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Xiaojing Xi
Wilfrid Laurier University

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by

Xiaojing Xi

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Abstract

In this thesis we focus on the development of a new class of stochastic models for asset price processes and their application to option pricing and hedging. The asset price process involves analytical treatments for calculating first-hitting (or first-passage) times for a regular diffusion with killing in combination with Markov state-switching. The dynamics is naturally dictated by the underlying diffusion process itself rather than arising from some additional exogenous process. To date, this class of asset pricing models appears to be novel in the literature and, moreover, offers a significant extension to the standard geometric Brownian motion commonly used in the original Black-Scholes (BS) model. In the class of models to be developed and analyzed in this thesis, model-specific first-hitting times are introduced (e.g. for upper and lower barriers) for the underlying asset price diffusion conditional on the given economic state. These combine to give asset price diffusion in continuous time with an additionally embedded two-state (extendible to multiple-state) continuous-time Markov chain. In each state, the asset price process obeys geometric Brownian *motion with a constant volatility corresponding to the variation of the process. Upper and lower state switching barriers are then introduced as natural barriers separating these regions. The random (stopping) times for the process to switch from one state to another are its first-hitting times (i.e. barrier crossing times from below and from above). We then investigate the analytical tractability for computing transition probability densities, first-passage time densities and for computing European option pricing formulae. Model calibration analysis is also studied.*

Keywords*: Black-Scholes model, Calibration, European option, First-passage time, Implied volatility, Regime switching.*

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Motivation

In financial modelling, it is well known that the parameters of financial data tend to change over time. In an influential paper, Hamilton (1989) suggested Markov switching techniques as a method of modelling non-stationary time series. In Hamilton's approach, the parameters are viewed as the outcome of a discrete-state Markov process. For example, expected returns in the stock market may be subject to occasional, discrete shifts. By using the technique proposed by Hamilton (1989), Schwert (1989) also considered a model in which the returns may have either a high or a low variance, and that switches between these return distributions are determined by a two-state Markov process. Turner, Startz and Nelson (1989) considered a Markov switching model in which either the mean, the variance, or both may differ between two regimes.

Hamilton and Susmel (1994) proposed the Markov switching ARCH model. In a partial equilibrium model, Turner, Startz and Nelson (1989) formulated a switching model of excess returns in which returns switch *exogenously* between a Gaussian low variance regime and a Gaussian high variance regime. The observed volatility clustering of many stock return time series points towards models of stock price dynamics that incorporate regime switching. In the mathematical finance literature, various Markov switching models have been proposed to describe the behavior of business cycles or volatility regimes. There is considerable evidence in the long run business life that asset price correlations are asymmetric: market volatilities are larger when the markets move downwards than when they move upward. This is especially true for extreme downside moves. The mathematical problem largely boils down to obtaining joint density functions for the asset price processes, and in some models these can be easily worked out due to the independency between the current asset price process and its history. The conditional transition probabilities for a Markov switching process can then be used to derive option pricing formulas.

An important assumption in usual Markov switching models is that the probability of the stock process leaving the current state follows an exponential distribution. That is, the regime switching is a memoryless process that is not directly dependent on the stock price process. Although the value of the rate parameters in the exponential distributions can be obtained by some calibration methods, such exogenous distributions cannot fully represent the relation between the asset price movements and the information of state-switching. In this thesis, we introduce a novel asset pricing model by adjoining the Black-Scholes diffusion model for stock fluctuation with volatility regime-switching via first passage time. Specifically, we assume that the asset prices are generated by a time-homogeneous diffusion (e.g. geometric Brownian motion), with volatility as a random process taking on different values depending on the level of the asset price. The process is conditional on an initial state. We also assume that for each state, there is a corresponding switching barrier level. The state-switching occurs at the moment that the asset price hits the switching barrier. Hence the time of switching follows a first-passage (hitting) time density that is driven by the underlying diffusion. We call this model a *Black-Scholes model with Regime switching via first hitting time*, or a *self-switch model.* This gives rise to a new class of "structural" regime-switching models. The key difference between such models and more usual regime-switching models is that the switching process for the volatility regimes does not follow a simple exogenous Markov chain. This class of models consist with the following important features. The martingale probability measure exists under the model. This condition leads to the fact that the asset pricing under the model is arbitrage-free and the model is a complete market model. An asset price process is a Markovian process only condition on stationary or a local Markovian process.

Much research has been devoted to developing volatility regime-switching for representing dynamic volatility. Bollen (1998) studies a discrete-time regime-switching lattice-based model. He takes an approach to construct a lattice that represents the possible future paths of a regime-switching variable. The regime persistence is dependent on the time period between observations. And he uses Monte Carlo simulation to determine the degree of certainty with which regimes can be revealed for a range of regime persistence values and observation frequencies. In our self-switching model, we derive an explicit closed-form expression for the transition density function and also an analytical solution for European-style options. The range of regime persistence can be controlled by the level of switching barriers and the value of the volatility parameters.

Previous studies have also documented strong evidence for regime switching in other relevant financial processes. Bollen, Gray and Whaley (2000) investigate the ability of regime-switching models to capture the time series properties of exchange rates. The results of their analysis suggest that regime-switching models may have practical implication for investors and that they capture the dynamics of the exchange rate better than alternative time series models. Kalimipalli and Susmel (2004) introduce a two-factor volatility regime-switching model to explain the behavior of short-term interest rates. Based on our methodology, the asset price volatility switching can also alternatively be driven by the exchange rate or interest rate dynamics. That is, the time of switching follows a first-hitting time density driven by the exchange rate process or interest rate process. This gives rise to a different type of correlation between the asset price process and relevant financial observables.

The variance or volatility of the asset price process represents the risk level. As seen in Figure 1.1, geometric Brownian motion (GBM) with different volatilities exhibits different variation, i.e. the higher the risk, the higher the variation. From the long run, each business cycle has a corresponding risk level. The contraction, meaning a declining economy, normally leads stock prices to fall. As a result the companies may not be able to meet their credit obligations and hence the default risk increases. Also, investors tend to be more uncertain about the future growth rate of the economy. From this point of view, we can assume a higher volatility of asset returns. Figure 1.2 shows the asset price process generated by geometric Brownian motion switching from contraction to expansion. As we can see, in the contraction period, due to the high volatilities, the stock price declines rapidly and thus involves very high risk. On the other hand, since the change of business cycle normally will not happen in a short period, the different regimes have different meaning. From the short run, the risk level is tightly related to the value of the asset. As the asset prices increases, investors tend to expect higher rates of return. And thus we assume a higher volatility of asset return in high value regime. Figure 1.3 illustrates the stock price generated by a GBM switching from "low risk regime" to "high risk regime". Another important aspect of regime switching models is their ability to capture both volatility clustering as well

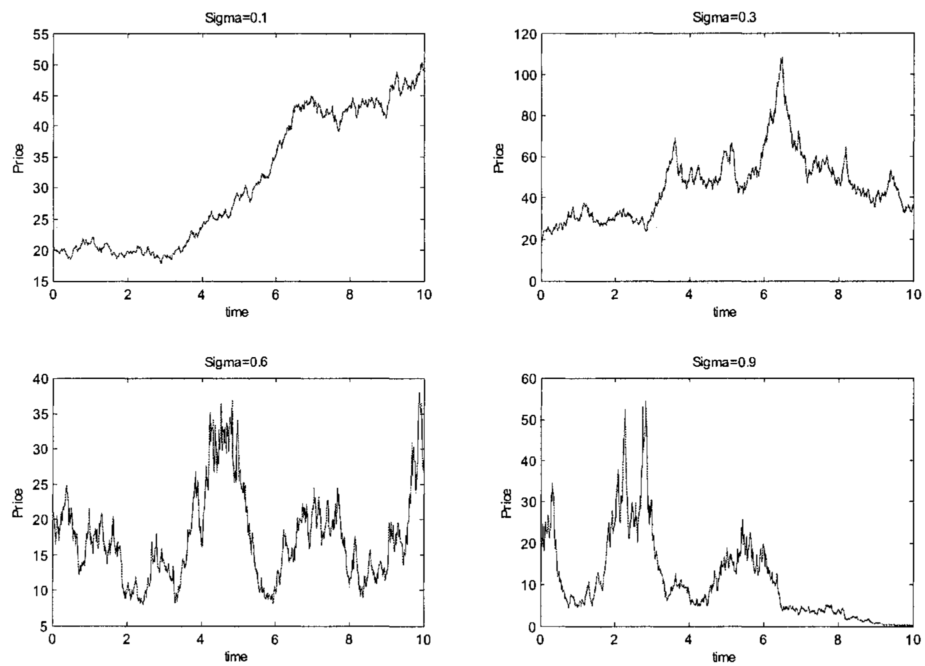

Figure 1.1: GBM with different volatilities

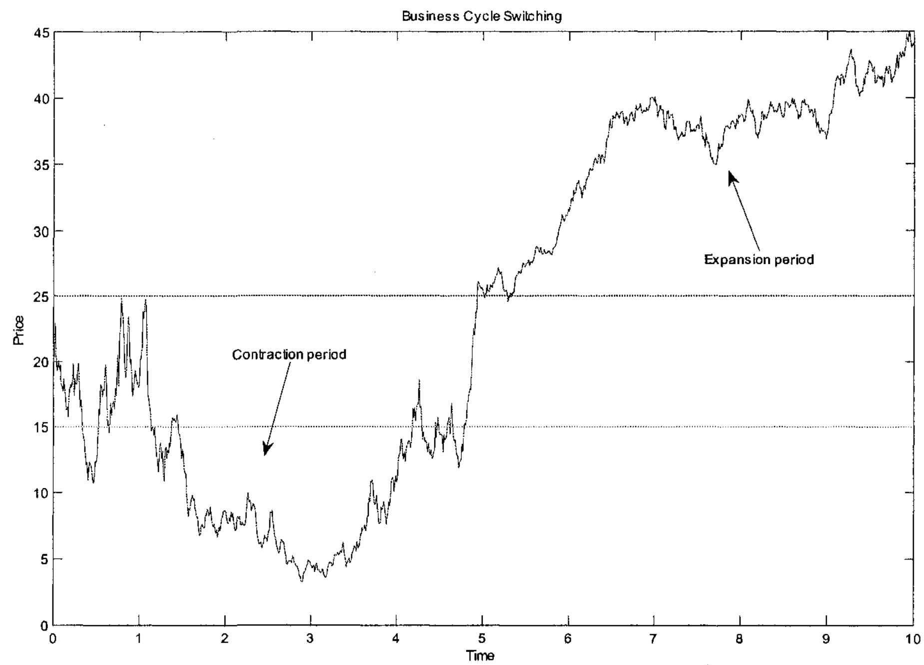

Figure 1.2: Bussiness cycle switches from contraction to expansion

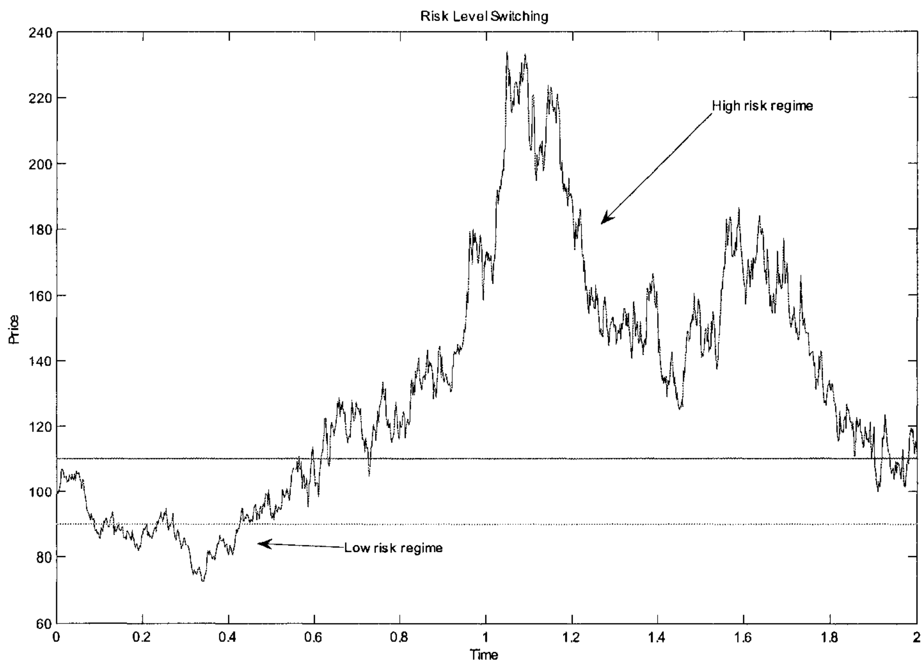

Figure 1.3: Stock price switches from low risk level to high risk level

as leptokurtosis (i.e. heavier tails) observed in most asset return data.

This thesis is organized as follows. In the first chapter, we review some basic concepts of regular diffusions, GBM and first passage time. In Chapter Two, we give a theoretical formulation of our new asset pricing models and derive corresponding transition densities for self-switching processes. Starting with the simplest *single-switch model*, we derive a general formula for the transition density for a self-switching diffusion with a finite number of switching times. In Chapter Three, we obtain analytical integral representations for European style option pricing and use such formulas to investigate the implied volatility surfaces. The calibration problem under the *double-switch model* is also studied. Finally, in Chapter Four, we introduce further applications of the newly formulated self-switching models to pricing exotic options.

1.2 Regular Diffusion and First Passage Time

1.2.1 Brownian Motion and Martingales

The standard one-dimensional Brownian motion $(B_t)_{0\leq t}$ serves as the basic underlying process for the cumulative effect of pure noise. The probability density for the zero-drift standard Brownian motion at time t with $B_{t=0} = B_0 = 0$ is given by

$$P(b,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{b^2}{2t}} \tag{1.1}$$

at $B_t = b$. Consider the process X_t satisfying the following stochastic differential equation (SDE):

$$dX_t = rdt + \sigma dB_t,\ X_0 = x_0. \tag{1.2}$$

Then, X_t is a Brownian motion with constant drift r and constant volatility $\sigma > 0$. The transition probability density function for the process X_t with free motion on the entire real line is a Gaussian density given by

$$g_{0,r}(x, x_0; t) = \frac{e^{-(x-x_0-rt)^2/2\sigma^2 t}}{\sigma\sqrt{2\pi t}}. \tag{1.3}$$

In what follows, we will need to make use of the (risk-neutral) transition probability for drifted Brownian motion with absorption (killing) at lower or upper barrier level. The way to derive this density is as follows. Let x_t denote the driftless Brownian motion starting at $x_0 < x_H$ at initial time t_0 with upper absorbing (killing) barrier at $x = x_H$. Let $\tilde{x}_t$ denote the same Brownian motion but with no barrier with transition density

$$u_0(x_t, x_0, \tau) := \frac{e^{(x-x_0)^2/2\tau}}{\sqrt{2\pi\tau}},\ \text{at } \tilde{x}_t = x, \tag{1.4}$$

$t_0 \leq \tau \leq t$. The probability of a path x_s, $t_0 \leq s \leq t$, having value X or less at time t, where $X < x_H$, is given by:

$$P\{x_t \leq X\} = P\left\{\tilde{x}_t \leq X,\ \sup_{t_0 \leq s \leq t} \tilde{x}_s < x_H\right\}. \tag{1.5}$$

From first principles, the total probability of the event

$$\{\tilde{x}_t \leq X\} = \left\{\tilde{x}_t \leq X,\ \sup_{t_0\leq s\leq t} \tilde{x}_s < x_H\right\} \cup \left\{\tilde{x}_t \leq X,\ \sup_{t_0\leq s\leq t} \tilde{x}_s \geq x_H\right\} \tag{1.6}$$

is given by the sum of the probabilities of the two mutually exclusive events:

$$P\{\tilde{x}_t \le X\} = P\left\{\tilde{x}_t \le X, \sup_{t_0 \le s \le t} \tilde{x}_s < x_H\right\} + P\left\{\tilde{x}_t \le X, \sup_{t_0 \le s \le t} \tilde{x}_s \ge x_H\right\}. \tag{1.7}$$

Let $t_H < t$ denote the time at which a path first hits x_H. Then the probability density that a Brownian path at x_H at time t_H subsequently attains the value at X at terminal time t is the same as that for a (reflected) path starting at x_H at time t_H and attaining a value $2x_H - X$ at time t. And thus for both paths this probability density is

$$u_0(X, x_H; t - t_H) = u_0(2x_H - X, x_H; t - t_H). \tag{1.8}$$

Using this, the second term in equation (1.7) and using (1.5) becomes

$$\begin{aligned} P\left\{\tilde{x}_t \le X, \sup_{t_0 \le s \le t} \tilde{x}_s \ge x_H\right\} &= P\left\{\tilde{x}_t \ge 2x_H - X, \sup_{t_0 \le s \le t} \tilde{x}_s \ge x_H\right\} \\ &= P\{\tilde{x}_t \ge 2x_H - X\}. \end{aligned} \tag{1.9}$$

Note that the last probability obtains from the fact that $\tilde{x}_t \ge 2x_H - X$ implies $\tilde{x}_t \ge x_H$ for any value $X \le x_H$, i.e. the supremum condition is redundant. Substituting this result into equation (1.7) gives

$$P\{x_t \le X\} = P\{\tilde{x}_t \le X\} - P\{\tilde{x}_t \ge 2x_H - X\} \tag{1.10}$$

for all $X \le x_H$.

We extend the result in (1.10) to the case of a constant drifted Brownian motion $X_t := x_0 + rt + \sigma B_t$ (i.e. $dX_t = rdt + \sigma dB_t$). The result from [1] is used to show that the transformation of transition density of a driftless Brownian motion into the one of corresponding drifted Brownian motion can be written as

$$u_r(x, x_0; \tau) = e^{\frac{r}{\sigma^2}(x-x_0)-\frac{r^2}{2\sigma^2}\tau} u_0(x, x_0; \tau), \tag{1.11}$$

where u_0 is given in (1.4). The transition density for the drifted Brownian motion, denoted by $u_r = g_r^u$, on the domain x, $x_0 \in (-\infty, x_H]$ is then given by

$$\begin{aligned} g_r^u(x_H, x, x_0; t) &= \frac{e^{\frac{r}{\sigma^2}(x-x_0)-\frac{r^2}{2\sigma^2}t}}{\sigma\sqrt{2\pi t}} \left(e^{-(x-x_0)^2/2\sigma^2 t} - e^{-(x+x_0-2x_H)^2/2\sigma^2 t}\right) \\ &= g_{0,r}(x, x_0; t) - e^{-\frac{2r}{\sigma^2}(x_H - x_0)} g_{0,r}(x, 2x_H - x_0; t) \\ &= g_{0,r}(x, x_0; t)\left[1 - e^{-\frac{2}{\sigma^2 t}(x_H^2 + x x_0 - x_H(x+x_0))}\right], \end{aligned} \tag{1.12}$$

where the function $g_{0,r}$ is defined by equation (1.3).

Definition *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability measure space with filtration $(\mathcal{F}_t)_{0\leq t\leq T}$, $\mathcal{F}_T = \mathcal{F}$. A real-valued $\mathcal{F}_t$-adapted continuous-time process $(X_t)_{t\geq 0}$ is said to be a $\mathbb{P}$-martingale if the boundedness condition $E\left[|X_t|\right] < \infty$ holds for all $t \geq 0$ and*

$$X_t = E_t[X_T] := E[X_T \mid \mathcal{F}_t] \ a.s. \tag{1.13}$$

for $0 \leq t \leq T \leq \infty$, where $E\left[\cdot\right]$ is the expectation with respect to $\mathbb{P}$-measure.

This definition implies that the conditional expectation for the value of a martingale process at a future time T, given all previous history up to the current time t (i.e. adapted to a filtration $\mathcal{F}_t$), is its current time t value. Our best prediction of future values of such a process is therefore the presently observed value. For example, simple driftless Brownian motion of the form

$X_t = \sigma B_t$, $t \geq 0$, is a martingale process under the assume $\mathbb{P}$-measure for which $(B_t)_{t\geq 0}$ is a $\mathbb{P}$-Brownian motion. As we will see later, martingales play an important role in derivative pricing.

1.2.2 Geometric Brownian Motion

Univariate geometric Brownian motion $(S_t)_{t\geq 0}$ with generally time dependent coefficients is characterized by the SDE of the form

$$dS_t = r(t)S_t dt + \sigma(t)S_t dB_t, \tag{1.14}$$

with initial condition $S_{t=0} = S_0$, where $r(t)$ and $\sigma(t)$ are deterministic functions of time t. This process is usually used to model the price of a stock or other risky asset. S_t represent a positive price of an asset, $r(t)$ is the riskless interest rate of return on a money-market account, $\sigma(t)$ is the volatility of the asset price and B_t is standard Brownian motion *under the risk-neutral measure* $\mathbb{Q}$ with $\exp(\int_0^t r(s)ds)$ as numeraire asset. The strong solution to (1.14) is

$$S_t = S_0 e^{(\bar{r}(t) - \frac{\bar{\sigma}^2(t)}{2})t + \bar{\sigma}(t)B_t}, t \geq 0, \tag{1.15}$$

where $\bar{r}(t)$ and $\bar{\sigma}(t)$ are the *time-averaged drift* and volatility defined by

$$\bar{r}(t) \equiv \frac{1}{t}\int_0^t r(s)ds \tag{1.16}$$

and

$$\bar{\sigma}(t) \equiv \frac{1}{t}\int_0^t \sigma(s)ds. \tag{1.17}$$

S_0 is the asset price at current time $t = 0$. Under the Black-Scholes model, the risk-free interest rate $r(t) = r$ and the volatility $\sigma(t) = \sigma$ are defined as constants. In this case, the strong solution to (1.15) for all $t \geq 0$ is

$$S_t = S_0 e^{(r-\frac{\sigma^2}{2})t+\sigma B_t}. \tag{1.18}$$

The constant volatility parameter σ may not be realistic from a practical point of view. This is the main reason we are going to investigate extensions of this model.

GBM without Absorption

The transition density in (1.3) can be used to provide exact pricing kernels for which the underlying asset price process S_t at time t is assumed to obey the standard Black-Scholes (B-S) model with constant interest rate and volatility. Note that a pricing kernel refers to the transition density of the asset price process in the risk-neutral measure. The case of zero-boundary conditions which give regular geometric Brownian motion on the entire half-line S, $S_0 \in (0, \infty)$. The pricing kernel for the regular case of no absorbing barrier, denoted by U, is obtained via equation (1.3), with the variable transformation $x = X(S) = \log S$, $x_0 = X(S_0) = \log S_0$, and sending $r \to r - \frac{\sigma^2}{2}$ to give:

$$\begin{aligned} U(S, S_0; t) &= \frac{1}{S} g_{0,r-\frac{\sigma^2}{2}}(\log S, \log S_0, t) \\ &= \frac{1}{\sigma S\sqrt{2\pi t}} e^{-\frac{(\log \frac{S}{S_0} - (r - \frac{\sigma^2}{2})t)^2}{2\sigma^2 t}}. \end{aligned} \tag{1.19}$$

Note: throughout the thesis, $\log x$ denotes the natural logarithm of x ($\ln x$).

GBM with Absorption at a Single Barrier

To obtain the pricing kernel for the case of a single absorbing barrier at $S = H > 0$, the transition density must satisfy zero boundary conditions at $S = H$ and $S = \infty$. By applying the barrier levels $x_H = X(H) = \log H$, and applying the above change of variables $x = X(S)$ and $x_0 = X(S_0)$ into the density in (1.12) gives

$$\begin{aligned} U(H, S, S_0; t) &= \frac{1}{S}[g_{0,r-\frac{1}{2}\sigma^2}(\log S, \log S_0; t) \\ &\quad - \left(\frac{H}{S_0}\right)^{\frac{2r}{\sigma^2}-1} g_{0,r-\frac{1}{2}\sigma^2}(\log S, \log\left(\frac{H^2}{S_0}\right); t)] \\ &= U_0(S, S_0; t) - \left(\frac{H}{S_0}\right)^{\frac{2r}{\sigma^2}-1} U_0\left(S, \frac{H^2}{S_0}; t\right) \\ &= U_0(S, S_0; t)\left[1 - \exp\left(-\frac{\log\left(\frac{S}{H}\right)\log\left(\frac{S_0}{H}\right)}{\frac{1}{2}\sigma^2 t}\right)\right], \end{aligned} \tag{1.20}$$

where U_0 is given by equation (1.19).

This single-barrier kernel is valid for *either lower* domain, S, $S_0 \in (0, H]$, *or upper* domain, S, $S_0 \in [H, \infty)$,with H being an upper barrier or lower barrier, respectively. This pricing kernel satisfies the appropriate forward and backward time-homogeneous Kolmogorov equations in S, S_0, which can be respectively written as

$$\frac{\partial U}{\partial t} = \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U}{\partial S^2} - rS\frac{\partial U}{\partial S}, \tag{1.21}$$

and

$$\frac{\partial U}{\partial t} = \frac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 U}{\partial S_0^2} + rS_0\frac{\partial U}{\partial S_0}. \tag{1.22}$$

This pricing kernel also satisfies zero boundary conditions at the barrier value for both $S = H$ and $S_0 = H$, and initial condition $\lim_{t\to 0} U(S, S_0; t) = \delta(S - S_0)$, the Dirac delta function.

European Barrier Option Formulas

The pricing kernel in (1.20) can be used to obtain exact analytical formulas for various types of single-barrier European-style options under GBM (i.e. Black-Scholes model). Given an arbitrary payoff function $\Lambda(S)$ at maturity time T, the fair value at current time $t_0 \leq T$ and spot price $S_0 > H$ of a down-and-out option with barrier level H is given by the discounted risk-neutral expectation over the domain above the barrier:

$$V^{DO}(S_0, t) = e^{-rt} \int_H^\infty U(H, S, S_0; t)\Lambda(S)dS, \tag{1.23}$$

where the option price is considered as a function of $t = T - t_0$, the time to maturity. The value of the corresponding up-and-out option with spot price $S_0 < H$ is given by the discounted risk-neutral expectation over the domain below the barrier:

$$V^{UO}(S_0, t) = e^{-rt} \int_0^H U(H, S, S_0; t)\Lambda(S)dS. \tag{1.24}$$

The values of the up-and-in and down-and-in options follow simply by (knock-in)-(knock-out) symmetry:

$$V^{UI} + V^{UO} = V^{DI} + V^{DO} = V, \tag{1.25}$$

where

$$V(S_0, t) = e^{-rt} \int_0^{\infty} U(S, S_0; t)\Lambda(S)dS \tag{1.26}$$

is the value of the plain European option with no barrier.

1.2.3 First Passage Time

The *first-passage* or *first-hitting time* of a diffusion process is the first time at which a process achieves a particular value . In formulating self-switching models, it is of importance to find the probability distribution for the first-hitting time of a diffusion. Consider the case of an *upper* barrier with current asset price $S_0 < H$, and let $\tau = t - t_0 > 0$ be the amount of time spent from current time t_0 until the barrier is first attained at calendar time t. Then, the *first hitting time up* at level $H > 0$ defined by the random variable $\tau_H^{(-)} := \inf\{\tau \geq 0 : S_\tau = H, S_0 < H\}$ has cumulative density:

$$P(\tau_H^{(-)} \leq \tau) := \Phi^{(-)}(H, S_0, \tau) = 1 - \int_0^H U(H, S, S_0; \tau)dS. \tag{1.27}$$

This represents the probability that the asset price process has attained the upper barrier H and has been absorbed by time τ, where $U(H, S, S_0; \tau)$ is given in equation (1.20). Since Φ is a cumulative function of the hitting time time $\tau_H^{(-)}$, the probability density function $f^{(-)}$ for the hitting time must be given by the derivative

$$f^{(-)}(H, S_0, \tau) = \frac{\partial \Phi^{(-)}(H, S_0, \tau)}{\partial \tau}. \tag{1.28}$$

The case where $S_0 > H$, i.e. H is a *lower* barrier, is treated similarly. Now $\tau_H^{(+)} := \inf\{\tau \geq 0 : S_\tau = H,\ S_0 > H\}$ is the *first hitting time down* at level $H > 0$ with cumulative density

$$P(\tau_H^{(+)} \leq \tau) := \Phi^{(+)}(H, S_0, \tau) = 1 - \int_H^\infty U(H, S, S_0; \tau) dS. \tag{1.29}$$

Differentiating with respect to τ gives the first hitting time up density:

$$f^{(+)}(H, S_0, \tau) = \frac{\partial \Phi^{(+)}(H, S_0, \tau)}{\partial \tau}. \tag{1.30}$$

By applying the forward Kolmogorov equation in (1.21), one readily shows that the expressions for $f^{(\pm)}$ are equivalent. Hence, in what follows we simply denote $f^{(\pm)}(H, S_0, \tau) = f(H, S_0, \tau)$. Explicit closed-form expressions for this density will be given and discussed in Chapter 2.

Chapter 2

Regime Switching Models

In this chapter, we formulate extensions of geometric Brownian motion. In the first part of this chapter, we review the first passage time for the GBM diffusion and introduce the self-switching model. The latter part of the chapter focuses on the derivation and simplification of transition density functions. We start with the single-switch model and then generate the density function for mutiple-switch models.

2.1 First Passage Time for Regular Diffusion

As mentioned in the previous chapter, the cumulative density function for first passage time is given in the form of (1.27) for lower killing barrier or (1.29) for upper killing barrier. For the case of geometric Brownian motion with constant drift r and volatility σ, we obtain exact formulas:

$$\begin{aligned}\Phi^{(-)}(H, S_0, \tau) &= N\left(-d_-(\frac{S_0}{H})\right) + \left(\frac{H}{S_0}\right)^{\frac{2r}{\sigma^2}-1} N\left(d_-(\frac{H}{S_0})\right) \\ &= 1 - N\left(\frac{\log\frac{S_0}{H} + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \\ &\quad + \left(\frac{H}{S_0}\right)^{\frac{2r}{\sigma^2}-1} N\left(\frac{-\log\frac{S_0}{H} + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \end{aligned} \tag{2.1}$$

for $S_0 > H$, and

$$\begin{aligned}\Phi^{(+)}(H, S_0, \tau) &= N\left(d_-(\frac{S_0}{H})\right) + \left(\frac{H}{S_0}\right)^{\frac{2r}{\sigma^2}-1} N\left(-d_-(\frac{H}{S_0})\right) \\ &= N\left(\frac{\log\frac{S_0}{H} + (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \\ &\quad + \left(\frac{H}{S_0}\right)^{\frac{2r}{\sigma^2}-1} N\left(\frac{\log\frac{S_0}{H} - (r - \frac{1}{2}\sigma^2)\tau}{\sigma\sqrt{\tau}}\right) \end{aligned} \tag{2.2}$$

for $S_0 < H$. The first hitting time density f is a partial derivative of Φ with respect to τ. The exact formula can be written as an inverse Gaussian density:

$$f(H, S_0, \tau) = \frac{\left|\log\frac{S_0}{H}\right|}{\sigma\tau^{3/2}\sqrt{2\pi}} e^{-\frac{\left[\log\frac{S_0}{H} + (r - \frac{\sigma^2}{2})\tau^2\right]^2}{2\sigma^2\tau}}, \tag{2.3}$$

for all S_0, $H > 0$. This density is valid for both hitting time down or up at a barrier H.

2.2 Black-Scholes Model with Self-Switching

In financial time series, such as stock price data, it is common to observe states of differing volatility. In fact, volatility clusters into periods of "low" and "high" volatility. This is the phenomenon of heteroscedasticity. We incorporate this economic phenomenon by modelling the fluctuation of a single stock price process $(S_t)_{t\geq 0}$, using the following SDE:

$$\frac{dS_t}{S_t} = \mu_{\epsilon_t} dt + \sigma_{\epsilon_t} dB_t \tag{2.4}$$

where $\epsilon_t \equiv \epsilon(S_t)$ defined by

$$\epsilon_t = \begin{cases} 0, & \text{when the process is in a high volatility state,} \\ 1, & \text{when the process is in a low volatility state,} \end{cases} \tag{2.5}$$

is a stochastic process representing the state of a business cycle, i.e. is depending on the path of S_t, B_t is the standard Brownian motion which is independent of ϵ_t. For each state, the drift μ_{ϵ_t} and the volatility σ_{ϵ_t} are known (deterministic). In this thesis, we assume (within the risk-neutral measure $\mathbb{Q}$) that $\mu_{\epsilon_t} \equiv \mu = r$ is a constant for all $t \geq 0$. If the stock pays a dividend $q > 0$, then the drift rate is $\mu = r - q$. We also assume that $\sigma_0 > \sigma_1$.

We introduce respective upper and lower state-switching barriers e.g. H and L, $H > L$, as the natural barriers separating the two regions of low and high volatility. Each of these states represents the associated volatility cycle. The switch happens when the stock price with current state, hits the corresponding switching barrier. For example, the state is high volatility, i.e. $\sigma_{\epsilon_t} = \sigma_0$, then the corresponding switching barrier is the lower level L. Assume that after time $\tau = \tau_L > 0$ the stock price falls to L, where τ_L is the first hitting time down at L. The stock process then enters into a state of lower volatility, and we have $\sigma_{\epsilon_{t+\tau}} = \sigma_1$. Furthermore, since the initial state depends on the previous history of the stock price, we have that with $S_0 \geq H$ the process automatically starts from state 1; and when $S_0 \leq L$, the process automatically starts from state 0. If $L < S_0 < H$, then the process can start in either low volatility σ_1, if $\epsilon_0 = 1$, or high volatility σ_0, if $\epsilon_0 = 0$.

We note that an equivalent representation of the process $\epsilon_t \equiv \epsilon(S_t)$ is given by

$$\epsilon_t = \begin{cases} 0, \text{ when the stock is low risk, at time } t \geq 0, \\ 1, \text{ when the stock is in high risk, at time } t \geq 0, \end{cases} \tag{2.6}$$

In this case the two regions represent the different risk levels for stock price, and we assume $\sigma_1 > \sigma_0$. In what follows we will focus on this case in the study of self-switching models.

The switching of the diffusion process has certain switching probabilities. We assume that the probability of switching from one state into another, given that the current state is ϵ_i is the same as the probability that the asset price will hit the appropriate switching barrier: H if $\epsilon_i = 0$ or L if $\epsilon_i = 1$. In particular, the conditional cumulative probability of the first switch from state 0 to state 1 occurring by time t_1, given that the initial state is 0, can be written as:

$$\begin{aligned} & P\left(\epsilon_{t=t_1} = 1 \mid S_{t=0} = S_0, \epsilon_{t=0} = 0\right) \\ = \; & P\left(\tau_H < t_1 \mid S_{t=0} = S_0, \epsilon_{t=0} = 0\right) \\ = \; & \Phi^{(0)}(H, S_0, t_1) \end{aligned} \tag{2.7}$$

where $0 < t_1 < T$, $T =$ fixed maturity, and $\Phi^{(0)}$ is given in equation (2.1) with volatility $\sigma = \sigma_0$. The probability density for this switch obtains as

$$f^{(0)}(H, S_0, t_1) = \frac{\partial \Phi^{(0)}(H, S_0, t_1)}{\partial t_1}. \tag{2.8}$$

In similar fashion the switching probability for $1 \to 0$ by time t_1 is

$$\begin{aligned} & P\left(\epsilon_{t=t_1} = 0 \mid S_{t=0} = S_0, \epsilon_{t=0} = 1\right) \\ = \; & P\left(\tau_L < t_1 \mid S_{t=0} = S_0, \epsilon_{t=0} = 1\right) \\ = \; & \Phi^{(1)}(L, S_0, t_1). \end{aligned} \tag{2.9}$$

The density is simply

$$f^{(1)}(L, S_0, t_1) = \frac{\partial \Phi^{(1)}(L, S_0, t_1)}{\partial t_1}. \tag{2.10}$$

All subsequent switches at later times are determined similarly.

2.3 Self-switching Model with One Switch

We first consider the case in which only one state-switch is allowed up to maturity T. Let L_s denote the switching barrier. The total probability for stock price to attain value in $[S, S + \Delta), \Delta > 0$, at time T, conditional on an initially known price S_0 and state $\epsilon_0 = \epsilon(S_{t=0}) \in \{0, 1\}$ at time 0, is the sum of the probabilities of the two mutually exclusive events:

$$\begin{aligned} & P\{S_T \in [S, S+\Delta) \mid S_{t=0} = S_0, \epsilon_0 = \epsilon(S_0)\} \\ = \; & P\{S_T \in [S, S+\Delta), S_\tau = L_s, for\ some\ \tau \in [0,T]\} \\ & +P\{S_T \in [S, S+\Delta), S_\tau \neq L_s, \forall \tau \in [0,T]\}. \end{aligned} \tag{2.11}$$

The first term represents the probability that the process switches once, and the second term represents the probability that no state-switch occurs. Let's focus on the case that the process $(S_t)_{t \geq 0}$ starts from state 0, i.e. $\epsilon(S_{t=0}) = 0$. Let $U_1(S, S_0, L_s, T \mid \epsilon(S_{t=0}) = 0)$ denote the *single-switch probability density* of the process given initial state 0, and therefore with upper barrier $L_s = H$. From (2.11), the transition density

$$U_1 = \lim_{\Delta \to 0^+} \frac{P\{S_T \in [S, S+\Delta) \mid S_{t=0} = S_0, \epsilon_0 = \epsilon(S_0)\}}{\Delta} \tag{2.12}$$

is given by:

$$\begin{aligned} & U_1(S, S_0, H, T \mid \epsilon(S_{t=0}) = 0) \\ = \; & U_{OH}(S, S_0, H, T) + U_{NH}(S, S_0, H, T), \end{aligned} \tag{2.13}$$

for all $0 < S_0, S < H$. The transition density functions for the two cases, "One-Hit" and "No-Hit", are denoted as U_{OH} and U_{NH}, respectively, and are defined as follows:

$$U_{OH}(S, S_0, H, T) = \int_0^T f^{(0)}(H, S_0, t_1) U^{(1)}(S, H, T - t_1) dt_1, \tag{2.14}$$

$$
\begin{aligned}
& U_{NH}(S, S_0, H, T) \\
= \; & U^{(0)}(H, S, S_0, T)\mathbb{I}_{\{0<S<H\}} \\
= \; & \left[U^{(0)}(S, S_0, T) - (\frac{H}{S_0})^{(\frac{2r}{\sigma_0^2}-1)} U^{(0)}(S, \frac{H^2}{S_0}, T) \right] \mathbb{I}_{\{0<S<H\}}. \qquad (2.15)
\end{aligned}
$$

where $f^{(0)}$, defined as in equation (2.8) with $\sigma = \sigma_0$, is the first hitting time density for the stock price attaining the value of upper switching barrier, and $U^{(1)}$ is defined in equation (1.19) with $\sigma = \sigma_1$.

Recall from Chapter 1 that the knock-in options have zero value unless the asset price S_t attains the barrier at a time before maturity time T. From this point, we can consider U_{OH} in equation (2.14) as an infinitesimally narrow butterfly up-and-in option with state-switching. Similarly, U_{NH} from (2.15) can be considered as an infinitesimally narrow butterfly (or up-and-out option) where $0 < S, S_0 < H$. Hence we can use the (knock-in)-(knock-out) symmetric relation defined in (1.25) to re-cast the single-switch kernel in (2.13) as:

$$
\begin{aligned}
& U_1 \left(S, S_0, H, T \mid \epsilon(S_{t=0}) = 0\right) \\
= \; & \int_0^T f^{(0)}(H, S_0, t_1) U^{(1)}(S, H, T - t_1) dt_1 + U^{(0)}(H, S, S_0, T) \\
= \; & \int_0^T f^{(0)}(H, S_0, t_1)[U^{(1)}(S, H, T - t_1) - U^{(0)}(S, H, T - t_1)] dt_1 \\
& + \int_0^T f^{(0)}(H, S_0, t_1) U^{(0)}(S, H, T - t_1) dt_1 + U^{(0)}(H, S, S_0, T)
\end{aligned}
$$

$$= \int_0^T f^{(0)}(H, S_0, t_1)[U^{(1)}(S, H, T - t_1) - U^{(0)}(S, H, T - t_1)]dt_1 + U^{(0)}(S, S_0, T) \tag{2.16}$$

Note that the second term in the second step represents an infinitesimally narrow butterfly up-and-in option without state-switching and the third term represents a corresponding infinitesimally narrow butterfly up-and-out option with the same volatility. By (knock-in)-(knock-out) symmetric relation, the sum of these two terms is the transition density of regular geometric Brownian motion with fixed volatility.

Similarly, by reversing the states, the transition density for the single-switch stock price process starting from state 1, i.e. $\epsilon(S_{t=0}) = 1$, and for a lower switching barrier at L is

$$\begin{aligned} & U_1 \left(S, S_0, L, T \mid \epsilon(S_{t=0}) = 1\right) \\ = & \int_0^T f^{(1)}(L, S_0, t_1)[U^{(0)}(S, L, T - t_1) - U^{(1)}(S, L, T - t_1)]dt_1 \\ & + U^{(1)}(S, S_0, T). \end{aligned} \tag{2.17}$$

This single-switch kernel satisfies the backward time-homogeneous Kolmogorov equation given in (1.22). The proof is given in Appendix A.

Note that in the special case where the state remains the same during the whole time period, i.e. with $\sigma_0 = \sigma_1$, then we simply recover the no-switch transition kernel for the diffusion process with given volatility in state

$\epsilon_0 \in \{0,1\}, L_s \in \{L,H\}:$

$$\begin{aligned}
& U_1(S, S_0, L_s, T \mid \epsilon_0 \in \{0,1\}) \\
= & \int_0^T f^{(\epsilon_0)}(L_s, S_0, t_1)[U^{(\epsilon_0)}(S, L_s, T-t_1) - U^{(\epsilon_0)}(S, L_s, T-t_1)]dt_1 \\
& + U^{(\epsilon_0)}(S, S_0, T) \\
= & U^{(\epsilon_0)}(S, S_0, T). \qquad (2.18)
\end{aligned}$$

Moreover, we can show that, under the risk-neutral measure, the discounted stock price for the one-switch process satisfies the *martingale property.* This follows from the fact that the stock price process for a given (fixed) state is a martingale under the risk-neutral measure. The *conditional* expectation with respect to the risk-neutral measure, where $(e^{-rt}S_t)_{t\geq 0}$ represents the discounted price process with one switch $\epsilon_0 \rightarrow \epsilon_1$, being allowed, is given by:

$$\begin{aligned}
& E^{\mathbb{Q}}\left[e^{-rT}S_T \mid S_{t=0} = S_0\right] \\
= & e^{-rT}\int_0^\infty U_1(S, S_0, L_s, T \mid \epsilon_0 = \epsilon(S_0))\, SdS \\
= & \int_0^\infty e^{-rT}\{\int_0^T f^{(\epsilon_0)}(L_s, S_0, t_1)[U^{(\epsilon_1)}(S, L_s, T-t_1) \\
& - U^{(\epsilon_0)}(S, L_s, T-t_1)]dt_1 + U^{(\epsilon_0)}(S, S_0, T)\}SdS \\
= & \int_0^T e^{-rt_1} f^{(\epsilon_0)}(L_s, S_0, t_1)[E^{\mathbb{Q}}\left[e^{-r(T-t_1)}S_{t_1} \mid S_{t_1} = L_s\right] \\
& - E^{\mathbb{Q}}\left[e^{-r(T-t_1)}S_{t_1} \mid S_{t_1} = L_s\right]]dt_1 + E^{\mathbb{Q}}\left[e^{-rT}S_T \mid S_{t=0} = S_0\right] \\
= & \int_0^T e^{-rt_1} f^{(\epsilon_0)}(L_s, S_0, t_1)[L_S - L_S]dt_1 + S_0 \\
= & S_0. \qquad (2.19)
\end{aligned}$$

Note that in the 2^{nd} last step we used the martingale property of the no-switch process in the risk-neutral measure:

$$\begin{aligned}
& e^{-r(T-t_1)}\int_0^\infty U^{(\epsilon_1)}\left(S, L_s, T-t_1\right) SdS \\
= \; & e^{-r(T-t_1)} E^{\mathbb{Q}}\left[S_T \mid S_{t_1} = L_s, \epsilon_1 \in \{0,1\}\right] \\
= \; & L_s
\end{aligned}$$

Since the risk-neutral measure is a martingale measure, the arbitrage-free option price of derivative contracts can be expressed as discounted expectations of future payoffs. We will cover option pricing in the next chapter.

2.4 Self-switching Model with Two Switches

Now let's consider the case in which a maximum of two state switches are allowed before maturity. Let $L_s^{(\epsilon_i)}$ denote the i^{th} switching barrier. Similar to the one-switch case, the total probability that $S_t \in [S, S+\Delta)$, $\Delta > 0$ at time T is the sum of the probabilities of *three* mutually exclusive events that include paths with: two hits, one hit and no hit. Let's first focus on the case that the process starts from state 0, i.e. $\epsilon(S_0) = 0$. For two switches we assume two barrier levels where $L_S^{(\epsilon_i)} \in \{H, L\}$, $0 < L < H$. Let $U_2(S, S_0, H, L, T \mid \epsilon(S_{t=0}) = 0)$ denote the transition density (kernel) of the process that takes into account up to two switches before time T, given initial state is state 0.

Then U is given by

$$\begin{aligned} & U_2\left(S, S_0, H, L, T \mid \epsilon(S_0)=0\right) \\ = \; & U_{TH}(S, S_0, H, L, T) + U_{OH}(S, S_0, H, T) \\ & + U_{NH}(S, S_0, H, T). \end{aligned} \tag{2.20}$$

Here U_{TH}, U_{OH} and U_{NH} are, respectively, the transition densities for the three possible events: Two-Hit, One-Hit and No-Hit. Note that since $\epsilon_0 = 0$, then switching occurs in the sequence of hitting H first, then L. From the first hitting time densities for the respective lower (L) and upper (H) barriers, we readily derive:

$$\begin{aligned} & U_{TH}(S, S_0, H, L, T) \\ = \; & \int_0^T f^{(0)}(H, S_0, t_1) \left[\int_{t_1}^T f^{(1)}(L, H, t_2) U^{(0)}(S, L, T-t_2) dt_2 \right] dt_1, \end{aligned} \tag{2.21}$$

$$U_{OH}(S, S_0, H, T) = \int_0^T f^{(0)}(H, S_0, t_1) U^{(1)}(S, H, T-t_1) dt_1, \tag{2.22}$$

and where $U_{NH}(S, S_0, H, T)$ is given by (2.15). Applying the same method as in last section (see (2.16)), we can simplify U_2 by iteration as follows:

$$\begin{aligned} & U_2\left(S, S_0, H, L, T \mid \epsilon(S_0)=0\right) \\ = \; & \int_0^T f^{(0)}(H, S_0, t_1) [\int_{t_1}^T f^{(1)}(L, H, t_2 - t_1) U^{(0)}(S, L, T-t_2) dt_2 \\ & + U^{(1)}(L, S, H, T-t_1)] dt_1 + U^{(0)}(H, S, S_0, T) \\ = \; & \int_0^T f^{(0)}(H, S_0, t_1) \{ \int_{t_1}^T f^{(1)}(L, H, t_2 - t_1) [U^{(0)}(S, L, T-t_2) \end{aligned}$$

$$
\begin{aligned}
&-U^{(1)}(S,L,T-t_2)]dt_2+U^{(1)}(S,H,T-t_1)\}dt_1+U^{(0)}(H,S,S_0,T)\\
=\;&\int_0^T f^{(0)}(H,S_0,t_1)\int_{t_1}^T f^{(1)}(L,H,t_2-t_1)[U^{(0)}(S,L,T-t_2)\\
&-U^{(1)}(S,L,T-t_2)]dt_2dt_1+\int_0^T f^{(0)}(H,S_0,t_1)[U^{(1)}(S,H,T-t_1)\\
&-U^{(0)}(S,H,T-t_1)]dt_1+U^{(0)}(S,S_0,T) \qquad (2.23)
\end{aligned}
$$

where $f^{(i)}$ is the first-passage time density for $S_0 < H$ for GBM, given in (2.3) with volatility σ_i, and $U^{(i)}$ is the lognormal density function for unconstrained GBM with volatility σ_i given in (1.19). Note that the first term in (2.23) involves stock price paths that hit level H first, and then $L < H$ (i.e. $(0) \rightarrow (1)$ state). The other two terms correspond to the transition density of single-switch model given in (2.16). The same formulation applies to the "double-switching" stock price process assume to start in state $\epsilon_0 = 1$. In this case, the first switch occurs at level $S_{t_1} = L$, while a second switch occurs when $S_{t_2} = H$. Following the derivation in (2.20)–(2.23) we have:

$$
\begin{aligned}
&U_2\left(S,S_0,H,L,T \mid \epsilon(S_{t=0})=1\right)\\
=\;&\int_0^T f^{(1)}(L,S_0,t_1)\int_{t_1}^T f^{(0)}(H,L,t_2-t_1)\\
&[U^{(1)}(S,H,T-t_2)-U^{(0)}(S,H,T-t_2)]dt_2dt_1\\
&+\int_0^T f^{(1)}(L,S_0,t_1)\left[U^{(0)}(S,L,T-t_1)-U^{(1)}(S,L,T-t_1)\right]dt_1\\
&+U^{(1)}(S,S_0,T). \qquad (2.24)
\end{aligned}
$$

Note that (2.24) follows by interchanging states $(0) \leftrightarrow (1)$ in (2.23).

2.5 Self-switching Model with Multiple Switches

2.5.1 Three-Switch Model

Before generalizing to multiple switches, let's consider one more step with up to three state switches allowed. From the simplification method given in sections (2.3)-(2.4), the transition density function, given initial state 0, for the switching process $(S_t)_{t\geq 0}$ that now takes into account up to a maximum of three regime switches is:

$$
\begin{aligned}
& U_3\left(S, S_0, H, L, T \mid \epsilon(S_{t=0}) = 0\right) \\
= & \int_0^T f^{(0)}(H, S_0, t_1) \int_{t_1}^T f^{(1)}(L, H, t_2 - t_1) \int_{t_2}^T f^{(0)}(H, L, t_3 - t_2) \\
& [U^{(1)}(S, H, T - t_3) - U^{(0)}(S, H, T - t_3)] dt_3 dt_2 dt_1 \\
& + \int_0^T f^{(0)}(H, S_0, t_1) \int_{t_1}^T f^{(1)}(L, H, t_2 - t_1) [U^{(0)}(S, L, T - t_2) \\
& - U^{(1)}(S, L, T - t_2)] dt_2 dt_1 \\
& + \int_0^T f^{(0)}(H, S_0, t_1) \left[U^{(1)}(S, H, T - t_1) - U^{(0)}(S, H, T - t_1)\right] dt_1 \\
& + U^{(0)}(S, S_0, T).
\end{aligned}
\tag{2.25}
$$

Note that the first (triple integral) term involves stock price paths that hit level H first, then $L < H$, and finally H again (i.e. $(0) \rightarrow (1) \rightarrow (0)$ state). The other combined terms correspond to the density $U_2\left(S, S_0, H, L, T \mid \epsilon_0 = 0\right)$ in

(2.23), which takes into account paths that can switch states up to a maximum of two times. The analogue of (2.25) for the process starting in state $\epsilon_0 = 1$ (and possibly hitting L first, etc.) is:

$$
\begin{aligned}
& U_3\left(S, S_0, H, L, T \mid \epsilon(S_{t=0}) = 1\right) \\
= \; & \int_0^T f^{(1)}(L, S_0, t_1) \int_{t_1}^T f^{(0)}(H, L, t_2 - t_1) \int_{t_2}^T f^{(1)}(L, H, t_3 - t_2) \\
& [U^{(0)}(S, L, T - t_3) - U^{(1)}(S, L, T - t_3)] dt_3 dt_2 dt_1 \\
& + \int_0^T f^{(1)}(L, S_0, t_1) \int_{t_1}^T f^{(0)}(H, L, t_2 - t_1) [U^{(1)}(S, H, T - t_2) \\
& - U^{(0)}(S, H, T - t_2)] dt_2 dt_1 \\
& + \int_0^T f^{(1)}(L, S_0, t_1) \left[U^{(0)}(S, L, T - t_1) - U^{(1)}(S, L, T - t_1)\right] dt_1 \\
& + U^{(1)}(S, S_0, T), \qquad (2.26)
\end{aligned}
$$

where $f^{(i)}$ is the first-passage time density for $S_0 < H$ for GBM, given in (2.3) with volatility σ_i, and $U^{(i)}$ is the lognormal density function for unconstrained GBM with volatility σ_i given in (1.19).

2.5.2 Multiple-Switch Model

In this subsection, we derive a general formula for a two-state switching diffusion model with up to a total of $N \geq 1$ switches allowed before maturity time. Let $L_{\epsilon_i} \in \{L, H\}$ and $\{\epsilon_i\}_{i \geq 0} \in \{0, 1\}$ denote the i^{th} switching barriers the states at time $t \in [t_i, t_{i+1})$, where t_i denotes the time when the i^{th} switch occurs. The transition density function such a process, given an initial state

$\epsilon(S_{t=0}) = \epsilon_0$, is obtained by iterating the above derivations.

In particular, the density that takes into account no switching, only one switch when $S_{t_1} = H$, two switches when $S_{t_1} = H$, $S_{t_2} = L$, three switches for $S_{t_1} = H$, $S_{t_2} = L$, $S_{t_3} = H$, etc., is given by:

$$
\begin{aligned}
& U_N\left(S, S_0, H, L, T \mid \epsilon(S_{t=0}) = \epsilon_0\right) \\
= & \sum_{j=2}^{n-1} \int_0^T \int_{t_1}^T \cdots \int_{t_j}^T f^{(\epsilon_0)}(L_{\epsilon_1}, S_0, t_1) \prod_{i=2}^{j} f^{(\epsilon_{i-1})}(L_{\epsilon_i}, L_{\epsilon_{i-1}}, t_{i+1} - t_i) \\
& f^{(\epsilon_j)}(L_{\epsilon_{j+1}}, L_{\epsilon_i}, T - t_{j+1})[U^{(\epsilon_{j+1})}(S, L_{\epsilon_{j+1}}, T - t_{j+1}) \\
& -U^{(\epsilon_j)}(S, L_{\epsilon_{j+1}}, T - t_{j+1})]dt_j dt_{j-1} \cdots dt_2 dt_1 \\
& + \int_{t_1}^{T} f^{(\epsilon_0)}(L_{\epsilon_1}, S_0, t_1) \left[U^{(\epsilon_1)}(S, L_{\epsilon_1}, T - t_1) - U^{(\epsilon_0)}(S, L_{\epsilon_1}, T - t_1)\right] dt_1 \\
& + U^{(\epsilon_0)}(S, S_0, T), \qquad (2.27)
\end{aligned}
$$

where $f^{(i)}$ is the first-passage time density for $S_0 < H$ for GBM, given in (2.3) with volatility σ_{ϵ_i}, and $U^{(i)}$ is the lognormal density function for unconstrained GBM with volatility σ_{ϵ_i} given in (1.19). In practice, this model for the stock price process is rich enough to capture the empirical phenomena of financial data series. Computationally tractable and leads to implied volatility surfaces that can be calibrated fairly well to market option data. Hence, we only consider the case of multiple regime-switching with two regimes (i.e. two barriers H and L and two states $\epsilon_i \in \{0, 1\}$).

An important property of the above model with transition density (2.27) is that the discounted process $(e^{-rt}S_t)_{t \geq 0}$ is a martingale under the risk-neutral measure, $\mathbb{Q}$, i.e. $E^{\mathbb{Q}}\left[e^{-rT}S_T \mid S_{t=0} = S_0, \epsilon 0 = \epsilon(S_0)\right] = S_0, \; \forall T \geq 0$. This follows by integrating S with the density in (2.27) and the derivation is similar to the one given at the end of Section 2.3. Moreover, all paths are continuous diffusions, i.e. GBM's conditional on given states. Hence, by the Martingale Representation Theorem of Itô diffusions the model is a complete market model (for a single risky asset in this case).

Chapter 3

Option Pricing and Model Calibration

In this chapter we focus on the application of the two-state first passage self-switching model with up to two switches allowed. In the first two parts we obtain semi-analytical results for European style options and discuss the corresponding empirical performance of our models. In the last section, we use a nonlinear least square method to obtain a practical solution to the calibration problem.

3.1 European Option Pricing

Now we provide the derivation of semi-analytical pricing formulas for standard European style options. A standard European-style option with maturity (expiration) $T > 0$ is defined by its payoff function $\Lambda(S)$ on the terminal price $S_T = S$ at the maturity time. For example,

$$\Lambda(S,K) = \begin{cases} Max(S-K,0), \text{ for European call option,} \\ Max(K-S,0), \text{ for European put option,} \end{cases} \tag{3.1}$$

where $K > 0$ is a strike price. We first consider the the case with given initial state 0, i.e. $\epsilon(S_{t=0}) = 0$. Under the risk-neutral measure, $\mathbb{Q}$, the discounted stock price process satisfies the martingale property. The value of the corresponding European option at current time $t = 0$ is given by the discounted risk-neutral expectation of the payoff at maturity time T. Let V_2 denote the option value under the double-switch model, then:

$$\begin{aligned} & V_2\left(S_0, K, H, L, T \mid \epsilon(S_{t=0}) = 0\right) \\ = \; & e^{-rT} E\left[\Lambda(S_T, K) \mid S_{t=0} = S_0, \epsilon(S_{t=0}) = 0\right]. \end{aligned} \tag{3.2}$$

Throughout we assume a constant interest rate r. By using the risk-neutral transition density in (2.23), formula is reduced to:

$$\begin{aligned} & V_2\left(S_0, K, H, L, T \mid \epsilon(S_{t=0}) = 0\right) \\ = \; & e^{-rT} \int_0^{\infty} U_2\left(S, S_0, H, L, T \mid \epsilon(S_{t=0}) = 0\right) \Lambda(S,K) dS \\ = \; & \int_0^T e^{-rt_1} f^{(0)}(H, S_0, t_1) \int_{t_1}^T e^{-r(t_2-t_1)} f^{(1)}(L, H, t_2 - t_1) \\ & \Bigg[e^{-r(T-t_2)} \int_0^{\infty} U^{(0)}(S, L, T-t_2) \Lambda(S,K) dS \end{aligned}$$

$$
\begin{aligned}
& -e^{-r(T-t_2)}\int_0^\infty U^{(1)}(S,L,T-t_2)\Lambda(S,K)dS\Bigg] dt_2 dt_1 \\
& +\int_0^T e^{-rt_1} f^{(0)}(H,S_0,t_1)\Bigg[e^{-r(T-t_1)}\int_0^\infty U^{(1)}(S,H,T-t_1)\Lambda(S,K)dS \\
& -e^{-r(T-t_1)}\int_0^\infty U^{(0)}(S,H,T-t_1)\Lambda(S,K)dS\Bigg] dt_1 \\
& +e^{-rT}\int_0^\infty U^{(0)}(S,S_0,T)\Lambda(S,K)dS \\
= & \int_0^T e^{-rt_1} f^{(0)}(H,S_0,t_1)\int_{t_1}^T e^{-r(t_2-t_1)} f^{(1)}(L,H,t_2-t_1) \\
& \left[V^{(0)}(L,K,T-t_2)-V^{(1)}(L,K,T-t_2)\right] dt_2 dt_1 \\
& +\int_0^T e^{-rt_1} f^{(0)}(H,S_0,t_1)\left[V^{(1)}(H,K,T-t_2)-V^{(0)}(H,K,T-t_2)\right] dt_1 \\
& +V^{(0)}(S_0,K,T). \qquad (3.3)
\end{aligned}
$$

$V^{(i)}$ is the exact option pricing formula under the standard Black-Scholes model with volatility σ_i. Let C and P denote call and put option vales, respectively. Then the Black-Scholes formulas are:

$$C^{(i)}(S_0,K,T) = S_0 N\left(d_+^{(i)}(\frac{S_0}{K})\right) - e^{-rT} K N\left(d_-^{(i)}(\frac{S_0}{K})\right), \qquad (3.4)$$

$$P^{(i)}(S_0,K,T) = e^{-rT} K N\left(-d_-^{(i)}(\frac{S_0}{K})\right) - S_0 N\left(-d_+^{(i)}(\frac{S_0}{K})\right), \qquad (3.5)$$

where

$$d_\pm^{(i)}(x) = \frac{\log x + (r \pm \frac{\sigma_i^2}{2})T}{\sigma_i\sqrt{T}}. \qquad (3.6)$$

As mentioned earlier, if the spot price is in the region between the two switching barriers, the process can start with either states. Figure (3.1) shows

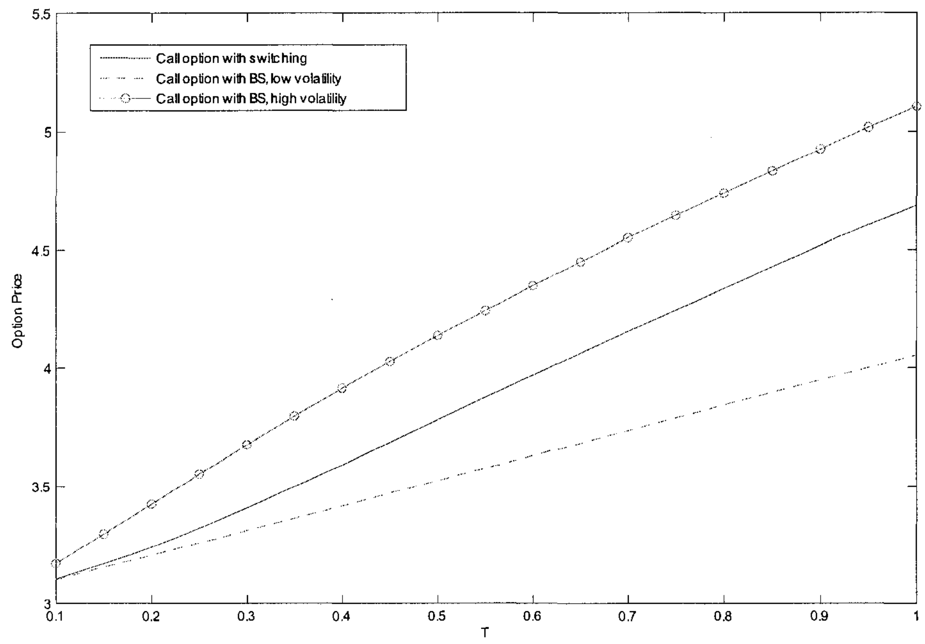

Figure 3.1: Option pricing with BS model and two-switching model with initial volatility σ_0.

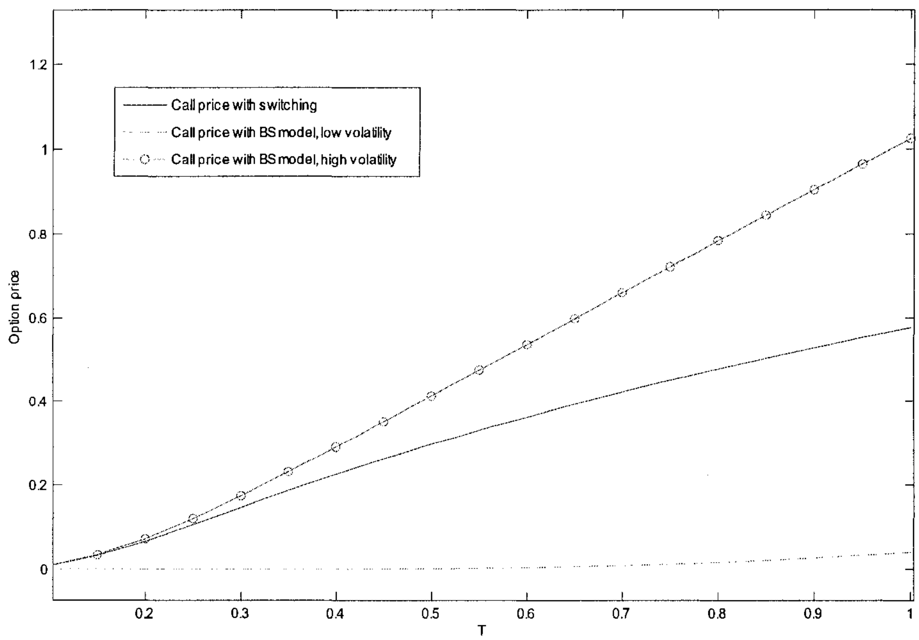

Figure 3.2: Option pricing with BS model and two-switching model with initial volatility σ_1.

some paths of option prices under the two-switch model with initial volatility σ_0, and also under the Black-Scholes (GBM) model. The parameters values are: $S_0 = 24$, $K = 21$, $H = 25$, $L = 15$, $\sigma_0 = 0.1$, $\sigma_1 = 0.3$, $r = 0.05$. Since the spot price S_0 is very close to the switching barrier H, the probability of switching is high. As we can see, the option price under the two-switch model is close to the one under standard B-S model with low volatility. As time passed, as a result of state-switching, the stock price tends to move upwards. Note that the option price is within the two classical B-S models for all $t \geq 0$. In Figure (3.2), we show the paths of stock price under the two-switch model with initial volatility σ_1, and also under the Black-Scholes model for comparison with parameters: $S_0 = 17$, $K = 21$, $H = 25$, $L = 15$, $\sigma_0 = 0.1$, $\sigma_1 = 0.3$, $r = 0.05$. We can see a similar behavior under two-switch model: the option price starts at where it is close to the value under B-S model with high volatility, and tends to decrease as the time passes. Again we can see that this option price is bounded by the two values under the two separate standard B-S models, for all $t \geq 0$.

Furthermore, since the discounted stock process S_t is a martingale under risk-neutral probability measure, for any admissible replicating strategy with $V(0) = 0$ we must have $E_0[V(T)] = 0$. Therefore there is no arbitrage, and thus the put-call parity is satisfied:

$$P_2 = C_2 - S + Ke^{-rT}. \tag{3.7}$$

In fact, this is also proven by (3.3).

We will study the empirical performance in the next section which focuses on pricing European call options. The put-call parity can be used to study the corresponding prices and behavior of puts.

3.2 Implied Volatility Surface

Now that we have pricing formulas under the two-switch model, it is of great interest to look at the types of implied volatility surfaces that can be generated by option prices produced by the two-switch model. Due to the difficulties of finding the closed-form inverse function of the B-S model, we use a root finding technique to solve the equation:

$$V_{BS}(S_0, K, T; \sigma) - V_2\left(S_0, K, H, L, T \mid \epsilon_0 \in \{0, 1\}\right) = 0, \tag{3.8}$$

where V_2 is the option value produced by the two-switch model with σ_0, σ_1 volatilities, and $\sigma_I = \sigma_I(K, T)$ is the corresponding B-S implied volatility. There are many techniques for finding roots, two of the most commonly used are Newton's method and Brent's method. Because options prices can move very quickly, it is often important to use the most efficient method when calculating implied volatilities. Since the Black-Scholes model yields a closed-form solution for the first partial derivative of the option's theoretical value with respect to volatility, i.e. *vega*, Newton's method can provide rapid convergence

. Hence we use Newton's method to generate the implied volatility surface for various strikes and times to maturity.

Let's assume the model parameters are as follows: a constant interest rate, $r = 0.05$, the volatilities for high risk regime and low rick regime are $\sigma_1 = 0.6$, $\sigma_0 = 0.3$, respectively. Figure (3.3) shows the implied volatility surfaces of process with different spot prices given initial state 0. And the implied volatility surfaces of process stars from state 1 are given in Figure (3.4). In both cases, the implied volatility surfaces show strong skew in short time to maturity, and the skewness is less pronounced in long time to maturity. A stock price process switching from state 0 (low volatility regime) to state 1 (high volatility regime) leads to an increase in implied volatility. And thus we observe concave surfaces in Figure (3.3). Similarly, a switching from a high volatility state to a low volatility state leads to a decrease in the implied volatility. Hence the stock price process with initial state 1 creates convex implied volatility surface. Note that the values for the implied volatility are bounded by σ_0 and σ_1. This supports the phenomenon that the option price under the two-switch model is within the two standard B-S model prices for all $t > 0$. See Figures (3.1) and (3.2). By fixing time to maturity and strike price, we can create other types of surfaces. Figures (3.5) and (3.6) show the implied volatility surfaces as a function in H and L with initial volatilities σ_0 and σ_1, respectively. Here we set $K = 100$ and $T = 1.5$. Assuming that the spot price is between H and L, the ranges of these two barriers are hence

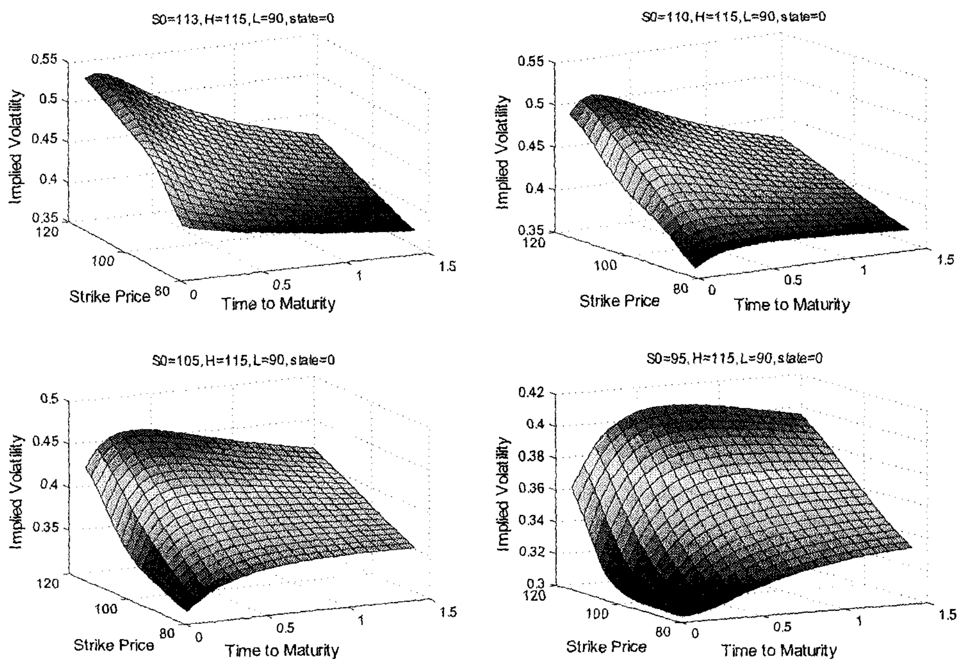

Figure 3.3: Implied volatility surfaces for inital state 0

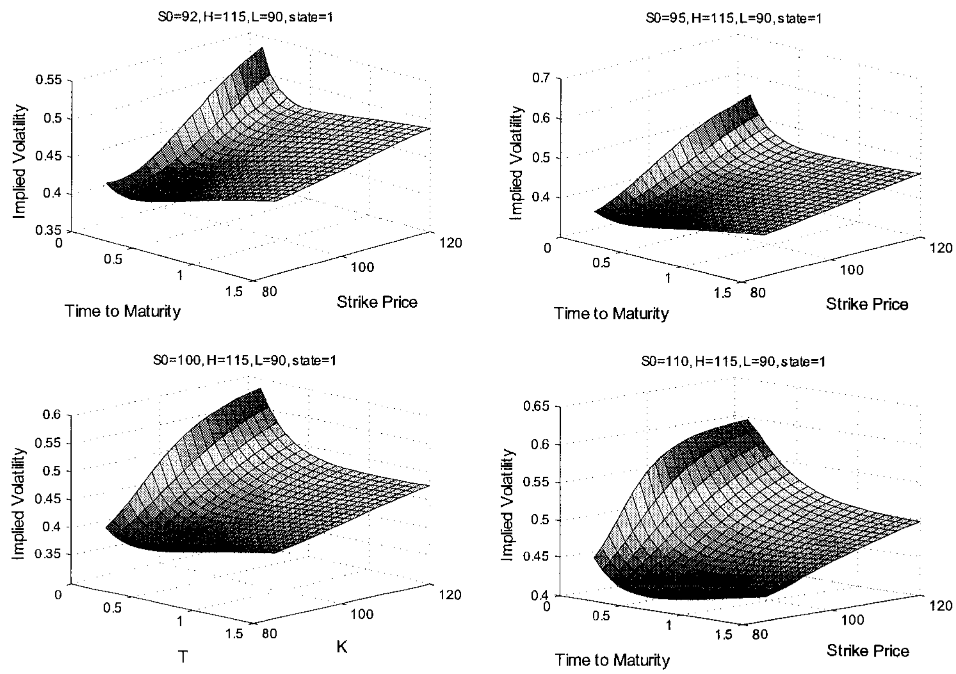

Figure 3.4: Implied volatility surfaces for initial state 1

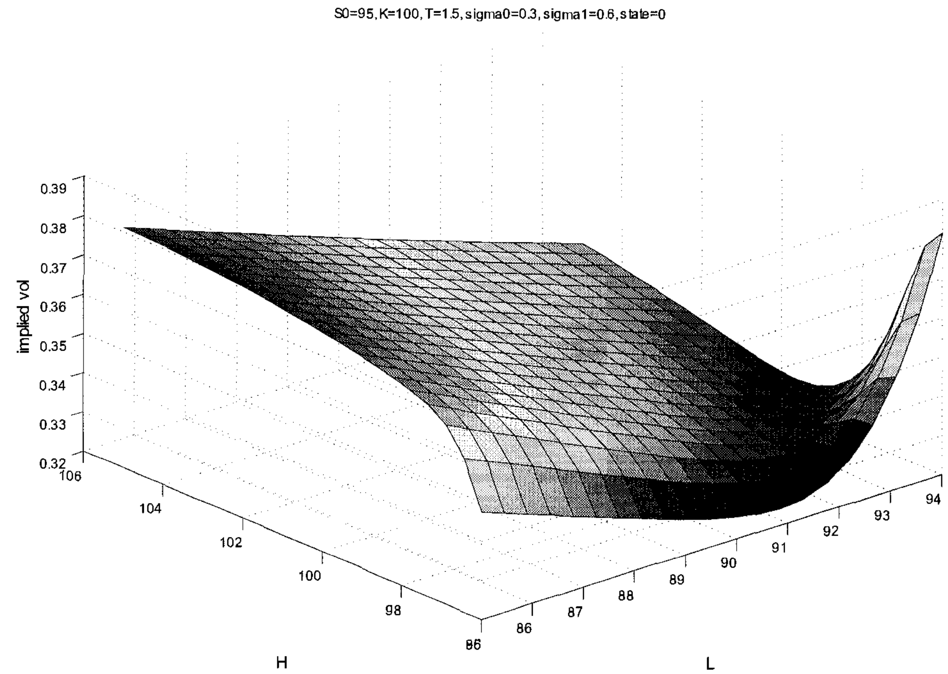

Figure 3.5: Implied volatilies changing in the values of H and L with initial state 0

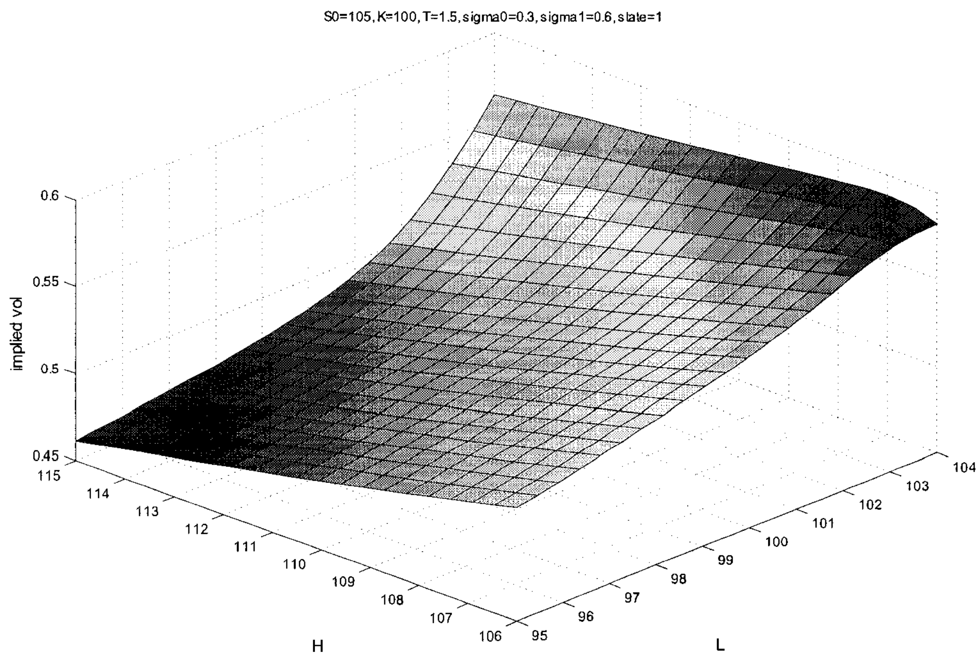

Figure 3.6: Implied volatilies changing in the values of H and L with initial state 1

different as S_0 changes. The two-switch model hence gives rise to many types of smiles and skews observed in the option markets.

3.3 Model Calibration

To test the ability of the two-switch model to reproduce observed option prices, we calibrate it to market data. We first introduce the method for model calibration.

3.3.1 Nonlinear Least Square Method

In order to obtain a practical solution to the calibration problem, we need to minimize the in-sample quadratic pricing error. More precisely, in the least squares method one must solve the following calibration problem:

Problem (Least-squares calibration) *Given a two-state self-switching model $(H(\theta), L(\theta), \sigma_0(\theta), \sigma_1(\theta))$ and observed prices C_i of call options for maturities T_i and strikes K_j, $1 \leq i \leq N_T, 1 \leq j \leq N_K$, find*

$$\theta^* = \arg\min_{\mathbb{Q}_\theta \in \Omega} \sum_{i=1}^{N_T} \sum_{j=1}^{N_K} \omega_{ij} \left| C^\theta(T_i, K_j) - C_i \right|^2, \tag{3.9}$$

where C^θ denotes the call option price computed for the self-switching model with parameters $(H(\theta), L(\theta), \sigma_0(\theta), \sigma_1(\theta))$ and Ω is the set of martingale measures.

The relative weights ω_{ij} of option prices in the functional to be minimized should reflect our confidence in individual data points, which is determined by the liquidity of a given option. This can be assessed from the bid-ask spreads:

$$\omega_{ij} = \frac{1}{\left|C_{ij}^{bid} - C_{ij}^{ask}\right|^2}. \tag{3.10}$$

Therefore, we only use options with bid/ask price quotes for our model calibration.

3.3.2 Calibration

We obtain the market data for Research In Motion Limited on November 17^{th}, 2007 from Yahoo Finance. The market data consists with European call ask and bid prices with spot price \$103.01, time to maturity ranging from 0.1 to 1.15 in years and 18 different strike prices from \$60 to \$190. The data is given in appendix B. The market implied volatility surface is shown in Figure (3.7). Note that this surface shows a smile in the implied volatility for short time to maturity and also a downward sloping term structure. Since the market implied volatility has higher value for smaller strike price in short time to maturity, we assume that the market has a higher volatility for lower stock price regime (i.e. $\sigma_0 > \sigma_1$).

We first optimize the model subject to two variables: H and L. In this case we choose the minimum and the maximum values of market implied volatilities as the variances to insure that the model implied volatilities fall in this range, where $\sigma_0 = 0.97591$ and $\sigma_1 = 0.6634$. The optimum parameters are: $H^* = 104.0101$, $L^* = 40$, obtained with initial state 1. The implied volatility surface

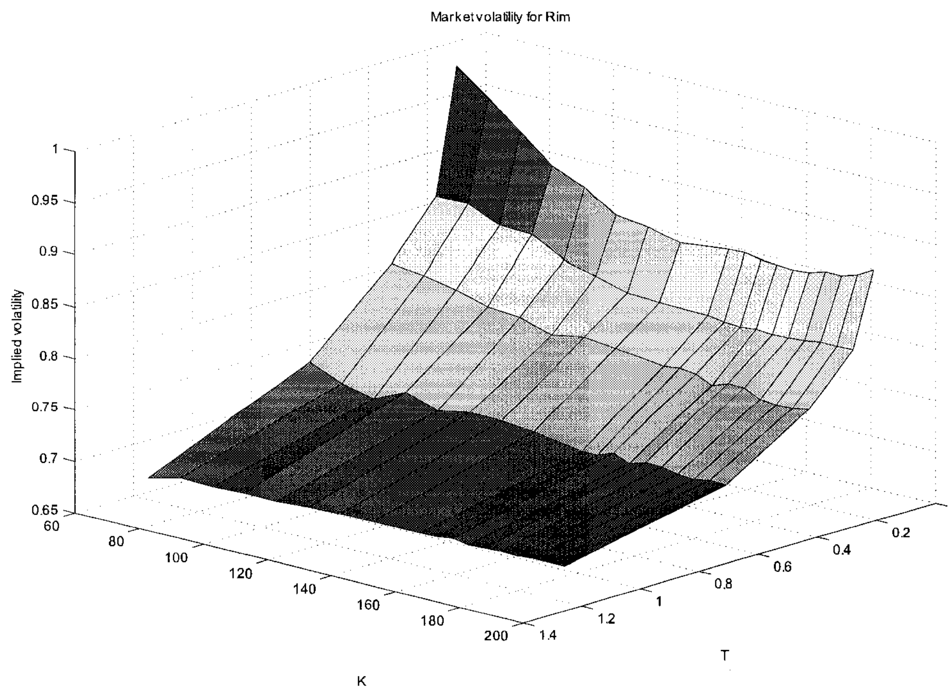

Figure 3.7: Market impled volatility surface for RIM on November 17, 2007

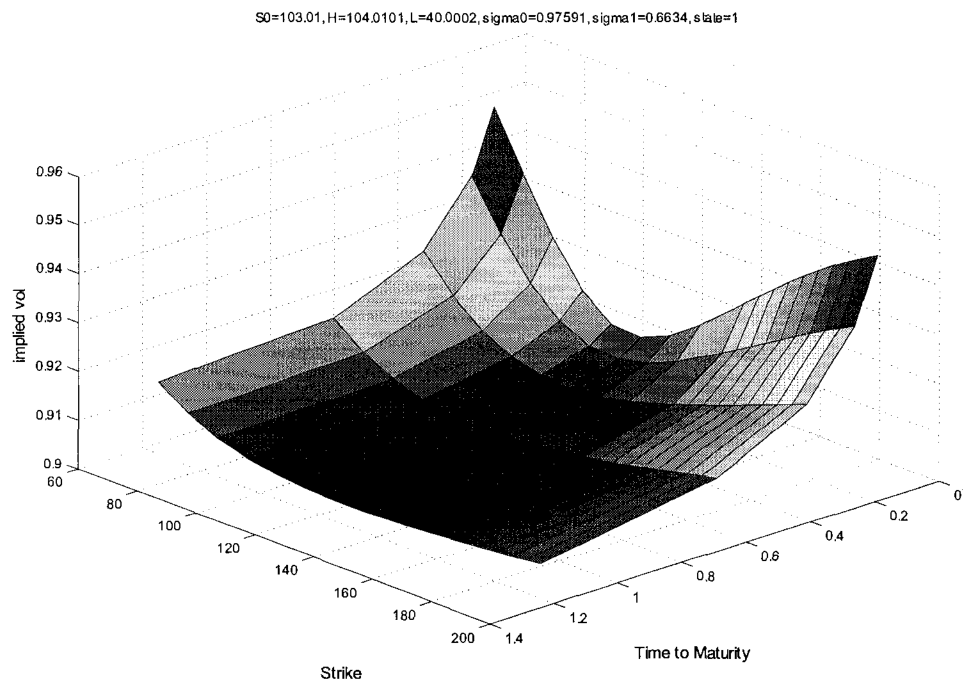

Figure 3.8: Implied volatility surface with optimized value of H and L

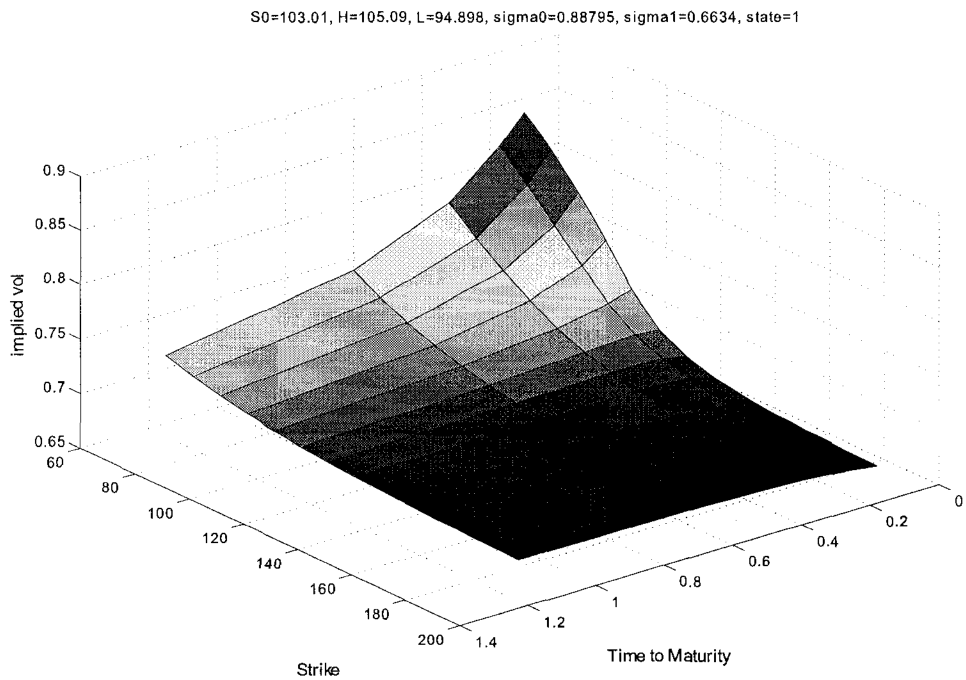

Figure 3.9: Implied volatility surface with optimized value of H, L, σ_0 and σ_1

with two optimized parameters is given in Figure (3.8). The implied volatility surface of optimizing the model subject to four variables: H, L, σ_0 and σ_1, is shown in Figure (3.9). The values of optimum are: $H^* = 105.09$, $L^* = 94.898$, $\sigma_0^* = 0.88795$ and $\sigma_1^* = 0.6634$ obtained with initial state 1. As in the case of two optimum parameters, the implied volatility generates a smile surface in short time to maturity. The smile flattens as time to maturity increases. In the case of four optimum parameters, the implied volatility surface produces a strong skew in short time to maturity, the implied volatility decreasing as a function of strike price. For long time to maturity the skew is less pronounced.

Chapter 4

Exotic Option Pricing

In this Chapter we derive pricing formulas for double-barrier knock-in style options. In particular,we obtain pricing formulas for two models: with and without state-switch when the stock price attains the barrier.

4.1 Double-barrier Knock-in Option without State-Switching

For double-barrier knock-in style option with upper barrier H and lower barrier L, the contract has nonzero payoff only if the stock price attains both barriers H and L at least once before or at maturity time T. We first consider the case where no state-switching takes place upon reaching the barrier, i.e. volatility is a constant for all $t > 0$. Let t_H and t_L denote the upper and lower hitting times. The probability of the stock price $S_T \in [S, S + \Delta)$, $\Delta > 0$, starting at

$S_{t=0} = S_0$ and attains both barrier values (H and L) is a sum of of probabilities of two mutually exclusive events:

$$\begin{aligned} & P\left(S_T \in [S, S+\Delta) \mid S_{t=0} = S_0, S_{t_H} = H \text{ and } S_{t_L} = L\right) \\ = \;& P\left(S_T \in [S, S+\Delta), t_H < t_L \leq T\right) \\ & + P\left(S_T \in [S, S+\Delta), t_L < t_H \leq T\right). \end{aligned} \tag{4.1}$$

The first term on the right hand side represents the probability of all stock paths that hit H first, and the second term represents the probability that L is attained before H. Dividing (4.1) by Δ, and taking the limit $\Delta \to 0^+$, leads to the transition density for a double knock-in. Based on the first hitting time density under a fixed state, the risk-neutral price of a double knock-in is:

$$\begin{aligned} V\left(S_0, H, L, K, T\right) = & \int_0^T e^{-rt_1} f(H, S_0, t_1) \int_{t_1}^T e^{-r(t_2 - t_1)} f(L, H, t_2 - t_1) \\ & V(L, K, T - t_2) dt_2 dt_1 \\ & + \int_0^T e^{-rt_1} f(L, S_0, t_1) \int_{t_1}^T e^{-r(t_2 - t_1)} f(H, L, t_2 - t_1) \\ & V(H, K, T - t_2) dt_2 dt_1, \end{aligned} \tag{4.2}$$

where f is defined in equation (2.3) and V is defined by (3.4) for a call option or (3.5) for a put option. Note that in (4.2) we used

$$V(L_s, K, T - t_2) = e^{-r(T - t_2)} E^{\mathbb{Q}}\left[\Lambda(S_T, K) \mid S_{t_2} = L_s\right], \tag{4.3}$$

for $L_s = H$ and $L_s = L$.

4.2 Double-Knock-In Option with State-Switching

We now consider a double-barrier knock-in style option where state-switching occurs once the stock price attains barriers at the two switching times. Let's focus on the case where process starts from lower state, i.e. $\sigma_{\epsilon_0} = \sigma_0$. In this case, the upper barrier H is the first switching barrier and L is the second switching barrier. As discussed in last section, generally the process is allowed to hit either barrier first, we have double-switch paths (H first, L second) and also single-switch paths, i.e. hitting L first and then H. Note that in the latter case the state remains as $\epsilon_0 = 0$ upon hitting L. The value of the double knock-in option given starting state $\epsilon_0 = 0$ is:

$$\begin{aligned} & V\left(S_0, H, L, K, T \mid \epsilon(S_{t=0}) = 0\right) \\ = & \int_0^T e^{-rt_1} f^{(0)}(H, S_0, t_1) \int_{t_1}^T e^{-r(t_2-t_1)} f^{(1)}(L, H, t_2 - t_1) \\ & V^{(0)}(L, K, T - t_2) dt_2 dt_1 \\ & + \int_0^T e^{-rt_1} f^{(0)}(L, S_0, t_1) \int_{t_1}^T e^{-r(t_2-t_1)} f^{(0)}(H, L, t_2 - t_1) \\ & V^{(1)}(H, K, T - t_2) dt_2 dt_1, \end{aligned} \tag{4.4}$$

where $f^{(i)}$ is defined in equation (2.3) and $V^{(i)}$ is defined by (3.4) for a call option or (3.5) for a put option with volatility σ_i. The analogues pricing formula for initial state $\epsilon_0 = 1$ is then:

$$V\left(S_0, H, L, K, T \mid \epsilon(S_{t=0}) = 1\right)$$

$$= \int_0^T e^{-rt_1} f^{(1)}(L, S_0, t_1) \int_{t_1}^T e^{-r(t_2-t_1)} f^{(0)}(H, L, t_2 - t_1)$$
$$V^{(1)}(H, K, T - t_2) dt_2 dt_1$$
$$+ \int_0^T e^{-rt_1} f^{(1)}(H, S_0, t_1) \int_{t_1}^T e^{-r(t_2-t_1)} f^{(1)}(L, H, t_2 - t_1)$$
$$V^{(0)}(L, K, T - t_2) dt_2 dt_1, \tag{4.5}$$

where $f^{(i)}$ is defined in equation (2.3) and $V^{(i)}$ is defined by (3.4) for a call option or (3.5) for a put option with volatility σ_i. Note that the second term in (4.5) represents the case that the state remains as $\epsilon_0 = 1$ upon hitting H.

Chapter 5

Conclusion and Future Work

In this thesis we introduced a class of asset pricing model by adjoining continuous diffusion (i.e. GBM) model for the stock fluctuation with volatility regime-switching via first passage time. we referred to these models as self-switching models. We gave a theoretical formulation of this class of models and derived the corresponding transition densities for such processes. Starting with the simplest single-switch model, we derived a general formula for the transition density of self-switching diffusions with a finite number of switching times. Analytical integral representations for European style option prices were given, and we used such formulas to investigate the implied volatility surfaces. We studied the calibration problem under a double-switch model with four adjustable parameters. Finally we introduced some applications of the self-switching models to pricing exotic options. The following are some conclusions drawn from this thesis.

- For all volatility self-switching models, there exists a risk-neutral probability measure $\mathbb{Q}$ such that the discounted asset (stock) price process $(S_t)_{0\leq t\leq T}$ is a martingale under $\mathbb{Q}$. This leads to arbitrage-free pricing of derivative contracts. Moreover, our self-switching models form a class of complete market models consisting of a risky asset and a money market account.

- The transition density function satisfies the backward Kolmogorov equation.

- The option price under the volatility self-switching model is bounded by the prices given by the two standard B-S models with low and high volatilities.

- The model can produce different kinds of skewed (and smile) implied volatility surfaces by choosing different parameters.

- The current volatility regime (i.e. state) of an asset can be extracted by solving the calibration problem.

The self-switching models give rise to a number of further applications. First, the model can be extended to multiple-state switching. And in this case the models can be used as alternate level-dependent models. Second, there are many types of option pricing formula that haven't been given under this model. Since the first-passage time problem is involved, one can also considered look-back style option pricing. Finally, we have only consider one-dimensional volatility self-switching geometric Brownian motion in this thesis. In fact, the model can be extended to two-dimensional volatility self-switching geometric Brownian motions (e.g. two underlying stocks). In this case, we need to study the joint density of first-passage time for two stocks conditional on the initial state of each stock price process. Our methodology can also be adapted to cases in which the switching process is dictated by other financial observables such as the interest rate and exchange rate dynamics.

Appendix A

The transition density for the multi-switch model satisfies the backward Kolmogorov equation (with respect to either starting state). It suffices to show this for the one-switch diffusion case starting in state 0, since the proof follows similarly for the more general multi-switch case.

The conditionally stationary transition density function of a stock price process under one switch model is given in (2.13):

$$\begin{aligned} & U_1\left(S, S_0, H, T \mid \epsilon\left(S_0\right)=0\right) \\ = \ & U_{OH}(S, S_0, H, T)+U_{NH}(S, S_0, H, T) \\ = \ & \int_0^T f^{(0)}(H, S_0, t_1) U^{(1)}(S, H, T-t_1) d t_1 \\ & +U^{(0)}(H, S, S_0, T). \end{aligned} \tag{1}$$

Recall that $U^{(0)}(H, S, S_0, T)$ is the transition density for GBM with a single absorbing barrier at H. Hence, it satisfies the backward Kolmogorov equation(BKE).

Defining $\mathcal{L}_{S_0}$ as:

$$\mathcal{L}_{S_0} U^{(0)}=\frac{1}{2} \sigma^2 S_0^2 \frac{\partial^2 U^{(0)}}{\partial S_0^2}+r S_0 \frac{\partial U^{(0)}}{\partial S_0}, \tag{2}$$

with $\sigma=\sigma_0$. Then BKE can be written as:

$$\frac{\partial U^{(0)}}{\partial T}=\mathcal{L}_{S_0} U^{(0)}. \tag{3}$$

Since the density of the one-hit process in (1) involves two probability density functions, we first need to verify that each of them satisfies the BKE. From the definition of $\Phi^{(+)}$ given in (1.29), we know that $\Phi^{(+)}(H, S_0, \tau)$ satisfies the BKE:

$$\frac{\partial \Phi^{(+)}(H, S_0, \tau)}{\partial \tau} = \mathcal{L}_{S_0} \Phi^{(+)}(H, S_0, \tau) \tag{4}$$

Since the transition density for the first hitting time is the partial derivative of $\Phi^{(+)}(\cdot)$ with respect to τ :

$$f(H, S_0, \tau) = \frac{\partial \Phi^{(+)}(H, S_0, \tau)}{\partial \tau}, \tag{5}$$

we have

$$\begin{aligned} \frac{\partial f(H, S_0, \tau)}{\partial \tau} &= \frac{\partial}{\partial \tau} \left(\frac{\partial \Phi^{(+)}(H, S_0, \tau)}{\partial \tau} \right) \\ &= \frac{\partial}{\partial \tau} \left(\mathcal{L}_{S_0} \Phi^{(+)}(H, S_0, \tau) \right) \\ &= \mathcal{L}_{S_0} \left(\frac{\partial \Phi^{(+)}(H, S_0, \tau)}{\partial \tau} \right) \\ &= \mathcal{L}_{S_0} f(H, S_0, \tau). \end{aligned} \tag{6}$$

Hence, $f(H, S_0, \tau)$ also satisfies the time-homogeneous BKE. The proof for $U^{(1)}(S, H, T - t_1)$ is trivial, since it is a pricing kernel.

Now the derivative of U_{OH} with respect to T can be written as:

$$\begin{aligned} \frac{\partial U_{OH}(S, S_0, T)}{\partial T} &= f^{(0)}(H, S_0, T) U^{(1)}(S, H, 0^+) \\ &+ \int_0^T f(H, S_0, t_1) \frac{\partial U^{(1)}(S, H, T - t_1)}{\partial T} dt_1, \end{aligned}$$

where $U^{(1)}(S,H,0^+) = 0$ since $S < H$. Using $\frac{\partial U^{(1)}}{\partial T}(S,H,T-t_1) = -\frac{\partial U^{(1)}}{\partial t_1}(S,H,T-t_1)$, and integrating by parts:

$$\begin{aligned}
\frac{\partial U_{OH}(S_0,H,T)}{\partial T} &= -\int_0^T f(H,S_0,t_1)\frac{\partial U^{(1)}(S,H,T-t_1)}{\partial t_1}dt_1 \\
&= -f^{(0)}(H,S_0,t_1)U^{(1)}(S,H,T-t_1)|_{t_1=0}^{t_1=T} \\
&\quad + \int_0^T \frac{\partial f^{(0)}(H,S_0,t_1)}{\partial t_1}U^{(1)}(S,H,T-t_1)dt_1 \\
&= \int_0^T \mathcal{L}_{S_0}\left[f^{(0)}(H,S_0,t_1)\right]U^{(1)}(S,H,T-t_1)dt_1 \\
&= \mathcal{L}_{S_0}\left[\int_0^T f^{(0)}(H,S_0,t_1)U^{(1)}(S,H,T-t_1)dt_1\right] \\
&= \mathcal{L}_{S_0}\left[U_{OH}(S,S_0,H,T)\right]. \qquad (7)
\end{aligned}$$

Therefore, the pricing kernel under the one-switch model satisfies BKE. However, we note that, in contrast to the (standard) no-switching diffusion, the forward Kolmogorov equation is not necessarily satisfied. This is not surprising since the stock price switching process is itself not Markovian.

Appendix B: Market Data

Market data for Research In Motion Limited on November 17, 2007, are presented in the following tables. This data was used in Section 3.3.3 for model calibration.

Strike\Time to Maturity	0.10	0.17	0.32	0.60	1.15
60	43.85	44.5	46.3	49.1	53.55
70	34.7	35.85	38.45	41.9	47.7
80	26.35	28.05	31.5	35.55	42.35
90	19.1	21.5	25.45	30.55	37.7
100	13.3	15.9	20.45	25.65	33.55
110	8.8	11.55	16.15	21.9	30
120	5.7	8.2	12.95	18.55	26.85
130	3.55	5.8	10.2	15.65	24.05
145	1.76	3.4	7.05	12	20.45
150	1.38	2.81	6.3	11.05	19.4
155	1.06	2.33	5.55	10.35	18.5
160	0.82	1.95	4.8	9.35	17.45
165	0.63	1.61	4.35	8.75	16.6
170	0.49	1.34	3.85	8.05	15.8
175	0.39	1.12	3.3	7.35	15
180	0.3	0.94	2.9	6.75	14.3
185	0.24	0.78	2.57	6.3	13.6
190	0.2	0.65	2.31	5.8	12.95

Table 1: Market Data: European Call ask price from Yahoo Finance on November 17, 2007. Spot price 103.01.

Strike\Time to Maturity	0.10	0.17	0.32	0.60	1.15
60	43.55	44.15	45.85	48.35	52.9
70	34.35	35.55	38.05	41.35	47.1
80	26	27.75	31.15	35.2	41.85
90	18.85	21.1	25.2	29.85	37.2
100	13.1	15.8	20.2	25.25	33.05
110	8.65	11.35	16.05	21.3	29.5
120	5.6	8	12.65	17.95	26.4
130	3.45	5.6	9.95	15.15	23.6
145	1.61	3.2	6.8	11.7	20.05
150	1.3	2.67	6	10.75	19
155	0.94	2.19	5.3	9.9	17.9
160	0.73	1.83	4.6	9.05	17.05
165	0.54	1.48	4.05	8.3	16.2
170	0.4	1.23	3.55	7.6	15.35
175	0.32	1.01	3.1	7.05	14.45
180	0.24	0.88	2.74	6.45	13.8
185	0.18	0.69	2.41	5.9	13.05
190	0.14	0.56	2.11	5.45	12.5

Table 2: Market Data: European Call bid price from Yahoo Finance on November 17, 2007. Spot price 103.01

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