy if the other players play theirs.

Formally, a strategy profile $(x_1, \ldots, x_n) \in S$ in an n-person strategic-form game Γ is a (pure-strategy) Nash equilibrium if and only if $\Pi_i(x_1, \ldots, x_n) \geq \Pi_i(x_1, \ldots, x_n | x_i')$ for all $i \in N$ and $x_i' \in S_i$ [7]. Here $(x_1, \ldots, x_n | x_i')$ is the same strategy profile as $(x_1, \ldots, x_n)$ except that player i uses x_i' in place of

x_i. If the inequality is strict for all $x_i' \neq x_i$, the strategy profile is known as a strict Nash equilibrium.

Example 1.1. Let the following matrix describe a game between two players where elements of Player 1's strategy set correspond to rows and elements of Player 2's strategy set correspond to columns. The first entry in each position of the matrix is the payoff to Player 1, and the second to Player 2. Games in this form are called *bimatrix games.*

	C_1	C_2	C_3
R_1	$(5,2)$	$(2,3)$	$(0,6)$
R_2	$(2,4)$	$(1,1)$	$(1,3)$
R_3	$(0,3)$	$(4,4)$	$(3,0)$

There is a Nash Equilibrium at position (R_3, C_2) with a payoff of 4 to each player. This can be seen to be a Nash equilibrium by examining the payoffs to either player should the other player's strategy remain fixed. Neither player would do better by switching. In fact, the unique Nash equilibrium of Example 1.1 is (R_3, C_2)

The concept of Nash equilibrium does not provide a clear solution to all games. Nash equilibria may fail to exist, or there may be more than one Nash equilibrium in a game.

Example 1.2. The following bimatrix game has no Nash equilibria.

	C_1	C_2
R_1	$(3,5)$	$(2,4)$
R_2	$(5,0)$	$(0,8)$
R_3	$(7,2)$	$(1,3)$

Example 1.3. The following bimatrix game has Nash equilibria at (R_1, C_1) and (R_2, C_3).

	C_1	C_2	C_3
R_1	$(6,5)$	$(2,2)$	$(2,2)$
R_2	$(2,4)$	$(0,3)$	$(4,7)$

Note that the players' preferences over the two Nash equilibria are opposed.

A Nash equilibrium may require choices that do not seem rational.

Example 1.4. [8] The only Nash Equilibrium in this game is at position (R_2, C_2) with a payoff of 2 to each player.

	C_1	C_2	C_3
R_1	$(1,3)$	$(2,0)$	$(3,1)$
R_2	$(0,2)$	$(2,2)$	$(0,2)$
R_3	$(3,1)$	$(2,0)$	$(1,3)$

This outcome, however, requires the players to choose strategies R_2 and C_2, which can be argued to be irrational. For Player 1, strategies R_1 and R_3 always provide at least as great a payoff as strategy R_2, and in two out of three cases, pay strictly more. It is thus said that strategy R_2 is weakly dominated by strategy R_1 and by strategy R_3. The situation is identical for

Player 2's strategies. Thus the principle that a rational player would never choose a dominated strategy seems counter to the view that Nash equilibria describe rational play.

A strategy $\hat{x}_i$ is a *weakly dominated strategy* if and only if there is a different strategy x_i^* such that

(1) $\Pi_i(x_1, \ldots, x_n | x_i^*) \geq \Pi_i(x_1, \ldots, x_n | \hat{x}_i)$ for all $(x_1, \ldots, x_n) \in S$, and

(2) for some $(x_1, \ldots, x_n) \in S$, $\Pi_i(x_1, \ldots, x_n | x_i^*) > \Pi_i(x_1, \ldots, x_n | \hat{x}_i)$ [7].

Thus in the above example, R_2 and C_2 are weakly dominated strategies.

1.2. **Strict-Dominance Solvability.** Example 1.4 showed a weakly dominated strategy. If, instead, a strategy is always better than a second strategy, it is said that the first strategy *strictly dominates* the second. As illustrated next, it is possible for iterative elimination of strictly dominated strategies to remove all but a single choice for all players, in which case we say that the game is *strict-dominance solvable.* The solution of a strict-dominance solvable game with finite strategy spaces provided by strict dominance must be a Nash equilibrium; moreover, a prediction based on strict-dominance solvability does not depend on a players' beliefs about the behaviour of others, unlike predictions based on Nash equilibrium. (For a formal definition of strict-dominance solvability, see Chapter 2 for symmetric games and Chapter 3 for asymmetric games, such as Example 1.5 below.)

Example 1.5. Consider the following bimatrix game.

	C_1	C_2	C_3
R_1	$(4, 2)$	$(2, 0)$	$(3, 4)$
R_2	$(1, 3)$	$(1, 3)$	$(2, 1)$
R_3	$(2, 1)$	$(2, 1)$	$(2, 2)$

Player 1 always receives a strictly better payoff from strategy R_1 than from strategy R_2, so R_2 can be eliminated from the game. The resulting bimatrix is

	C_1	C_2	C_3
R_1	$(4, 2)$	$(2, 0)$	$(3, 4)$
R_3	$(2, 1)$	$(2, 1)$	$(2, 2)$

Player 2's strategies C_1 and C_2 are now strictly dominated by C_3, and so both C_1 and C_2 are eliminated from the game. Notice that neither C_1 nor C_2 were dominated in the original game. The game is now

	C_3
R_1	$(3, 4)$
R_3	$(2, 2)$

Now R_1 strictly dominates R_3. After this round of elimination the game is reduced to a single choice for each player,

	C_3
R_1	$(3, 4)$

Therefore, the game is strict-dominance solvable, and the solution is (R_1, C_3) resulting in a payoff of $(3, 4)$.

The concept of strict dominance thus provides a technique that can be used normatively to predict the play of games based only on a very strict criterion of individual rationality.

1.3. **The Snowdrift Game.** One of the more famous examples of non-cooperative game theory, the Snowdrift (SD) game models an interaction between two drivers who are blocked on either side of an intervening snow-drift [4]. Both players have the options of removing the snowdrift or staying in the car and doing nothing. The latter option is the best choice as it leaves all of the work up to the other player; however, if that player also opts to stay in the car, then neither will ever get to their destination. This model can similarly be used to represent situations where cooperation by one of the players creates a common good that can be exploited by others, but where a player may be better off cooperating if no one else is, despite the prospect of being exploited. This game is also identical to the game Chicken [7]. In matrix notation, the discrete (two strategy) Snowdrift game appears as

$$\begin{array}{ccc} & C & D \\ C & (b-\frac{c}{2}, b-\frac{c}{2}) & (b-c, b) \\ D & (b, b-c) & (0,0) \end{array}$$

where b, the benefit to cooperating, outweighs c, the cost ($b > c > 0$). Thus the game is differentiated from the famous (two strategy) Prisoner's Dilemma (PD) by the fact that it is always better to cooperate when your opponent defects. Also, unlike the Prisoner's Dilemma, the SD game does not have a symmetric Nash equilibrium in pure strategies (see Definition 2.3 and Example 2.4 below) and so is not strictly dominance solvable.

1.4. **Continuous Strategy Spaces.** The previous examples are all games in which there is a finite number of strategies for each player, but often games with *continuous strategy spaces* are used to fashion models. These strategy spaces are typically intervals of the real line, and thus offer an uncountably infinite number of possible strategy choices to each player.

The use of continuous strategy spaces brings many new problems, as most of the ideas discussed above depend on finite strategy spaces. If there are an uncountable number of strategies, can a solution be found by strict dominance in a finite (or countable) number of steps? Is the solution provided by strict dominance guaranteed to be a Nash equilibrium?

1.5. **The Snowdrift Game in Continuous Strategy Spaces.** Consider the case if cooperative investments are allowed to vary continuously in the SD game. Investments should yield a benefit not only to the opponents, but also to the investing player. If $x, y \in [0, 1]$ are the investments of players 1 and 2 respectively, then the payoff to player 1 is $\Pi_1(x, y) = B_1(x + y) - C_1(x)$ and the payoff to player 2 is $\Pi_2(x, y) = B_2(x + y) - C_2(y)$, where $B_i(x+y)$ specifies the benefit to player i obtained from the total cooperative investment made by both players, and $C_i(x)$ specifies the cost incurred to player i because of their own investment [4]. In section 2.1.3 it will be shown that for certain benefit and cost functions, the continuous SD game becomes strict-dominance solvable.

1.6. **Goals.** The concept of strict dominance is well known. It is not known, however, under which conditions certain classes of games are strict-dominance solvable, particularly in the case of games with continuous strategy spaces. This thesis will provide conditions under which specific classes of games are strict-dominance solvable. Further, there is the problem of how to identify rational behaviour in a finite game, a problem that is even more difficult for continuous games. This thesis explores one answer to this question: When a game is strict-dominance solvable to a strategy x^*, then x^* is the only possible outcome under rational play, and therefore there can be no ambiguity about what choices are 'best' for rational players.

2. Symmetric Games

A game is called *symmetric* if permuting the players cannot change the strategic problem facing any player. The SD game in Section 1.3 is an example of a two strategy symmetric game. In general, a two-player symmetric game $\Gamma = \langle 2; S_1 = S, S_2 = S; \Pi_1 = \Pi, \Pi_2 = \Pi$ is defined by a strategy set, S, and a payoff function $\Pi : S \times S \longrightarrow \mathbb{R}$ where $\Pi(x, z)$ is the payoff to strategy x when the opponent chooses strategy z. Let $T \subseteq S$. In a two-player symmetric game, strategy $x \in S$ strictly dominates strategy $y \in S$ with respect to T if and only if $\Pi(x, z) > \Pi(y, z)$ for all $z \in T$. Note that if $T = S$, x strictly dominates y.

Definition 2.1. A symmetric two-player game Γ, with payoff function Π and pure strategy set S is *strict-dominance solvable* (SDS) [2] to $x^* \in S$ if there is a finite or countable sequence of subsets of S, $S = S_0 \supset S_1 \supset S_2 \supset \ldots$, called a *reduction sequence* for x^*, such that

(1) S_{i+1} is a proper subset of S_i.

(2) For every $y \in S_i \setminus S_{i+1}$ there is an $x \in S_{i+1}$ such that x strictly dominates y with respect to S_i.

(3) $\bigcap S_i = \{x^*\}$.

Example 2.2. Consider the following symmetric bimatrix game with $S = S_0 = \{1, 2, 3\}$.

	C_1	C_2	C_3
R_1	$(4,4)$	$(2,2)$	$(3,1)$
R_2	$(2,2)$	$(1,1)$	$(1,3)$
R_3	$(1,3)$	$(3,1)$	$(0,0)$

Strategy 2 is strictly dominated by strategy 1, and once it is removed, the resulting game looks like

	C_1	C_3
R_1	$(4,4)$	$(3,1)$
R_3	$(1,3)$	$(0,0)$

Now strategy 3 is strictly dominated with respect to the new strategy space $S_1 = \{1, 3\}$ by strategy 1. Note that it is not true that strategy 3 is strictly dominated with respect to the original strategy space $S = S_0 = \{1, 2, 3\}$. When strategy 3 is removed the game is reduced to

	C_1
R_1	$(4,4)$

Hence, by Definition 2.1, this game is SDS to strategy 1 with payoff $(4,4)$ by the reduction sequence $S_0 = \{1, 2, 3\}$, $S_1 = \{1, 3\}$, and $S_2 = \{1\}$.

Definition 2.3. If (x, x) is a pure strategy Nash Equilibrium for a symmetric game Γ, i.e. if $\Pi(x, x) \geq \Pi(y, x)$ for all $y \in S$, then x is called a *symmetric Nash equilibrium* (NE). The strategy x is called a *strict Nash equilibrium* (SNE) if the inequality is strict whenever $y \neq x$. Note that there are other definitions of Nash equilibrium concepts, but this is the only one that will be used here.

Example 2.4. Consider the two strategy SD game.

	C	D
C	$(b-\frac{c}{2}, b-\frac{c}{2})$	$(b-c, b)$
D	$(b, b-c)$	$(0,0)$

Assuming that $b > c > 0$, both (D, C) and (C, D) are Nash equilibria, however, neither are symmetric Nash equilibria.

Theorem 2.5. *If Γ is a symmetric game that is SDS to $x^* \in S$, and x^* is a NE, then x^* is a SNE.*

Proof. Assume that x^* is not a SNE. Therefore for some $y \in S$ with $y \neq x^*$, $\Pi(y, x^*) \geq \Pi(x^*, x^*)$. Let S_i be the final subset in a reduction sequence for x^* such that $y \in S_i$. Hence there exists a $z \in S_{i+1}$ such that $\Pi(z, x^*) > \Pi(y, x^*) \geq \Pi(x^*, x^*)$ which contradicts the assumption that x^* is a NE. $\square$

Even more than this, Moulin proves in [8] that if Γ is SDS to x^* for any game with continuous payoffs on compact strategy spaces, then x^* is a NE. Thus by the above theorem, if Γ is SDS to x^* in such a game, then x^* is a SNE, and by the Theorem below, it is the unique NE. For examples where x^* is not a NE please refer to [5].

Theorem 2.6. *If Γ is SDS to x^*, and $\hat{x}$ is a NE of Γ, then $x^* = \hat{x}$.*

Proof. Assume that $x^* \neq \hat{x}$. Thus $\hat{x}$ must be eliminated at some stage k in the reduction sequence, and so there exists a $y \in S_k$ such that $\Pi(y, \hat{x}) > \Pi(\hat{x}, \hat{x})$, contradicting the fact that $\hat{x}$ is a NE. □

Corollary 2.7. *If x is a NE of Γ, then x is a member of every subset in any reduction sequence for x of Γ.*

2.1. Symmetric Games With Quadratic Payoff Functions and One Dimensional Strategy Spaces.

2.1.1. *Interior Symmetric NE.* All of the results of this section assume that the strategy set S is of the form $[-l, k]$ for some $l, k > 0$, that the NE is $x^* = 0$, and that the payoff function is quadratic. This can be achieved for any game Γ with a quadratic payoff function, a compact one-dimensional strategy space and an interior NE, by a translation of coordinates so that the coefficient of the linear term in a player's own strategies is equal to 0.

Theorem 2.8. *Fix $k, l > 0$ and let $\Gamma_{k,l}$ be the symmetric game with $S = [-l, k]$ and payoff function $\Pi : S \times S \to \mathbb{R}$ given by $\Pi(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$, where a, b, c, d, e, and f are constants. Then:*

1. *$x = 0$ is a NE for $\Gamma_{k,l}$ if and only if $d = 0$ and $a \leq 0$.*
2. *If $a = d = 0$, then there are no $x, y \in S$ such that x strictly dominates y.*

Thus a necessary condition for $\Gamma_{k,l}$ to be SDS to $x = 0$ is $d = 0$ and $a < 0$. In this case $x = 0$ is the unique NE.

Proof. (1) Assume that $x = 0$ is a NE. This means that $\Pi(0,0) \geq \Pi(y,0)$ for all $y \in S$, or equivalently, that $0 \geq y(ay+d)$. For small values of $|y|$, if $d \neq 0$, the d term dominates the ay term, so the only value of d that guarantees $0 \geq dy$ is $d = 0$. Assuming $d = 0$, $0 \geq y(ay+d)$ becomes $0 \geq ay^2$. Because y^2 is nonnegative, only $a \leq 0$ guarantees that $0 \geq ay^2$ for all y.

(2) Assume that $a = d = 0$, and that there exists some $x, y \in S$ such that $\Pi(x,z) > \Pi(y,z)$ for all $z \in S$. This gives $bxz + cz^2 + ez + f > byz + cz^2 + ez + f$ which simplifies to $bxz > byz$. Substituting $z = 0$ into this inequality yields the obvious contradiction $0 > 0$.

The remainder of the proof follows from Corollary 2.7 □

Now the necessary and sufficient conditions for a game of this class to be SDS to $x = 0$ are found.

Theorem 2.9. *Fix $k, l > 0$ and let $\Gamma_{k,l}$ be the symmetric 2 player game with continuous payoff function $\Pi : [-l,k] \times [-l,k] \to \mathbb{R}$ where $\Pi(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$, and a, b, c, d, e, and f are constants. Then $\Gamma_{k,l}$ is SDS to 0 and 0 is a NE if and only if $a < 0$, $d = 0$, and*

$$2 > \left|\frac{b}{a}\right| \tag{2.1}$$

Proof. Assume that (2.1) holds, and that $a < 0$ and $d = 0$. By Theorem 2.8, 0 is a NE. Define $S = S_{0,0}$. Let $k \geq l$ and let $y \geq k - \frac{1}{2}\varepsilon k = k_\varepsilon$, where $\varepsilon > 0$ will be chosen later. For $\Pi(k(1-\varepsilon), z) > \Pi(y,z)$ to be true for every

$z \in [-l, k]$ and every $y \geq k_\varepsilon$,

$$a(k(1-\varepsilon))^2 + bk(1-\varepsilon)z > ay^2 + byz, \tag{2.2}$$

or

$$y^2 - (k(1-\varepsilon))^2 > \frac{bz}{|a|}(y - k(1-\varepsilon)).$$

Because $y \geq k - \frac{1}{2}\varepsilon k$, $y - k(1-\varepsilon) > 0$. Therefore, dividing both sides of the inequality by $y - k(1-\varepsilon)$ shows that for all $z \in [-l, k]$ (2.2) is equivalent to

$$y + k(1-\varepsilon) > \frac{b}{|a|}z. \tag{2.3}$$

To show that (2.3) is true, note that the left side is at least $k\left(1 - \frac{1}{2}\varepsilon\right) + k(1-\varepsilon) = 2k - \frac{3}{2}\varepsilon$, and the right side is at most $\left|\frac{b}{a}\right| k \leq \left|\frac{b}{a}\right| k$ if $b \geq 0$ since it is maximized by $z = k$, and at most $\left|\frac{b}{a}\right| l$ if $b < 0$ since it is maximized by $z = -l$. Hence (2.2) is true if

$$k\left(1 - \frac{1}{2}\varepsilon\right) + k(1-\varepsilon) > \left|\frac{b}{a}\right| k,$$

which is true provided $\varepsilon < \frac{2}{3}\left(2 - \left|\frac{b}{a}\right|\right)$.

In particular, if $2 - \left|\frac{b}{a}\right| > \frac{3}{2}\varepsilon > 0$ then each strategy $y \geq k - \frac{1}{2}\varepsilon k$ is dominated by the strategy $x = k(1-\varepsilon)$. Fix an ε with $0 < \varepsilon < \frac{2}{3}(2 - \left|\frac{b}{a}\right|)$ and let the subset $S_{0,1}$ be the interval $\left[-l, k\left(1 - \frac{\varepsilon}{2}\right)\right]$. It has been shown that every $y \in S_{0,0} \setminus S_{0,1}$ is strictly dominated by $x = k(1-\varepsilon) \in S_{0,1}$.

Since k can have any positive value, the procedure can be repeated. Therefore, define $S_{0,i} = \left[-l, k\left(1 - \frac{\varepsilon}{2}\right)^i\right]$, and continue until $l \geq k\left(1 - \frac{\varepsilon}{2}\right)^{i_0}$ for some $i_0 \in \mathbb{N}$. Define $S_{1,0} = [-l, l]$

Similar calculations show that for the same value of ε the strategies $y \leq -l + \frac{1}{2}\varepsilon l$ are dominated by the strategy $x = -l(1-\varepsilon)$ and $y \geq l - \frac{1}{2}\varepsilon l$ are dominated by the strategy $x = l(1-\varepsilon)$. Accordingly, let the subset $S_{1,1}$ be the interval $\left[-l\left(1-\frac{\varepsilon}{2}\right), l\left(1-\frac{\varepsilon}{2}\right)\right]$. Each $y \in S_{1,0} \setminus S_{1,1}$ is strictly dominated by either $x = l(1-\varepsilon) \in S_{1,1}$ or $x = -l(1-\varepsilon) \in S_{1,1}$.

Since l can have any positive value, this procedure can also be repeated. Define $S_{1,i} = \left[-l\left(1-\frac{\varepsilon}{2}\right)^i, l\left(1-\frac{\varepsilon}{2}\right)^i\right]$. Since $(\bigcap_{i=1}^{i_0} S_{0,i}) \bigcap (\bigcap_{i=1}^{\infty} S_{1,i}) = \{0\}$, $\Gamma_{k,l}$ is SDS to 0. The same result can be obtained starting with $l > k$.

Conversely, assume $\Gamma_{k,l}$ is SDS to 0 and 0 is an NE, but that (2.1) fails. Theorem 2.8 shows that $a < 0$ and $d = 0$. Let $l \geq k$ and let S_i be a reduction sequence for $x^* = 0$.

Let S_j be the last subset of S including both k and $-k$. Suppose k is dominated by $w \in S_j$ with respect to S_j. Then $w < k$ and $\Pi(w,z) > \Pi(k,z)$ for all $z \in S_j$. In particular, if $z = 0$, we must have

$$aw^2 > ak^2.$$

Dividing both sides by a gives

$$w^2 < k^2,$$

that is, $|w| < |k|$.

Further, for strategy k to be strictly dominated, it must be true that $\Pi(w,z) > \Pi(k,z)$ for $z = k$. Expanding this inequality gives

$$aw^2 + bwk + ck^2 + ek + f > ak^2 + bk^2 + ck^2 + ek + f,$$

or more simply,

$$|a|(k^2 - w^2) > (bk)(k - w).$$

Because $k - w > 0$, $\Pi(w, k) > \Pi(k, k)$ is equivalent to

$$k + w > \frac{bk}{|a|}$$

or, because $2k > k + w$,

$$2 > \frac{1}{k}(k + w) > \frac{b}{|a|}.$$

The assumption that (2.1) is false then implies that $b < 0$.

However, for k to be dominated by w, it must also be true that $\Pi(w, -k) > \Pi(k, -k)$. Expanding and simplifying this inequality exactly as above produces

$$2 > \frac{1}{k}(w + k) > -\frac{b}{|a|}$$

which, by the same argument, implies $b > 0$, a contradiction. A similar contradiction can be reached by considering when the strategy $-k$ is dominated.

Hence the strict-dominance reduction sequence cannot proceed past S_j. Thus $\Gamma_{k,l}$ cannot be strictly dominance solvable to $x = 0$. □

This result is important as it gives a definite and precise requirement for a symmetric 2 player game with quadratic payoff functions on compact interval strategy spaces to be SDS to an interior Nash equilibrium. For games $\Gamma_{k,l}$ that satisfy the conditions of the above theorem, an interior Nash equilibrium is a very strong candidate for a solution to $\Gamma_{k,l}$ because there is no doubt as to its rationality. Here such games will be considered solved.

It is interesting to note that the conditions for such a game to be SDS to an interior NE rely only on the terms in the payoff function that concern a player's own strategy, and do not depend on those which concern only the strategic choice of the opponent. As will be shown below, this observation generalizes to other games.

2.1.2. *Boundary Symmetric NE.* The discussion in the previous section concerns cases where the game is strictly dominance solvable to a strategy in the interior the strategy space. The conditions are slightly relaxed for a game to be strictly dominance solvable to an endpoint.

Theorem 2.10. *Suppose Γ_k is a symmetric game with payoff $\Pi : S \times S \to \mathbb{R}$ given by $\Pi(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$ where a, b, c, d, e, and f are constants, and $S = [0, k]$ for some $k > 0$. Then:*

(1) *If $x = 0$ is a NE for all Γ_k, then $d \leq 0$ and $a \leq 0$.*

(2) *If $a = 0$ and Γ_k is SDS to 0 for every $k > 0$, then $d < 0$.*

Thus if Γ_k is SDS to $x = 0$ for all k, then $a \leq 0$ and $d \leq 0$ but $a = 0$ and $d = 0$ cannot both be true.

Proof. (1) Assume that $x = 0$ is a NE. This means that $\Pi(0, 0) \geq \Pi(y, 0)$, or equivalently, that $0 \geq y(ay + d)$. Since $y \geq 0$, $ay + d \leq 0$ for all $y \neq 0$. If $d > 0$, there exists a $0 < y \leq k$ such that $ay + d > 0$. Thus $d \leq 0$. Now assume that $a > 0$. Given the value of d, there can exist a positive value of y such that $ay+d>0$ given large enough k. Thus $a \leq 0$.

(2) The proof that a and d cannot both be equal to 0 is exactly the same as the analogous proof in Theorem 2.8. The fact that a cannot be greater than 0 follows from the first part of this theorem and the fact that, if Γ_k is SDS to 0, then 0 must be a NE.

□

Statements (1) and (2) in the above theorem hold only for the class of games on $S = [0, k]$ for any k. Given a fixed value for k or a limit on k neither is necessarily true (see Theorem 2.14 below). For now, the discussion continues to assume that the conditions of Theorem 2.10 are to hold for every positive value of k.

Lemma 2.11. *Let Γ_k be a symmetric 2 player game with continuous payoff function $\Pi : [0,k] \times [0,k] \to \mathbb{R}$ for any $k > 0$ where $\Pi(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$, $a = 0$ and b, c, d, e, and f are constants. Then Γ_k is SDS to 0 for all k if and only if $d < 0$ and $b \leq 0$.*

Proof. Assume that $a = 0$, $d < 0$ and $b \leq 0$. Consider $\Pi(0,z) > \Pi(x,z)$. This inequality expands and simplifies to

$$0 > bxz + dx.$$

This is true for all $x, z \in S = [0,k]$ where $x \neq 0$. Therefore any strategy $x \in S$ such that $x \neq 0$ is strictly dominated with respect to S and Γ_k is SDS to 0 in one step.

Now assume that Γ_k is SDS to 0. The proof that if $a = 0$ then $d < 0$ is one conclusion of Theorem 2.10. Note that Γ_k is SDS to 0 so there must exist some strict dominance in Γ_k. Thus there exist some $x, y \in S = [0, k]$ such that $\Pi(x, z) > \Pi(y, z)$ for any $z \in S$. This gives

$$bxz + cz^2 + dx + ez + f > byz + cz^2 + dy + ez + f$$

which simplifies to

$$bz(x - y) > d(y - x).$$

Recall that $d < 0$, and so

$$bz(x - y) > |d|(x - y).$$

Assume $x > y$. It then follows that $bz > |d|$ which is false when $z = 0$. Now assume that $x < y$, and so $bz < |d|$. This can be true for any value of $0 < z \leq k$ only if $b \leq 0$ since k is arbitrarily large. □

Now the necessary and sufficient conditions for a game of this class to be SDS to $x = 0$.

Theorem 2.12. *Let Γ_k be a symmetric 2 player game with continuous payoff function $\Pi : [0, k] \times [0, k] \to \mathbb{R}$ for any $k > 0$ where $\Pi(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$, and a, b, c, d, e, and f are constants. Then Γ_k is SDS to 0 for all k if and only if one of the following cases hold*

(1) $a < 0$, $d \leq 0$, *and* $2 > \frac{b}{|a|}$.

(2) $a = 0$, $d < 0$, *and* $b \leq 0$.

Proof. The proof of the sufficient conditions for strict-dominance solvability will be divided into several cases where it is shown that Γ_k is SDS to 0 in each case.

Case 1a: $a < 0$, $d = 0$, $b > 0$, and $2 > \frac{b}{|a|}$. This case follows from the proof for Theorem 2.9.

Case 1b: $a < 0$, $d = 0$, $b \leq 0$. Let $y \geq \frac{1}{2}k$. thus $\Pi(0, z) > \Pi(y, z)$ for all $z \in S$ since

$$\Pi(y, z) - \Pi(0, z) = ay^2 + byz \leq ay^2 < 0.$$

Thus let $S_1 = [0, \frac{1}{2}k)$. This is true for any positive value of k, so similarly define $S_i = [0, (\frac{1}{2})^i k)$. With this definition, as $i \to \infty$, $\max\{|x| : x \in S_i\} \to 0$. Therefore Γ_k is SDS to 0.

Case 1c: $a < 0$, $d < 0$, $b \leq 0$. Consider $\Pi(0, z) > \Pi(x, z)$, which reduces to $0 > ax^2 + byx + dx$. This is also true for any $x \in S$ where $x \neq 0$, and Γ_k is reduced to the strategy 0 in one step.

Case 1d: $a < 0$, $d < 0$, $b > 0$, and $2 > \frac{b}{|a|}$. Let $y \geq k\left(1 - \frac{1}{2}\varepsilon\right)$, with $\varepsilon > 0$ to be defined later. Consider $\Pi(k(1-\varepsilon), z) > \Pi(y, z)$. This expands to

(2.4)

$$a(k(1-\varepsilon))^2 + b(k(1-\varepsilon))z + cz^2 + dk(1-\varepsilon) + ez + f > ay^2 + byz + cz^2 + dy + ez + f$$

and then simplifies to

$$y^2 - (k(1-\varepsilon))^2 > \frac{bz + d}{|a|}(y - k(1-\varepsilon)).$$

Note that for any choice of $y \geq k\left(1-\frac{1}{2}\varepsilon\right)$, $y-k(1-\varepsilon)$ is positive. Dividing the previous inequality on both sides by $y-k(1-\varepsilon)$ yields

$$y+k(1-\varepsilon) > \frac{bz+d}{|a|}. \tag{2.5}$$

which is equivalent to (2.4). The inequality (2.5) can be shown to be true by noting that the left side is at least $k\left(1-\frac{1}{2}\varepsilon\right)+k(1-\varepsilon) = 2k-\frac{3}{2}\varepsilon = 2k-\frac{3}{2}\varepsilon k$, and for $z \in [0,k]$, the right side is at most $\frac{bk+d}{|a|}$, resulting in

$$k\left(1-\frac{1}{2}\varepsilon\right)+k(1-\varepsilon) > \frac{bk+d}{|a|}.$$

Since $d < 0$, this inequality is true if $1-\frac{1}{2}\varepsilon+1-\varepsilon \geq \frac{b}{|a|}$. (i.e. $\frac{3}{2}\varepsilon \leq 2-\frac{b}{|a|}$). Let ε =min$\{1, \frac{2}{3}(2-\frac{b}{|a|})\}$.

Thus each strategy $y \geq k\left(1-\frac{1}{2}\varepsilon\right)$ is strictly dominated by $x = k(1-\varepsilon)$ and a new subset $S_1 = \left[0, k\left(1-\frac{1}{2}\varepsilon\right)\right]$ is created. Again, the value of k can be any positive real number, and this allows the procedure to be repeated. Therefore define $S_i = \left[0, k\left(1-\frac{1}{2}\varepsilon\right)^i\right]$. Under this definition, as $i \to \infty$, $\max\{|x| : x \in S_i\} \to 0$. Thus Γ_k is strictly dominance solvable to 0.

Case 2: $a = 0$, $d < 0$, $b \leq 0$. Consider $\Pi(0,z) > \Pi(x,z)$, which when expanded and simplified is equivalent to $0 > bxz + dx$. This is true for any $x \in S$ where $x \neq 0$, and so 0 strictly dominates all other strategies in $S = [0,k]$, and Γ_k is solved in one step.

To show necessity, assume that both cases given in the theorem fail and 0 is a NE, but Γ_k is SDS to 0 for all k. Let S_i be the last subset to contain k as a strategy, and let $w \in S_{i+1}$ be a strategy which dominates k with respect to S_i. Therefore $\Pi(w,k) > \Pi(k,k)$. Expanding this inequality,

$$aw^2 + bwk + ck^2 + dw + ek + f > ak^2 + bk^2 + ck^2 + dk + ek + f.$$

From Theorem 2.10 $a \leq 0$ and $d \leq 0$. Consider when $a < 0$. Thus,

$$|a|(k^2 - w^2) > (bk + d)(k - w).$$

It is clear that $k - w > 0$, and so,

$$k + w > \frac{bk + d}{|a|}$$

Further, $k + w < 2k$, and thus

$$2 > \frac{1}{k}(k + w) > \frac{b}{|a|} + \frac{d}{k|a|}.$$

If k is allowed to grow arbitrarily large, then the $\frac{d}{k|a|}$ approaches 0 and so the contradiction

$$2 > \frac{b}{|a|} \geq 2,$$

is reached. This shows that k cannot be strictly dominated. Therefore the strict-dominance reduction sequence cannot proceed past S_i. Hence Γ_k cannot be SDS to $x = 0$.

If $a = 0$ then to be SDS to 0 as hypothesized, Lemma 2.11 proves that the payoff function Π of Γ_k satisfies the second case of the theorem. □

Now, with the results from Theorem 2.12 and those of Theorem 2.9, any 2 player symmetric game with quadratic payoff functions on continuous strategy spaces can be easily classified as either SDS or not SDS provided the game is being considered on a strategy space that can be of any size.

The two cases of Theorem 2.12 place restrictions on the values of a and d to ensure that Γ_k is SDS to 0 for any positive value k. If the value of k has a limit, however, then games which do not satisfy the conditions of Theorem 2.12 may yet be strictly dominance solvable to 0 given a small enough value of k. When a specific value of k is used, it will be denoted by $\underline{k}$ and assumed to be positive.

Corollary 2.13. *Let Γ be a symmetric 2 player game with continuous payoff function $\Pi : [0, \underline{k}] \times [0, \underline{k}] \longrightarrow \mathbb{R}$ for some fixed $\underline{k} > 0$ where $\Pi(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$ with $a < 0$, $d \leq 0$, and b, c, e, and f are constants. Then Γ is SDS to 0 if and only if $2 > \frac{b}{|a|} + \frac{d}{\underline{k}|a|}$.*

Theorem 2.14. *Let Γ be a symmetric 2 player game with continuous payoff function $\Pi : [0, \underline{k}] \times [0, \underline{k}] \longrightarrow \mathbb{R}$ where $\Pi(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$ with $d \leq 0$, $a = 0$, $b > 0$, and c, e, and f are constants. Then Γ is SDS to 0 if and only if*

$$\underline{k} < \left| \frac{d}{b} \right|$$

Proof. The proof that Γ is not strictly dominance solvable if $\underline{k} \geq \left|\frac{d}{b}\right|$ is provided above in the proof of Lemma 2.11, by following the line of reasoning there that assumes that there are strategies $x, y \in S$ such that $\Pi(x, z) > \Pi(y, z)$ for any $z \in S$. This inequality was shown to be equivalent to

$$bz(x - y) > |d|(x - y),$$

and further, that for Γ to be SDS to 0 that $x < y$. Thus $bz < |d|$. Hence, if $\underline{k} \geq \left|\frac{d}{b}\right|$, then

$$|d| = b\left|\frac{d}{b}\right| \leq bz < |d|,$$

Therefore $\underline{k} < \left|\frac{d}{b}\right|$.

Now consider $\Pi(0,z) > \Pi(x,z)$, which expands and simplifies to $0 > (bz + d)x$. This is always true for $z < \left|\frac{d}{b}\right|$ and $x \neq 0$, and so Γ can be reduced to the single strategy 0 in one step if $\underline{k} < \left|\frac{d}{b}\right|$. □

Lemma 2.15. *Suppose Γ is a symmetric 2 player game with continuous payoff function $\Pi : [0,\underline{k}] \times [0,\underline{k}] \to \mathbb{R}$ where $\Pi(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$ with $a > 0$, $d \leq 0$ and b, c, e, and f constant. Then $x = 0$ is a NE if and only if $\underline{k} \leq \left|\frac{d}{a}\right|$.*

Proof. Assume that $x = 0$ is a NE. Thus $\Pi(0,0) \geq \Pi(y,0)$ which is equivalent to $0 \geq y(ay+d)$. Since $y \geq 0$, it follows that $ay + d \leq 0$, or equivalently, $ay \leq |d|$ for all y. The left side is maximized at $y = \underline{k}$, and therefore, $\underline{k} \leq \left|\frac{d}{a}\right|$.

Conversely, if $\underline{k} \leq \left|\frac{d}{a}\right|$, then $ay + d \leq 0$ for all $y \in [0,\underline{k}]$ and so 0 is a NE. □

Theorem 2.16. *Let Γ be a symmetric 2 player game with continuous payoff function $\Pi : [0,\underline{k}] \times [0,\underline{k}] \to \mathbb{R}$ where $\Pi(x,y) = ax^2 + bxy + cy^2 + dx + ey + f$ with $a > 0$, $d \leq 0$, and $\underline{k} \leq \left|\frac{d}{a}\right|$, and where b, c, e, and f are constants. Then*

(1) *if $b \geq 0$, then Γ is SDS to 0 if and only if $(a+b)\underline{k} < |d|$.*

(2) *if* $b < 0$, *then* Γ *is SDS to* 0 *if and only if* $a\underline{k} < |d|$.

Proof. (1) Assume that $a > 0$, $b \geq 0$, $d \leq 0$, $\underline{k} \leq \left|\frac{d}{a}\right|$, and $(a+b)\underline{k} < |d|$. Consider the inequality $\Pi(0,z) - \Pi(x,z) > 0$. Expanded and simplified, this inequality is equivalent to

$$-x(ax + bz + d) > 0. \tag{2.6}$$

Since $(ax + bz + d) < 0$ for all $x, z \in S$, (2.6) is true for all $x \in S$ where $x \neq 0$. Therefore all strategies x are strictly dominated and can be eliminated in one step, and thus Γ is SDS to 0.

Now assume that $(a+b)\underline{k} \geq |d|$ and that Γ is SDS to 0. Let S_i be the last subset to contain $\underline{k}$ and let $w \in S_{i+1}$ be the strategy which strictly dominates $\underline{k}$ with respect to S_i. Therefore $\Pi(w,z) - \Pi(\underline{k},z) > 0$ for all $z \in S_i$. Expanding and simplifying this inequality yields

$$a(w^2 - \underline{k}^2) + (bz + d)(w - \underline{k}) > 0.$$

Dividing through by $(w - \underline{k}) < 0$ then makes the inequality equivalent to

$$a(w + \underline{k}) + bz + d < 0.$$

The left side is maximized when $z = \underline{k}$, giving

$$aw + (a+b)\underline{k} + d < 0,$$

a contradiction. Thus the strategy $\underline{k}$ can never be strictly dominated. Hence the strict domination reduction sequence cannot proceed past S_i and thus Γ cannot be SDS to 0.

(2) Assume that $a > 0$, $b < 0$, $d \leq 0$, and $\underline{k} \leq \left|\frac{d}{a}\right|$. In the same manner as the proof in part (1), consider the inequality $\Pi(0, z) - \Pi(x, z) > 0$. Expanded and simplified, this inequality is equivalent to

$$-x(ax + bz + d) \geq -x(ax + d) > 0, \tag{2.7}$$

which is true for all $x \in S$, where $x \neq 0$. Therefore all strategies x are strictly dominated and can be eliminated in one step, and thus Γ is SDS to 0.

The proof that Γ is not SDS to 0 if $\underline{k} > \left|\frac{d}{a}\right|$ follows from the fact that for such a $\underline{k}$, Lemma 2.15 proves 0 is not a NE. Hence Γ cannot be SDS to 0.

Now assume that Γ is SDS to 0 and $\underline{k} = \left|\frac{d}{a}\right|$. Then $\Pi(x, y) = ax^2 + bxy + cy^2 - a\underline{k}x + ey + f$. Let S_i be the last subset to contain the strategy $\underline{k}$ and let $w \in S_{i+1}$ be the strategy which strictly dominates it with respect to S_i. Consider $\Pi(w, 0) > \Pi(\underline{k}, 0)$, which when expanded and simplified, gives

$$aw^2 > a\underline{k}^2$$

Dividing the left side by $a > 0$ makes the inequality equivalent to

$$w^2 > \underline{k}^2,$$

a contradiction. Therefore $\underline{k}$ cannot be strictly dominated and so the strict-dominance reduction sequence cannot progress past S_i. Thus Γ is not SDS to 0.

□

The following corollary summarizes the results from Corollary 2.13 to Theorem 2.16 which require that $\underline{k}$ be limited to certain values. Note that the first case of the corollary is slightly relaxed from its counterpart in Theorem 2.10. Also note that as in the case of interior NE, these conditions apply to the terms of the payoff function which depend on a player's own strategy and ignore those terms which only depend on the other player's.

Corollary 2.17. *Let Γ be a symmetric 2 player game with continuous payoff function $\Pi : [0, \underline{k}] \times [0, \underline{k}] \to \mathbb{R}$ for some fixed $\underline{k} > 0$ where $\Pi(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$ and a, b, c, d, e, and f are constants. Then Γ is SDS to 0 if and only if one of the following cases hold*

(1) $a < 0$, $d \leq 0$, *and* $2 > \frac{b}{|a|} + \frac{d}{\underline{k}|a|}$.

(2) $a = 0$, $d < 0$, *and* $b \leq 0$.

(3) $a = 0$, $d < 0$, $b > 0$, *and* $\underline{k} < \left|\frac{d}{b}\right|$.

(4) $a > 0$, $d \leq 0$, $b \leq 0$, *and* $\underline{k} \leq \left|\frac{d}{a}\right|$.

(5) $a > 0$, $d \leq 0$, $b \geq 0$, $(a + b)\underline{k} < |d|$.

The above results from Theorem 2.10 to Corollary 2.17 for games on strategy spaces of the type $S = [0, k]$ can easily be shown for games on strategy spaces of the type $S = [k, 0]$ with a minor change to the theorems. In each case replace the condition $d \leq 0$ with $d \geq 0$ and $d < 0$ with $d > 0$. This gives rise to the following theorem.

Theorem 2.18. *Fix $k, l > 0$ and let $\Gamma_{k,l}$ be a symmetric 2 player game with continuous payoff function $\Pi : [-l, k] \times [-l, k] \to \mathbb{R}$ where $\Pi(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$ and a, b, c, d, e, and f are constants. Then $\Gamma_{k,l}$ restricted to $[0, k] \times [0, k]$ and to $[-l, 0] \times [-l, 0]$ are both SDS to 0 if and only if $a < 0$, $d = 0$, and $2 > \frac{b}{|a|}$.*

Proof. For $\Gamma_{k,l}$ to be SDS to 0 on both $[0, k]$ and $[-l, 0]$ for any $k, l > 0$ it must be true that $d = 0$. Since $d = 0$, the only case that applies is part (1) of Theorem 2.12. Therefore $a < 0$, $d = 0$ and $2 > \frac{b}{|a|}$ if $\Gamma_{k,l}$ is SDS to 0 for both $[-l, 0]$ and $[0, k]$. Conversely, if $a < 0$, $d = 0$ and $2 > \frac{b}{|a|}$, then part (1) of Theorem 2.12 implies that $\Gamma_{k,l}$ is SDS to 0 for both $[-l, 0]$ and $[0, k]$. □

This result is interesting because it is identical with the conditions for $x = 0$ to be a *continuously stable strategy* (CSS) for the symmetric game [6], as will be shown.

Definition 2.19. A strategy $x^* \in S$ is a CSS if the following conditions are satisfied.

(1) $\Pi(x, x^*) \leq \Pi(x^*, x^*)$ for any $x \in S$.

(2) If $\Pi(x, x^*) = \Pi(x^*, x^*)$ then $\Pi(x, x) < \Pi(x^*, x)$ for any $x \in S$ where $x \neq x^*$.

(3) There exists a $\varepsilon > 0$ such that for any $y \in S$ with $|y - x^*| < \varepsilon$ there exists a $\delta > 0$ such that for all $z \in S$ with $|y - z| < \delta$, $\Pi(z, y) > \Pi(y, y)$ if and only if $|z - x^*| < |y - x^*|$.

The first two conditions are known as the *evolutionarily stable strategy* (ESS) conditions [9], and a strategy that satisfies both of them is know as an ESS. The third condition, also known as convergent stability, ensures that if a strategy y is close to an ESS strategy x^* then, for z close to y, z will perform better against y than y will against itself, if and only if z is closer to x^* than y.

Theorem 2.20. *Let $\Gamma_{k,l}$ be a symmetric 2 player game with continuous payoff function $\Pi : [-l, k] \times [-l, k] \rightarrow \mathbb{R}$ where $\Pi(x, y) = ax^2 + bxy + cy^2 + dx + ey + f$ and a, b, c, d, e, and f are constants. If $\Gamma_{k,l}$ restricted to $[0, k] \times [0, k]$ and to $[-l, 0] \times [-l, 0]$ are both SDS to 0 for any $k, l > 0$ then $x^* = 0$ is a CSS, and conversely.*

Proof. By Theorem 2.18, $a < 0$. Thus $\Pi(x, x^*) = ax^2 \leq 0 = \Pi(x^*, x^*)$ for all $x \in S$. Furthermore, $\Pi(x, x^*) < \Pi(x^*, x^*)$ if $x \neq x^*$, and so ESS conditions (1) and (2) are satisfied.

Let $x' \in S$ be a strategy which is slightly closer than x to $x^* = 0$, defined as $x' = x + h(x^* - x)$ where $h > 0$ and $h \cong 0$. Consider $\Pi(x', x) > \Pi(x, x)$. Expanded, this inequality is

$$a((1-h)x + hx^*)^2 + b((1-h)x + hx^*)x > ax^2 + bx^2. \tag{2.8}$$

The hx^* term can be omitted because $x^* = 0$. Thus we have

$$a(1-h)^2x^2 + b(1-h)x^2 > ax^2 + bx^2.$$

Expanding this inequality gives

$$a(h-2)hx^2 > bhx^2,$$

which is equivalent to (2.8) Divide both sides by $hx^2 > 0$ and expand to reach

$$ah + 2|a| = ah - 2a > b.$$

Thus, if $2|a| > b$, $\Pi(x', x) > \Pi(x, x)$ for all x' close to x. By similar reasoning with $h < 0$, $\Pi(x', x) < \Pi(x, x)$ if x' is close to x but farther from x^* than x. That is, if $2 > \frac{b}{|a|}$, then $x^* = 0$ satisfies (3), and thus, x^* is a CSS.

On the other hand, if $2 \leq \frac{b}{|a|}$, then $ah - 2a < b$ for all $h > 0$. Thus $\Pi(x', x) < \Pi(x, x)$ if x' is close to x and closer than x to x^*. That is, x^* is not a CSS □

Theorems 2.18 and 2.20 provide further evidence that, for a game that satisfies their conditions, the prediction of rational behaviour provided by strict-dominance is an excellent choice. In addition to its already robust nature, due to it being a Nash equilibrium and a survivor of any strict-dominance reduction sequence, it is also evolutionarily and continuously stable should the game be played more than once.

2.1.3. *The Snowdrift Game Revisited.* The continuous Snowdrift game, as noted earlier, can be SDS, but this behaviour depends on the form of the cost and benefit functions $B(x)$ and $C(x)$. The strategy set can be defined as $S = [0, 1]$, where $x = 0$ implies no cooperation at all and $x = 1$ implies total cooperation. If these two functions are quadratic, that is that $B(x) =$

$b_2x^2 + b_1x$ and $C(x) = c_2x^2 + c_1x$, then $\Pi(x,y) = B(x+y) - C(x) = (b_2 - c_2)x^2 + 2b_2xy + b_2y^2 + (b_1 - c_1)x + b_1y$. Thus, to compare this function with the results of the previous section, note that $a = (b_2 - c_2)$, $b = 2b_2$, $c = b_2$, $d = (b_1 - c_1)$, $e = b_1$ and $f = 0$. Therefore, by Theorem 2.12, it is clear that the game will be SDS to 0 if either

(1) $b_2 < c_2$, $b_1 \leq c_1$, and $|b_2 - c_2| > b_2$ or

(2) $b_2 = c_2$, $b_1 < c_1$, and $b_2 \leq 0$.

is true.

Thus example (d) of the continuous SD game in [4] can be shown to be SDS to 0 as well as being evolutionarily and dynamically stable. In the example, $b_2 = -1.5$, $b_1 = 7$, $c_2 = -1$ and $c_1 = 8$, which satisfy the first of the above conditions.

Example (e) from the same text shows that by simply changing one of the linear values, namely from $c_1 = 8$ to $c_1 = 2$ results not only in the game being evolutionarily stable to the opposite endpoint, but also removes the quality of strict-dominance solvability from the game.

Example (b), which represents an interior ESS [4] at $x = 0.6$, is also strict dominance solvable to $x = 0.6$ when considered on the strategy spaces $[0, 0.6]$ and $[0.6, 1]$. This can be seen by taking the original payoff function with values $b_2 = -1.5$, $b_1 = 7$, $c_2 = -1$ and $c_1 = 4.6$ resulting in $\Pi(x,y) = -0.5x^2 - 3xy - 1.5y^2 + 2.4x + 7y$ and then translating the coordinates by $x = \hat{x} + 0.6$ and $y = \hat{y} + 0.6$ to create the function $\hat{\Pi}(\hat{x},\hat{y}) = -0.5(\hat{x} + 0.6)^2 - 3(\hat{x} + 0.6)(\hat{y} + 0.6) - 1.5(\hat{y} + 0.6)^2 + 2.4(\hat{x} + 0.6) + 7(\hat{y} + 0.6) =$

$-0.5\hat{x}^2 - 3\hat{x}\hat{y} - 1.5\hat{y}^2 + 3.4\hat{y} + 3.84$ that has a NE at $\hat{x} = 0$. With values $a = -0.5$, $b = -3$ and $d = 0$, it then becomes clear that Theorem 2.18 is satisfied, and so by Theorem 2.20, $\hat{x} = 0$ is a CSS.

2.1.4. *A Game Solved in a Finite Number of Steps.* A closer look at the proofs in this thesis for when Γ is SDS to 0 on a compact interval reveals that the games are solved in either one step or in a countably infinite number of steps.

From the above results, it may appear that games are only solvable in either one step or in a countably infinite number of steps. Below is an example of a symmetric game which is strict-dominance solvable in five steps.

Example 2.21. Let Γ be the symmetric game on $S = [-1, 2]$ with payoff function $\Pi : S \times S \to \mathbb{R}$ defined by

$$\Pi(x, y) = \begin{cases} -x^2 & x \in [-1, 1], y \in [-1, 2]. \\ -\frac{1}{3}x^2 - \frac{1}{2}(x-1)y - \frac{2}{3} & x \in (1, 2], y \in [-1, 2]. \end{cases}$$

Note that $\Pi(x, y)$ is continuous in (x, y). Initially, this game can be reduced in the manner outlined in Theorem 2.9 with $\varepsilon = \frac{1}{3}$. Thus at each iteration $S_i = (-1, k(1 - \frac{1}{6})^i) = (-1, 2(\frac{5}{6})^i)$. After the third iteration, the strategy space is $S_3 = (-1, \frac{125}{108})$. Now consider $\Pi(1, z) > \Pi(y, z)$ for all z with $y > 1$ and $y, z \in S_3$. The inequality $\Pi(1, z) > \Pi(y, z)$ is equivalent to

$$-1 > -\frac{1}{3}y^2 - \frac{1}{2}(y-1)z - \frac{2}{3}$$

which simplifies to

$$-\frac{1}{3}(1-y^2) > \frac{1}{2}z(1-y).$$

When both sides are divided by $1-y<0$, $\Pi(1,z) > \Pi(y,z)$ is shown to be equivalent to

$$-\frac{1}{3}(1+y) < \frac{1}{2}z$$

Since $y>1$ is required, the left side of the last inequality must be less than $-\frac{1}{3}(1+1) = -\frac{2}{3}$. Minimize the right hand side by substituting $z = -\frac{125}{108}$. Thus,

$$-\frac{2}{3} < -\frac{125}{216},$$

and therefore the strategy $x=1$ strictly dominates any such strategy $y \in S_3$, and $S_4 = [-1,1]$ is found. In this fourth subset, notice that $\Pi(0,z) > \Pi(x,z)$ for all $x, z \in S_4$ and so the strategy space is reduced to 0 in one further step.

2.2. **Higher Order Payoff Functions in Symmetric Games.** Even in the well behaved and simplified case of symmetric games with one-dimensional strategy spaces, the introduction of cubic or higher order terms quickly adds complications.

2.2.1. *Interior Symmetric NE.* Similar to the above section on interior symmetric NE, this section assumes that the strategy set S is of the form $S=[-l,k]$ for some $l,k>0$ and that the NE is $x^*=0$.

Theorem 2.22. *Fix $k,l>0$. Let $\Gamma_{k,l}$ be a symmetric 2 player game with continuous payoff function $\Pi : S \times S \to \mathbb{R}$ given by $\Pi(x,y) = ax^2 + bxy +$*

$cy^2 + dx + ey + f + gx^3 + hx^2y + ixy^2 + jy^3$ where a, b, c, d, e, f, g, h, i and j are constants, and $S = [-l, k]$ for any $k, l > 0$. Then:

(1) *$x^* = 0$ is a NE for all $\Gamma_{k,l}$ if and only if $d = g = 0$ and $a \leq 0$.*

(2) *If $k, l > 0$ are fixed and $a = d = g = 0$, then there are no $x, y \in S$ such that x strictly dominates y in $\Gamma_{k,l}$.*

Thus if $\Gamma_{k,l}$ is SDS to $x^* = 0$ for all k, l, then the NE is $x = 0$, $d = g = 0$, and $a < 0$.

Proof. (1) Assume that $x = 0$ is a NE, and thus $\Pi(0, 0) \geq \Pi(x, 0)$. This inequality is equivalent to

$$0 \geq x(ax + d + gx^2). \tag{2.9}$$

For small values of $|x|$, if $d \neq 0$, the d term dominates the other terms in (2.9), and the only value which guarantees that $0 \geq dx$ is $d = 0$. Assuming $d = 0$ transforms (2.9) to

$$0 \geq x^2(a + gx), \tag{2.10}$$

where now for small values of $|x|$, if $a \neq 0$, the a term dominates the g term. Thus to guarantee $0 \geq ax^2$ it follows that $a \leq 0$. However, for large values of $|x|$, the g term dominates the a term in (2.10), and so if $g \neq 0$ there is no value of g which can guarantee that $0 \geq x^2(a + gx)$. Therefore $g = 0$.

Now assume that $a \leq 0$ and $d = g = 0$, and examine $\Pi(0, 0) \geq \Pi(x, 0)$. This inequality is equivalent to $0 > ax^2$ which is true for any $x \in S$. Therefore, $x = 0$ is a NE.

(2) Assume that $a = d = g = 0$, and that there exists some $x, y \in S$ such that $\Pi(x, z) > \Pi(y, z)$ for all $z \in S$. This inequality expands to

$$bxz+cz^2+ez+f++hx^2z+ixz^2+jz^3 > byz+cz^2+ez+f+hy^2z+iyz^2+jz^3$$

and simplifies to

$$bxz + hx^2z + ixz^2 > byz + hy^2z + iyz^2.$$

Substituting $z = 0$ into this inequality yields $0 > 0$, an obvious contradiction.

The remainder of the proof follows from Corollary 2.7. □

Theorem 2.23. *Fix $k, l > 0$. Let $\Gamma_{k,l}$ be a symmetric 2 player game with continuous payoff function $\Pi : S \times S \to \mathbb{R}$ given by $\Pi(x, y) = ax^2 + bxy + cy^2 + dx + ey + f + gx^3 + hx^2y + ixy^2 + jy^3$ where $a, b, c, d, e, f, g, h, i$ and j are constants and $S = [-l, k]$. If $a < 0$, $d = g = 0$, and $\left|\frac{b}{a}\right| < 2$ then $\Gamma_{k,l}$ is SDS to 0 if l and k are both positive and sufficiently close to 0.*

Proof. Assume that $\left|\frac{b}{a}\right| < 2$, $a < 0$ and $d = g = 0$. Thus by the above Theorem, $x = 0$ is a NE. Without loss of generality, assume that $k \geq l$. Define $S = S_{0,0}$ and let $y \geq k - \frac{1}{2}\varepsilon k$, where $1 > \varepsilon > 0$ will be chosen later. Examine the inequality $\Pi(k(1-\varepsilon), z) > \Pi(y, z)$. for this to be true for every $y, z \in S$, it follows that

(2.11)

$$a(k(1-\varepsilon))^2+bk(1-\varepsilon)z+h(k(1-\varepsilon))^2z+ik(1-\varepsilon)z^2 > ay^2+byz+hy^2z+iyz^2,$$

or equivalently,

$$y^2 - (k(1-\varepsilon))^2 > \frac{bz + iz^2}{|a| - hz}(y - k(1-\varepsilon)). \tag{2.12}$$

It is true that $y - k(1-\varepsilon) > 0$, so if both sides of the above inequality are divided by a factor of $y - k(1-\varepsilon)$, (2.12) becomes

$$y + k(1-\varepsilon) > \frac{bz + iz^2}{|a| - hz}. \tag{2.13}$$

If the values of l and k are small enough so that b is the dominant term in the numerator of the fraction in 2.13 and $|a|$ is the dominant term in the denominator, this can be rewritten as

$$y + k(1-\varepsilon) > \frac{b}{|a|}z.$$

The remainder of the proof then follows exactly that of Theorem 2.9 starting at inequality (2.3). □

Theorem 2.24. *Let $\Gamma_{k,l}$ be a symmetric 2 player game with continuous payoff function $\Pi : S \times S \to \mathbb{R}$ given by $\Pi(x,y) = ax^2 + bxy + cy^2 + dx + ey + f + gx^3 + hx^2y + ixy^2 + jy^3$ where $a, b, c, d, e, f, g, h, i$ and j are constants and $S = [-l, k]$ for any $k, l > 0$. If $\left|\frac{b}{a}\right| > 2$, then $\Gamma_{k,l}$ is not SDS to 0.*

Proof. Choose $0 < \hat{k} \leq \min(k, l)$ where $\hat{k}$ is small and to be determined below. Assume that $\Gamma_{k,l}$ is SDS to 0, and so, that 0 is a NE. Thus by Theorem 2.22, $d = g = 0$, and $a < 0$. Let $l \geq k$, and S_i be a reduction sequence for $x^* = 0$. Let S_j be the last subset that contains both $\hat{k}$ and $-\hat{k}$. For $\Gamma_{k,l}$ to be SDS to 0, it must be true that there is a strategy $w \in S_{i+1}$

which strictly dominates one of the strategies $\hat{k}$ or $-\hat{k}$ with respect to S_i. Assume it is $\hat{k}$. Therefore $\Pi(w, z) > \Pi(\hat{k}, z)$ for all $z \in S_j$. Let $z = 0$. The inequality then expands and simplifies to

$$aw^2 > a\hat{k}^2,$$

and dividing both sides by a then gives

$$w^2 < \hat{k}^2.$$

Therefore, $|w| < \hat{k}$.

It also follows that $\Pi(w, \hat{k}) > \Pi(\hat{k}, \hat{k})$. Expanding and simplifying this inequality shows

$$aw^2 + bw\hat{k} + hw^2\hat{k} + iw\hat{k}^2 > a\hat{k}^2 + b\hat{k}^2 + h\hat{k}^3 + i\hat{k}^3.$$

This inequality is equivalent to

$$(a + h\hat{k})(w^2 - \hat{k}^2) > (b\hat{k} + i\hat{k}^2)(\hat{k} - w).$$

Note that $\hat{k} - w > 0$, and so dividing both sides of the inequality gives

$$(|a| - h\hat{k})(\hat{k} + w) > b\hat{k} + i\hat{k}^2$$

Given small enough $\hat{k}$ it is true that $|a| - h\hat{k} > 0$, and thus dividing both sides of the previous inequality by that value gives

$$\hat{k} + w > \frac{b\hat{k} + i\hat{k}^2}{|a| - h\hat{k}}$$

It is clear that $2\hat{k} > \hat{k} + w$ and so

$$2\hat{k} > \frac{b\hat{k} + i\hat{k}^2}{|a| - h\hat{k}}$$

or, by dividing through by k,

$$2 > \frac{b + i\hat{k}}{|a| - h\hat{k}}.$$

Since $\lim_{\hat{k}\to 0} \frac{b+i\hat{k}}{|a|-h\hat{k}} = \frac{b}{|a|}$, when $\hat{k}$ is sufficiently small, it must be that $2 \geq \frac{b}{|a|}$ for w to strictly dominate such a $\hat{k}$. Similarly, to eliminate $\hat{k}$ as a strictly dominated strategy it must be that $\Pi(w, -\hat{k}) > \Pi(\hat{k}, -\hat{k})$ and from this inequality that $2 \geq \frac{-b}{|a|}$ for sufficiently small $\hat{k}$. The same inequalities must hold if $-\hat{k}$ is to be eliminated for sufficiently small $\hat{k}$. Therefore the strict dominance reduction sequence cannot proceed past S_i and thus $\Gamma_{k,l}$ is not SDS to 0. □

Corollary 2.25. *Let $\Gamma_{k,l}$ be a symmetric 2 player game with continuous payoff function $\Pi : S \times S \to \mathbb{R}$ where the coefficient of the x^2 term is a and the coefficient of the xy term is b where a and b are constants and $S = [-l, k]$ for any $k, l > 0$. If $a \neq 0$ and $|\frac{b}{a}| > 2$ then $\Gamma_{k,l}$ is not SDS to 0. On the other hand, if $a \neq 0$ and $\left|\frac{b}{a}\right| < 2$, then $\Gamma_{k,l}$ is SDS to 0 if $k, l > 0$ are sufficiently small.*

The proof of the corollary follows the same structure as the above proofs, relying on the fact that all terms besides those involving a and b will disappear for small values of k.

The results in this section on higher order payoff functions in relation to quadratic payoff functions in Section 2.1 when 0 is in the interior of S can be summarized in this fashion. By Corollary 2.25, the strict-dominance solvability of $\Gamma_{k,l}$ to 0 for small $k, l > 0$ is characterized through the quadratic terms of the payoff functions except in the threshold cases $a = 0$ or $\left|\frac{b}{a}\right| = 2$. That is, by restricting the strategy space to be near 0, higher order terms can be ignored when examining strict-dominance solvability to 0 (except in threshold cases).

2.3. **Symmetric Games with Higher Dimensional Strategy Spaces.** All of the games which have been considered thus far have had one dimensional strategy spaces for each player. In fact, it is the same strategy space for each player since the games considered have been symmetric. This is another restriction which can be relaxed to examine when a broader class of games is strict-dominance solvable. As always, the first consideration is to see when games of this class have a Nash equilibrium. In the discussion below, note that $\overline{x}$ is a column vector and D and E are row vectors and so expressions such as $D\overline{x}$ are the inner product of those two vectors.

Theorem 2.26. *Let Γ be a symmetric 2 player game with continuous payoff function $\Pi : S \times S \to \mathbb{R}$ given by $\Pi(\overline{x}, \overline{y}) = \overline{x}^T A\overline{x} + \overline{x}^T B\overline{y} + \overline{y}^T C\overline{y} + D\overline{x} + E\overline{y} + F$ where S is an n dimensional strategy space which has $\overline{0}$ in the interior and where A, B, and C are constant $n \times n$ matrices, D and E are constant n-dimensional vectors, and F is a constant. The vector $\overline{0}$ is a NE if and only if A is negative semi-definite and $D = \overline{0}$. Further, if Γ is SDS to $\overline{0}$ then A is negative definite.*

Proof. Assume that $\overline{0}$ is a NE. Thus $\Pi(\overline{0}, \overline{0}) \geq \Pi(\overline{x}, \overline{0})$ for any $\overline{x} \in S$. This inequality, when expanded and simplified is equivalent to $0 \geq \overline{x}^T A\overline{x} + D\overline{x}$. For small values of $\overline{x}$ the $D\overline{x}$ term dominates the right side of the inequality, and the only value of D which guarantees $0 \geq D\overline{x}$ is $D = \overline{0}$. Therefore, the inequality becomes $0 \geq \overline{x}^T A\overline{x}$ for all $x \in S$. Since 0 is in the interior of S, this implies $0 \geq \overline{x}^T A\overline{x}$ for all $\overline{x} \in \mathbb{R}^n$, which is the definition of A being a negative semi-definite matrix. The remainder of the proof follows from Theorem 2.5, which states that if Γ is SDS to $\overline{0}$ then it must be a SNE, and thus that $0 > \overline{x}^T A\overline{x}$ for all $\overline{x} \neq 0$, hence A is a negative definite matrix. □

The addition of other dimensions to the strategy space quickly compounds the difficulty in finding clear conditions for a game to be SDS. The following two examples show that the class of games with A negative definite and $D = \overline{0}$ contain examples that are SDS to 0 and others that are not SDS to 0 depending on the matrix B.

Example 2.27. Let Γ be the 2 player symmetric game with strategy space $S \times S$ where $S = \{(x, y)|x^2 + y^2 \leq 1\}$ and symmetric payoff functions $\Pi((\hat{x}, \hat{y}), (x, y)) = (\hat{x}, \hat{y})^T A(\hat{x}, \hat{y}) + (\hat{x}, \hat{y})^T B(x, y)$ where $A = -I$ and $B = I$. Then Γ is SDS to $\overline{0} = (0, 0)$.

Proof. Let $x_0^2 + y_0^2 = 1$. Then $\Pi((1-\varepsilon)(x_0, y_0), (x, y)) - \Pi((x_0, y_0), (x, y))$ is equivalent to

$$-(1-\varepsilon)^2(x_0^2 + y_0^2) + (1-\varepsilon)(xx_0 + yy_0) - [x_0^2 + y_0^2 + xx_0 + yy_0]$$

which simplifies to

$$(2\varepsilon - \varepsilon^2) - \varepsilon(x, y) \cdot (x_0, y_0)$$

Therefore $\Pi((1-\varepsilon)(x_0, y_0), (x, y)) - \Pi((x_0, y_0), (x, y)) \geq \varepsilon - \varepsilon^2 > 0$ if $0 < \varepsilon < 1$ since $(x, y) \cdot (x_0, y_0) \leq \|(x, y)\| \|(x_0, y_0)\| = \sqrt{x^2 + y^2} \leq 1$. Thus all

(x,y) near the boundary of S can be eliminated in one step to form a new $S_1 = \{(x,y)|x^2+y^2 \leq r\}$ for some $0 < r < 1$. By iterating this procedure a reduction sequence is formed, with $S_n = \{(x,y)|x^2+y^2 \leq r^n\}$ and so $\bigcap_{n=1}^{\infty} S_n = \{(0,0)\}$. □

The above clearly shows that some games with higher dimensional strategy spaces are SDS. This is not always the case, even if A is negative definite and $D = \overline{0}$, as the following counterexample shows.

Example 2.28. Let Γ be the 2 player symmetric game with strategy space $S \times S$ where $S = \{(x,y)|x^2+y^2 \leq 1\}$ and symmetric payoff functions $\Pi((\hat{x},\hat{y}),(x,y)) = (\hat{x},\hat{y})^T A(\hat{x},\hat{y}) + (\hat{x},\hat{y})^T B(x,y)$ where $A = -I$ and $B = 2I$. Then Γ is not SDS to $\overline{0}$.

Proof. Suppose that $x_0^2 + y_0^2 = 1$. For Γ to be SDS the strategy (x_0,y_0) must be strictly dominated in some subset of S. Therefore assume that there exists a $(x,y) \in S_{i+1}$ such that $\Pi((x,y),(\hat{x},\hat{y})) > \Pi((x_0,y_0),(\hat{x},\hat{y}))$ for all $(\hat{x},\hat{y}) \in S_i$. It follows then that $\Pi((x,y),(x_0,y_0)) > \Pi((x_0,y_0),(x_0,y_0))$. Expanding and simplifying this inequality shows

$$-(x^2+y^2) + 2(x,y)\cdot(x_0,y_0) > 1.$$

Recall that $(x,y)\cdot(x_0,y_0) \leq \|(x,y)\|\|(x_0,y_0)\| = \sqrt{x^2+y^2}$. thus, $-(x^2+y^2) + 2(x,y)\cdot(x_0,y_0) \leq -(x^2+y^2) + 2\sqrt{x^2+y^2} = \sqrt{x^2+y^2}(2-\sqrt{x^2+y^2})$. Since $\sqrt{x^2+y^2}(2-\sqrt{x^2+y^2}) \leq 1$ for all $0 \leq \sqrt{x^2+y^2} \leq 1$, $\Pi((x,y),(x_0,y_0)) \leq \Pi((x_0,y_0),(x_0,y_0))$ for all $(x,y) \in S$. That is, no such (x_0,y_0) can be eliminated as a strictly dominated strategy and so Γ is not SDS to $\overline{0}$. □

It is then clear that there must be some restrictions on a game of this class which apply to the values of A and B in the payoff functions. The exact requirement may be a strict inequality that generalizes the scalar inequality $\left|\frac{b}{a}\right| < 2$ to one involving matrices A and B, or may have requirements at a boundary which cannot be easily classified. This avenue of research warrants further investigation.

3. Asymmetric Games

Until now, all of the games considered have been symmetric. The following sections will deal with two-player games which have asymmetric payoff functions. A two-player asymmetric game Γ is defined by pure strategy sets S and T for Players 1 and 2 respectively, with payoff function $\Pi_1 : S \times T \to \mathbb{R}$ for Player 1 and $\Pi_2 : S \times T \to \mathbb{R}$ for Player 2. Similarly to a symmetric game, in a two-player asymmetric game a strategy pair $x \in S$ strictly dominates a strategy $\hat{x} \in S$ with respect to S if and only if $\Pi_1(x, z) > \Pi_1(\hat{x}, z)$ for all $z \in T$, and a strategy $y \in T$ strictly dominates a strategy $\hat{y} \in T$ with respect to T if and only if $\Pi_2(w, y) > \Pi_2(w, \hat{y})$ for all $w \in S$.

Definition 3.1. An asymmetric two-player game Γ, with payoff functions Π_1 and Π_2 and pure strategy sets S and T is *strict-dominance solvable* (SDS) to the strategy pair (x^*, y^*) if there is a finite or countable number of subsets of $S \times T$, $S \times T = U_0 \supset U_1 = S_1 \times T_1 \supset U_2 = S_2 \times T_2 \supset \ldots \supset U_i = S_i \times T_i \supset \ldots$, called a *reduction sequence* for (x^*, y^*), such that

(1) U_{i+1} is a proper subset of U_i.

(2) for every $x \in S_i \setminus S_{i+1}$ there is a $\hat{x} \in S_{i+1}$ such that $\hat{x}$ strictly dominates x with respect to S_i and for every $y \in T_i \setminus T_{i+1}$ there is a $\hat{y} \in T_{i+1}$ such that $\hat{y}$ strictly dominates y with respect to T_i.

(3) $\bigcap U_i = \{(x^*, y^*)\}$.

Example 3.2. Consider the game in Example 1.5. By the above definition, it is SDS to $\{(1,3)\}$ with $U_0 = \{(1,2,3) \times (1,2,3)\}$, $U_1 = \{(1,3) \times (1,2,3)\}$, $U_2 = \{(1,3) \times (3)\}$, and $U_3 = \{(1) \times (3)\}$.

Definition 3.3. A strategy pair $(x, y) \in S \times T$ is called a *Nash equilibrium* (NE) if $\Pi_1(x, y) \geq \Pi_1(\hat{x}, y)$ for all $\hat{x} \in S$ and $\Pi_2(x, y) \geq \Pi_2(x, \hat{y})$ for all $\hat{y} \in T$. If both of the previous inequalities are strict where $\hat{x} \neq x$ and $\hat{y} \neq y$, the strategy pair (x, y) is called a *strict Nash equilibrium* (SNE).

Theorem 3.4. *Let Γ be an asymmetric game that is SDS to a NE $(x, y) \in S \times T$. Then (x, y) is a SNE.*

Proof. Assume that (x, y) is not a SNE. Hence there exists some $(\hat{x}, y) \in S \times T$ with $(\hat{x}, y) \neq (x, y)$ such that $\Pi_1(\hat{x}, y) \geq \Pi_1(x, y)$ or some $(x, \hat{y}) \in S \times T$ with $(x, \hat{y}) \neq (x, y)$ such that $\Pi_2(x, \hat{y}) \geq \Pi_2(x, y)$. Without loss of generality, assume that $(\hat{x}, y) \in S \times T$ exists. Let U_i be the last subset in a reduction sequence for (x, y) such that $(\hat{x}, y)$ is a member. Therefore there exists a $(x^*, y) \in U_{i+1}$ such that $\Pi_1(x^*, y) > \Pi_1(\hat{x}, y) \geq \Pi_1(x, y)$. This contradicts the assumption that (x, y) is a NE. $\square$

Moulin's result states that if Γ is SDS to x then x is a NE holds regardless of whether Γ is symmetric or asymmetric, so long as every Π_i is continuous and S and T are compact. Thus the above theorem can then be read: Let Γ be an asymmetric game that is SDS to $(x, y) \in S \times T$. Then (x, y) is a SNE.

Theorem 3.5. *If Γ is SDS to (x^*, y^*) and $(\hat{x}, \hat{y})$ is a NE of Γ then $(x^*, y^*) = (\hat{x}, \hat{y})$.*

Proof. Assume that $(x^*, y^*) \neq (\hat{x}, \hat{y})$. Thus there must be some stage k in which $(\hat{x}, \hat{y})$ is strictly dominated. Therefore for some $(x, y) \in U_k$ either $\Pi_1(x, \hat{y}) > \Pi_1(\hat{x}, \hat{y})$ or $\Pi_2(\hat{x}, y) > \Pi_2(\hat{x}, \hat{y})$, which both contradict the assumption that $(\hat{x}, \hat{y})$ is a NE. □

Corollary 3.6. *If (x, y) is a NE of Γ and Γ is SDS to (x, y), then (x, y) is a member of every subset in any strict-dominance solvability reduction sequence of Γ.*

In summary, these three results show that the general theory of strict-dominance solvability for symmetric games at the beginning of Section 2 extends to two-player asymmetric games. The following section extends Section 2.1 to two player asymmetric games, first showing what conditions need be met for a game to have interior or boundary NE, and then conditions for SDS.

3.1. **Asymmetric Games With Quadratic Payoff Functions and One Dimensional Strategy Spaces.** This section will completely characterize what is necessary and sufficient for an asymmetric game Γ with compact strategy sets S and T and quadratic payoff functions Π_1 and Π_2 to be SDS to a strategy pair (x, y). Such games are guaranteed to have at least one mixed strategy NE [3].

3.1.1. *Interior NE.*

Theorem 3.7. *Suppose Γ_{k_1,l_1,k_2,l_2} is a game with payoff functions $\Pi_1 : S \times T \to \mathbb{R}$ and $\Pi_2 : S \times T \to \mathbb{R}$ given by $\Pi_i(x, y) = a_i x^2 + b_i xy + c_i y^2 + d_i x + e_i y + f_i$*

where a_i, b_i, c_i, d_i, e_i, and f_i are constants, and $S = [-l_1, k_1]$ and $T = [-l_2, k_2]$ for some $k_1, k_2, l_1, l_2 > 0$. The following then hold:

(1) *If $(x, y) = (0, 0)$ is a NE, then $d_1 = 0$ and $e_2 = 0$.*

(2) *If $(x, y) = (0, 0)$ is a NE, then $a_1 \leq 0$ and $c_2 \leq 0$.*

(3) *If $a_1 = d_1 = 0$, then there are no $x, \hat{x} \in S$ such that x strictly dominates $\hat{x}$. Similarly, if $c_2 = e_2 = 0$, then there are no $y, \hat{y} \in T$ such that y strictly dominates $\hat{y}$.*

Thus when considering games with quadratic payoffs on compact interval strategy spaces that are strictly dominance solvable to $(x, y) = (0, 0)$ in $S \times T = [-l_1, k_1] \times [-l_2, k_2]$ it can be assumed that $d_1 = e_2 = 0$ and both a_1 and c_2 are negative.

Proof. (1) Assume $(0, 0)$ is a NE and so $\Pi_1(0, 0) \geq \Pi_1(x, 0)$. Thus $0 \geq x(a_1 x + d_1)$. If $d_1 \neq 0$, then for small values of $|x|$ the d_1 term dominates the a_1 term, and $0 \geq d_1 x$ is only guaranteed for $d_1 = 0$. Similarly, if $(0, 0)$ is a NE then $\Pi_2(0, 0) \geq \Pi_2(0, y)$, and so $0 \geq y(c_2 y + e_2)$, and accordingly $e_2 = 0$.

(2) Assume $(0, 0)$ is a NE. Again this means that $0 \geq x(a_1 x + d_1)$, or by the above $0 \geq a_1 x^2$. Therefore $a_1 \leq 0$. A parallel argument shows $c_2 \leq 0$.

(3) Assume $a_1 = d_1 = 0$ and that there are strictly dominated strategies in S. Thus there exist some $x, \hat{x} \in S$ such that $\Pi_1(x, y) > \Pi_1(\hat{x}, y)$ for all $y \in T$. Expanded and simplified, this means that $bxy > b\hat{x}y$,

which is impossible at $y = 0$. A similar contradiction can be reached for $c_2 = e_2 = 0$.

$\square$

Theorem 3.8. *Fix $k_1, k_2, l_1, l_2 > 0$. Let Γ_{k_1,l_1,k_2,l_2} be a game with payoff functions $\Pi_1 : S \times T \to \mathbb{R}$ and $\Pi_2 : S \times T \to \mathbb{R}$ given by $\Pi_i(x, y) = a_i x^2 + b_i xy + c_i y^2 + d_i x + e_i y + f_i$ where a_i, b_i, c_i, d_i, e_i, and f_i are constants, and $S = [-l_1, k_1]$ and $T = [-l_2, k_2]$. Then Γ_{k_1,l_1,k_2,l_2} is SDS to $(x, y) = (0, 0)$ and $(0, 0)$ is a NE if and only if $a_1 < 0$, $c_2 < 0$, $d_1 = e_2 = 0$, and*

$$4 > \left|\frac{b_1}{a_1}\right| \left|\frac{b_2}{c_2}\right|. \tag{3.1}$$

Proof. To show that (3.1) combined with $a_1, c_2 < 0$ and $d_1 = e_2 = 0$ are sufficient for Γ_{k_1,l,k_2} to be SDS to $\overline{0}$, assume that all of those conditions hold. Let $\hat{x} \geq k_1 - \frac{1}{2}\varepsilon_1 k_1$, where $\varepsilon_1 > 0$ will be chosen later. Consider $\Pi_1(k_1(1 - \varepsilon_1), y) > \Pi_1(\hat{x}, y)$, that, if true for all $y \in T$, would mean that all of Player 1's strategies $x \geq \hat{x}$ are strictly dominated and may be eliminated. For this inequality to be true,

$$a_1(k_1(1 - \varepsilon_1))^2 + b_1 k_1(1 - \varepsilon_1)y > a_1\hat{x}^2 + b_1\hat{x}y. \tag{3.2}$$

This can be rearranged and simplified to

$$\hat{x} + k_1(1 - \varepsilon_1) > \frac{b_1}{|a_1|}y.$$

Minimize the left hand side by substituting $\hat{x} = k_1 - \frac{1}{2}\varepsilon_1 k_1$. For $y \in [-l_2, k_2]$ the right hand side of the last inequality is maximized by either $y = -l_2$ if

$b_1 < 0$ and $y = k_2$ if $b_1 \geq 0$. Hence (3.2) is equivalent to either

$$k_1\left(1 - \frac{1}{2}\varepsilon_1\right) + k_1(1 - \varepsilon_1) > \left|\frac{b_1}{a_1}\right| l_2$$

if $b_1 < 0$ or

$$k_1\left(1 - \frac{1}{2}\varepsilon_1\right) + k_1(1 - \varepsilon_1) > \left|\frac{b_1}{a_1}\right| k_2,$$

if $b_1 \geq 0$, that can be rearranged to

$$2 - \frac{3}{2}\varepsilon_1 > \left|\frac{b_1}{a_1}\right| \frac{l_2}{k_1}$$

and

$$2 - \frac{3}{2}\varepsilon_1 > \left|\frac{b_1}{a_1}\right| \frac{k_2}{k_1},$$

respectively. Similar conditions arise when considering $\Pi_1(-l_1(1-\varepsilon_1), y) > \Pi_1(x', y)$ with $x' \leq -l_1 + \frac{1}{2}\varepsilon_1 l_1$, $\Pi_2(x, k_2(1-\varepsilon_2) > \Pi_2(x, \hat{y})$ with $\hat{y} \geq k_2 - \frac{1}{2}\varepsilon_2 k_2$, and $\Pi_2(-l_2(1-\varepsilon_2), y) > \Pi_2(x, y')$ with $y' \leq -l_2 + \frac{1}{2}\varepsilon_2 l_2$.

The remainder of the proof follows the case when $b_1 \geq 0$ and $b_2 < 0$. The other cases follow a similar argument. Let $x' \leq -l_1 + \frac{1}{2}\varepsilon_1 l_1$, $\hat{y} \geq k_2 - \frac{1}{2}\varepsilon_2 k_2$, and $y' \leq -l_2 + \frac{1}{2}\varepsilon_2 l_2$.

In this case, if

$$2 - \frac{3}{2}\varepsilon_1 > \left|\frac{b_1}{a_1}\right| \frac{k_2}{k_1}. \tag{3.3}$$

then all strategies $x \geq \hat{x}$ can be eliminated as strictly dominated strategies. Similarly, if

$$2 - \frac{3}{2}\varepsilon_1 > \left|\frac{b_1}{a_1}\right| \frac{l_2}{l_1} \tag{3.4}$$

then all strategies $x \leq x'$ can be eliminated, if

$$2 - \frac{3}{2}\varepsilon_2 > \left|\frac{b_2}{c_2}\right| \frac{l_1}{k_2} \tag{3.5}$$

then all strategies $y \geq \hat{y}$ can be eliminated, and if

$$2 - \frac{3}{2}\varepsilon_2 > \left|\frac{b_2}{c_2}\right| \frac{k_1}{l_2} \tag{3.6}$$

then all strategies $y \leq y'$ can be eliminated.

Consider the case $k_1 \geq l_1$, $l_2 \geq k_2$. If this is true, then for sufficiently small values of ε_1 and ε_2 either (3.3) or (3.5) is true since, by (3.1)

$$\begin{aligned}\left(2 - \frac{3}{2}\varepsilon_1\right)\left(2 - \frac{3}{2}\varepsilon_2\right) &= 4 - 3(\varepsilon_1 + \varepsilon_2) + \frac{9}{4}\varepsilon_1\varepsilon_2 \\ &> \left|\frac{b_1 b_2}{a_1 c_2}\right| - 3(\varepsilon_1 + \varepsilon_2) + \frac{9}{4}\varepsilon_1\varepsilon_2 \\ &\geq \left|\frac{b_1 k_2}{a_1 l_1}\right| \left|\frac{b_2 l_1}{c_2 k_2}\right| - 3(\varepsilon_1 + \varepsilon_2) + \frac{9}{4}\varepsilon_1\varepsilon_2\end{aligned}$$

and so, strategies $x \geq \hat{x}$ or $y \geq \hat{y}$ can be eliminated as strictly dominated strategies. Continue this process using (3.3) or (3.5) until either $k_1 = l_1$ and $l_2 \geq k_2$ or $k_1 \geq l_1$ and $l_2 = k_2$. Without loss of generality, assume that the stage reached is that of $k_1 = l_1$ and $l_2 \geq k_2$.

Now, if $\frac{l_2}{k_1} > \sqrt{\left|\frac{b_2 a_1}{b_1 c_2}\right|}$, then

$$\left|\frac{b_2 k_1}{c_2 l_2}\right| < \left|\frac{b_2}{c_2}\right| \sqrt{\left|\frac{b_1 c_2}{b_2 a_1}\right|} = \sqrt{\left|\frac{b_1 b_2}{a_1 c_2}\right|} < 2,$$

and therefore (3.6) holds and strategies $y \leq y'$ can be eliminated as strictly dominated strategies by using a sufficiently small ε_2. This decreases $\frac{l_2}{k_1}$. If $l_2 \leq k_2$ at any stage, then l_2 can be kept equal to k_2 thereafter by eliminating $y \leq y'$ and $y \geq \hat{y}$.

If on the other hand $\frac{l_2}{k_1} < \sqrt{\left|\frac{b_2 a_1}{b_1 c_2}\right|}$, then

$$\left|\frac{b_1 k_2}{a_1 k_1}\right| < \left|\frac{b_1 l_2}{a_1 l_1}\right| < \left|\frac{b_1}{a_1}\right| \sqrt{\left|\frac{b_2 a_1}{b_1 c_2}\right|} = \sqrt{\left|\frac{b_1 b_2}{a_1 c_2}\right|} < 2,$$

and thus (3.3) and (3.4) hold and strategies $x \geq \hat{x}$ and $x \leq x'$ can be eliminated, keeping the new interval symmetric about 0. This increases $\frac{l_2}{k_1}$ since $k_1 = l_1$ decreases. Eventually a stage will be reached where $\frac{l_2}{k_1} = \sqrt{\left|\frac{b_2 a_1}{b_1 c_2}\right|}$, $l_2 = k_2$ and $l_1 = k_1$. Thereafter, all four inequalities (3.3), (3.4), (3.5) and (3.6) hold and all strategies $x \geq \hat{x}$, $x \leq x'$, $y \geq \hat{y}$ and $y \leq y'$ can be eliminated simultaneously in such a way to keep $\frac{l_2}{k_1} = \sqrt{\left|\frac{b_2 a_1}{b_1 c_2}\right|}$. The intersection of these rectangles is $\{(0,0)\}$ and thus Γ_{k_1,l_1,k_2,l_2} is SDS to $(0,0)$.

A comparable situation arises if any assumptions about b_1, b_2, or the relative lengths of l_1, k_1, l_2, or k_2 are changed.

Assume Γ_{k_1,l_1,k_2,l_2} is SDS to $\bar{0}$ and $\bar{0}$ is a NE. Then $a_1 < 0$, $c_2 < 0$, and $d_1 = 0 = e_2$ by Theorem 3.7. Assume that (3.1) fails, but Γ_{k_1,l,k_2} is SDS to $(0,0)$. Pick $0 < k \leq \min\{l_1, k_1\}$ and $0 < l \leq \min\{l_2, k_2\}$ such that $\frac{l}{k} = \sqrt{\left|\frac{b_2}{c_2}\right| \left|\frac{a_1}{b_1}\right|}$ and let $S_i \times T_i$ be the last subset of $S \times T$ to contain the strategies $\pm k$ and $\pm l$. To eliminate Player 1's strategy k it must be true that $\Pi_1(w, l) > \Pi_1(k, l)$ for some $w \in S_{i+1}$. If $\Pi_1(w, 0) > \Pi_1(k, 0)$ is considered it is clear to see that once expanded and simplified, that inequality becomes $a_1 w^2 > a_1 k^2$. Since $a_1 < 0$, the previous inequality becomes $w^2 < k^2$, or more simply, that $|w| < |k|$.

It must also be true that $\Pi_1(w, l) > \Pi_1(k, l)$. When this inequality is expanded and simplified it is equivalent to

$$k + w > \frac{b_1}{|a_1|} l.$$

If both sides are divided by k, then it is clear that

$$2 > \frac{1}{k}(k + w) > \frac{b_1}{|a_1|} \frac{l}{k}.$$

Substituting the given value of $\frac{l}{k}$ into the right side of above inequality gives

$$2 > \frac{1}{k}(k + w) > (*) \left| \frac{b_1}{a_1} \right| \sqrt{\left| \frac{b_2}{c_2} \right| \left| \frac{a_1}{b_1} \right|},$$

where $(*)$ is the sign of b_1. This inequality is equivalent to

$$2 > \frac{1}{k}(k + w) > (*) \sqrt{\left| \frac{b_2}{c_2} \right| \left| \frac{b_1}{a_1} \right|},$$

and so b_1 must be negative.

However, to eliminate the strategy k, it must also be true that $\Pi_1(w, -l) > \Pi_1(k, -l)$. This inequality, expanded and simplified, yields

$$k + w > -\frac{b_1}{|a_1|} l.$$

The same method as in the above paragraph the shows that

$$2 > \frac{1}{k}(k + w) > -(*) \sqrt{\left| \frac{b_2}{c_2} \right| \left| \frac{b_1}{a_1} \right|},$$

and so b_1 must be positive. Therefore it is impossible to eliminate the strategy k while Player 2 can still play strategies l and $-l$.

Parallel arguments show that it is impossible to eliminate $-k$ before l and $-l$, as well as it being impossible to eliminate l or $-l$ before k and $-k$. Therefore Γ_{k_1,l,k_2} is not SDS to $(0,0)$. □

Just as in the case of Theorem 2.9, this result provides a characterization of which quadratic games are SDS to an interior NE. In fact, the restrictions in the above theorem are exactly the same as those in Theorem 2.9 when applied to a symmetric game where $a_1 = c_2 = a$ and $b_1 = b_2 = b$.

Theorem 3.8 shows there is a clear line between games that satisfy the conditions of the theorem and those with interior Nash equilibria that do not. In [8], Moulin also provides a definite condition that must be satisfied for a continuous and compact game to be SDS, and the result of the above theorem coincides with that of Moulin, but extends past his result in the fact that it considers the game on any strategy space $S \times T$ and gives both sufficient and necessary condition for a game to be SDS to $\overline{0}$, whereas Moulin's theorem only provides a sufficient condition for "local" games (i.e. k, l close to $\overline{0}$). Moulin's theorem, however, does take into account a wider class of games, those with non-quadratic payoff functions. The fact that Moulin's theorem is true for such a large group of games accounts for its inability to show necessity when applied in a narrower view.

3.1.2. *Boundary NE.* Similar to the section on quadratic functions in symmetric games, games where the NE is on the boundary of one of the strategy sets will now be considered.

Theorem 3.9. *Fix $k_1, l, k_2 > 0$. Let Γ_{k_1,l,k_2} be a game with payoff functions $\Pi_1 : S \times T \to \mathbb{R}$ and $\Pi_2 : S \times T \to \mathbb{R}$ given by $\Pi_i(x, y) = a_i x^2 + b_i xy + c_i y^2 + d_i x + e_i y + f_i$ where a_i, b_i, c_i, d_i, e_i, and f_i are constants, and $S = [0, k_1]$ and $T = [-l, k_2]$ for any $k_1, k_2, l > 0$. The following are then true:*

(1) *If $(0,0)$ is a NE of Γ_{k_1,l,k_2} for all $k_1, l, k_2 > 0$, then $a_1 \leq 0$, $d_1 \leq 0$, $c_2 \leq 0$ and $e_2 = 0$.*

(2) *If $a_1 = d_1 = 0$, then for every $l, k_2 > 0$, there exists a $k_1 > 0$ such that no strategy in S is strictly dominated. Similarly, if $c_2 = e_2 = 0$, then there are no $y, \hat{y} \in T$ such that y strictly dominates $\hat{y}$.*

Therefore, if Γ_{k_1,l,k_2} is SDS to $(0,0)$ for all $k_1 > 0$, then $a_1 \leq 0$, $d_1 \leq 0$, $c_2 < 0$ and $e_2 = 0$, but $a_1 = 0$ and $d_1 = 0$ cannot both be true.

Proof. The proof of this theorem follows exactly Theorems 2.8 and 2.10 applied to T and S, respectively. □

Analogously to Theorem 2.10, these conditions only hold when k_1 is allowed initially to have any positive value. Given a limit on the size of k_1, neither is necessarily true. We continue to assume in the following two results that k_1 is allowed to have any positive value. The following result is the analogue of Lemma 2.11.

Theorem 3.10. *Fix $k_1, l, k_2 > 0$. Let Γ_{k_1,l,k_2} be a game with payoff functions $\Pi_1 : S \times T \to \mathbb{R}$ and $\Pi_2 : S \times T \to \mathbb{R}$ given by $\Pi_i(x, y) = a_i x^2 + b_i xy + c_i y^2 + d_i x + e_i y + f_i$ where $a_1 = 0$, $c_2 < 0$, $d_1 < 0$, $e_2 = 0$, a_2, b_i, c_1, d_2, e_1, and*

f_i are constants, and $S = [0, k_1]$ and $T = [-l, k_2]$ for any $k_1, k_2, l > 0$. Then Γ_{k_1,l,k_2} is SDS to $(0,0)$ for all choices of k_1, k_2, $l > 0$ if and only if $b_1 b_2 \leq 0$.

Proof. The proof of the sufficient conditions for strict-dominance solvability will be divided into two cases, where it is shown that Γ_{k_1,l,k_2} is SDS in each case.

Case 1: $a_1 = 0$, $c_2 < 0$, $d_1 < 0$, $e_2 = 0$, $b_1 \geq 0$ and $b_2 \leq 0$. The inequality $\Pi_2(x, 0) - \Pi_2(x, y) > 0$ is equivalent to $0 > c_2 y^2 + b_2 xy$, which is true for all $y \in T$ where $y > 0$. Any strategy $y \in T$ where $y > 0$ is therefore strictly dominated by 0 and can be eliminated, reducing T to $T_1 = [-l, 0]$ is one step. The resulting strategy space is $S_1 \times T_1 = [0, k_1] \times [-l, 0]$.

Now consider $\Pi_1(0, y) - \Pi_1(x, y) > 0$ which is equivalent to $0 > b_1 xy + d_1 x$ and is always true for $x \in S$ where $x \neq 0$ since $b_1 \geq 0$, $y \leq 0$, $x > 0$ and $d_1 < 0$. Therefore any strategy $x \in S$ where $x \neq 0$ is strictly dominated and can be eliminated, reducing S_1 to $S_2 = \{0\}$ in one step, resulting in the strategy space $S_2 \times T_2 = \{0\} \times [-l, 0]$.

Finally consider $\Pi_2(0, 0) - \Pi_2(0, y) > 0$. This inequality is equivalent to $0 > c_2 y^2$ which is true for all $y \in T_2$ where $y \neq 0$. Any strategy $y \in T_2$ where $y \neq 0$ is therefore strictly dominated and can be eliminated.

Thus Γ_{k_1,l,k_2} is SDS to $(0,0)$ in three steps.

Case 2: $a_1 = 0$, $c_2 < 0$, $d_1 < 0$, $e_2 = 0$, $b_1 \leq 0$ and $b_2 \geq 0$. This case follows a similar argument to the above case, however, in the first step any strategy $y \in T$ where $y < 0$ is eliminated, followed by the elimination of any strategy $x \in S_1$ where $x \neq 0$, and finishing with the elimination of the remaining strategies $y \in T_2$ where $y > 0$. Again Γ_{k_1,l,k_2} is SDS to $(0,0)$ in three steps.

Conversely, assume that Γ_{k_1,l,k_2} is SDS to $(0,0)$ and that $a_1 = 0$, $d_1 < 0$, $c_2 < 0$ and $e_2 = 0$, but that $b_1 b_2 > 0$. This portion of the proof will also have two cases. Both reach a contradiction which shows that Γ_{k_1,l,k_2} will not be SDS to $(0,0)$ if $b_1 b_2 > 0$.

Case 1: $b_1, b_2 > 0$. Let $S_i \times T_i$ be the last subset of $S \times T$ such that $S \subseteq S_i$ and $k_2, 0 \in T_i$. For a strategy $\hat{x} \in S_{i+1}$ to strictly dominate a strategy $x \in S_i$ with respect to S_i, it must be true that $\Pi_1(\hat{x}, y) - \Pi_1(x, y) > 0$ for all $y \in T_i$. This inequality expands and simplifies to

$$(b_1 y + d_1)(\hat{x} - x) > 0.$$

If $\hat{x} - x > 0$, then it must be true that $b_1 y + d_1 > 0$ for all $y \in T$, however $y = 0$ yields the contradiction $d_1 > 0$. Therefore $\hat{x} - x < 0$, and $\Pi_1(\hat{x}, y) - \Pi_1(x, y) > 0$ expands and simplifies to

$$b_1 y + d_1 < 0 \tag{3.7}$$

which is a contradiction at $y = k_2$ since k_2 can have any positive value. Therefore no strategy $x \in S_i$ can be eliminated before k_2.

Hence there is a $w \in T_{i+1}$ such that w strictly dominates k_2 with respect to T_i. Then $\Pi_2(x, w) > \Pi_2(x, k_2)$ for all $x \in S_i$. In particular, if $x = 0$, then

$$c_2 w^2 > c_2 k_2^2$$

which is equivalent to $|w| < |k_2|$.

Further, for strategy w to strictly dominate strategy k_2 it must be true that $\Pi_2(k_1, w) > \Pi_2(k_1, k_2)$. This inequality expands and simplifies to

$$c_2 w^2 + b_2 k_1 w > c_2 k_2^2 + b_2 k_1 k_2$$

which is equivalent to

$$c_2(w^2 - k_2^2) + b_2 k_1 (w - k_2) > 0.$$

Since $w - k_2 < 0$, dividing both sides of the above inequality by $w - k_2$ shows $\Pi_2(k_1, w) > \Pi_2(k_1, k_2)$ is equivalent to

$$c_2(w + k_2) + b_2 k_1 < 0$$

This can be rearranged to produce

$$\frac{1}{k_2}(w + k_2) > \frac{b_2 k_1}{|c_2| k_2}$$

and since this must hold for any positive value of k_1, $b_2 \leq 0$. This contradicts the assumption that $b_2 > 0$. Thus Γ_{k_1,l,k_2} is not SDS to $(0, 0)$.

Case 2: $b_1, b_2 < 0$. Let $S_i \times T_i$ be the last subset of $S \times T$ such that $S \subseteq S_i$ and $-l, 0 \in T_i$. For a strategy $\hat{x} \in S_{i+1}$ to strictly dominate a strategy $x \in S_i$ with respect to S_i, it must be true that $\Pi_1(\hat{x}, y) - \Pi_1(x, y) > 0$ for all $y \in T_i$. As in the above case, this can be shown to be equivalent to

$$b_1 y + d_1 < 0 \tag{3.8}$$

which is a contradiction at $y = -l$ since k_2 can have any positive value. Therefore no strategy $x \in S_i$ can be eliminated before $-l$.

Therefore there is a $-v \in T_{i+1}$ such that $-v$ strictly dominates $-l$ with respect to T_i. Then $\Pi_2(x, -v) > \Pi_2(x, -l)$ for all $x \in S_i$. In particular, if $x = 0$, then

$$c_2 v^2 > c_2 l^2$$

which is equivalent to $|v| < |l|$.

As well, for strategy $-v$ to strictly dominate strategy $-l$ it must be true that $\Pi_2(k_1, -v) > \Pi_2(k_1, -l)$. This inequality expands and simplifies to

$$c_2 w^2 - b_2 k_1 v > c_2 l^2 - b_2 k_1 l$$

which is equivalent to

$$c_2(v^2 - l^2) - b_2 k_1 (v - l) > 0.$$

Since $v - l < 0$, dividing both sides of the above inequality by $v - l$ shows $\Pi_2(k_1, -v) > \Pi_2(k_1, -l)$ is equivalent to

$$c_2(v + l) - b_2 k_1 < 0$$

This can be rearranged to produce

$$\frac{1}{l}(v + l) > \frac{-b_2 k_1}{|c_2| l}$$

and, as this must hold for any positive value of k_1, $b_2 > 0$. This contradicts the assumption that $b_2 < 0$. Thus Γ_{k_1, l, k_2} is not SDS to $(0, 0)$. □

The above theorem gives a sufficient and necessary condition for certain games to be SDS to a boundary NE, and also demonstrates the increasing complexity due to the fact that the game is no longer symmetric when compared with the corresponding Lemma (Lemma 2.11) in Section 2.1. The final result in this thesis (Theorem 3.11) is the analogue of Case (1) of Theorem 2.12.

Theorem 3.11. *Let Γ_{k_1,l,k_2} be a game with payoff functions $\Pi_1 : S \times T \to \mathbb{R}$ and $\Pi_2 : S \times T \to \mathbb{R}$ given by $\Pi_i(x,y) = a_i x^2 + b_i xy + c_i y^2 + d_i x + e_i y + f_i$ where a_i, b_i, c_i, d_i, e_i, and f_i are constants, and $S = [0, k_1]$ and $T = [-l, k_2]$ for all choices of $k_1, k_2, l > 0$. If $a_1 < 0$, $c_2 < 0$, $d_1 = 0$, $e_2 = 0$ then Γ_{k_1,l,k_2} is SDS to $\bar{0}$ if*

$$4 > \frac{b_1 b_2}{a_1 c_2}$$

Proof. This proof will be divided into three cases. Cases 1a and 1b, where $b_1 b_2 \leq 0$, follow a similar argument to that in Theorem 3.10. Case 2, where $b_1 b_2 > 0$, is a direct consequence of Theorem 3.8.

Case 1a: $b_1 \leq 0$ and $b_2 \geq 0$. Consider $\Pi_2(x, 0) > \Pi_2(x, y)$. Expanded and simplified, this inequality is equivalent to

$$0 > b_2 xy + c_2 y^2.$$

This is true for all $x \in S$ and all $y \in T$ such that $y < 0$. Therefore any strategy $y \in T$ such that $y < 0$ is strictly dominated by $y = 0$ and can be eliminated. Thus let $S_1 \times T_1 = [0, k_1] \times [0, k_2]$.

Next, consider $\Pi_1(0, y) > \Pi_1(x, y)$. This inequality is equivalent to

$$0 > a_1 x^2 + b_1 xy,$$

which is true for all $x \in S_1$ and all $y \in T_1$ provided $x \neq 0$. Therefore any strategy $x \in S_1$ where $x \neq 0$ is strictly dominated with respect to S_1 and can be eliminated, resulting in the strategy set $S_2 \times T_2 = \{0\} \times [0, k_2]$.

Finally, consider $\Pi_2(0,0) > \Pi_2(0,y)$. This inequality expands and simplifies to

$$0 > c_2 y^2.$$

This is true for all $y \in T_2$ such that $y \neq 0$, and so any strategy $y \in T_2$ where $y \neq 0$ can be eliminated as a strictly dominated strategy, reducing the strategy set to $S_3 \times T_3 = \{0\} \times \{0\}$. Thus Γ_{k_1,l,k_2} is SDS to $(0,0)$ in three steps.

Case 1b: $b_1 \geq 0$ and $b_2 \leq 0$. This case follows a similar argument to the above case, except that the change in the signs of b_1 and b_2 causes the order of elimination to differ. In the first step, any strategy $y \in T$ where $y > 0$ can be eliminated, followed by the elimination of any strategy $x \in S_1$ such that $x \neq 0$, and lastly, the elimination of any $y \in T_2$ where $y < 0$. Again Γ_{k_1,l,k_2} is SDS to $(0,0)$ in three steps.

Case 2: $b_1 b_2 > 0$. Since $\frac{b_1 b_2}{a_1 c_2} = \left|\frac{b_1 b_2}{a_1 c_2}\right|$, following the same steps (in one direction) of Theorem 3.8 produces the desired result that Γ_{k_1,l,k_2} is SDS to $(0,0)$ in a countable number of steps. □

4. Conclusion

Iterated strict dominance is one of the most basic principles in game theory [1]. As a tool for identifying rational behaviour in games, it is clear, robust and highly credible. Throughout this thesis, various necessary and sufficient conditions for the existence of a strict-dominance solution have been discovered and proven. In the simple case of 2 player symmetric games whose strategy space is a compact interval, a complete characterization of strict-dominance solvability to an interior NE is proven when payoff functions are quadratic (Theorem 2.9). The most important condition for this strict-dominance solvability is $2 > \left|\frac{b}{a}\right|$ (see Section 2.1 for the meanings of the parameters a and b in the quadratic payoff function) that shows that strict-dominance solvability is directly related to those terms in the payoff function concerning a player's own strategies, rather than those terms that concern the strategic choice of the opponent. This same phenomenon shows up again for games SDS to boundary NE (Theorem 2.12, as well as others in that section) and for interior NE in games with higher order payoff functions (Theorem 2.23). This trend appears to generalize to games with higher dimensional strategy spaces.

The requirements for a game to be strict dominance solvable to a boundary NE from either side coincide with the requirements for the game to have a continuously stable strategy, linking these two solution concepts (Theorems 2.18 and 2.20). Thus the answer provided by strict-dominance solvability is a very convincing prediction of rational behaviour in the game. Again, the most important condition, $2 > \frac{b}{|a|}$ (Theorem 2.18), is still comprised solely of those terms that concern a player's own strategy.

It is not surprising that small changes to the payoff functions of a game have little effect on the conditions for strict dominance. When the payoff function is of a higher order than quadratic, it is again the terms relating to a player's own strategies that must fulfill certain criteria for the game to be strict-dominance solvable. When the game is no longer symmetric, the same statement is true. The consistency of the requirements for strict-dominance solvability of asymmetric games with those for symmetric games can be seen from Theorem 3.1 (by setting $a_1 = c_2$ and $b_1 = b_2$, see Section 3.1 for the definition of the parameters).

It is clear from the analysis in Section 2.3 on games with multidimensional strategy spaces that much work remains to be done on this topic. While it appears plausible that a set condition governs whether a game of this class is strict-dominance solvable, the complexity of such games renders that condition beyond the scope of this thesis. Similarly, further analysis of games with higher-order payoff functions, as well as analysis of games with non-identical payoff functions for the players could be fruitful.

These new conditions, when found, can be tested for consistency with those already known for the simpler cases, and can be compared to other solution concepts such as continuously stable strategies or evolutionarily stable strategies.

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