2010

On the Complexity of Finding a Sun in a Graph

Chính T. Hoàng
Wilfrid Laurier University, choang@wlu.ca

SIAM J. DISCRETE MATH.
Vol. 23, No. 4, pp. 2156–2162

ON THE COMPLEXITY OF FINDING A SUN IN A GRAPH*

CHÍNH T. HOÀNG†

Abstract. The sun is the graph obtained from a cycle of length even and at least six by adding edges to make the even-indexed vertices pairwise adjacent. Suns play an important role in the study of strongly chordal graphs. A graph is chordal if it does not contain an induced cycle of length at least four. A graph is strongly chordal if it is chordal and every even cycle has a chord joining vertices whose distance on the cycle is odd. Farber proved that a graph is strongly chordal if and only if it is chordal and contains no induced suns. There are well known polynomial-time algorithms for recognizing a sun in a chordal graph. Recently, polynomial-time algorithms for finding a sun for a larger class of graphs, the so-called HHD-free graphs (graphs containing no house, hole, or domino), have been discovered. In this paper, we prove the problem of deciding whether an arbitrary graph contains a sun is NP-complete.

Key words. chordal graph, strongly chordal graph, sun

AMS subject classification. 68Q25

DOI. 10.1137/080729281

1. Introduction.

A *hole* is an induced cycle with at least four vertices. A graph is *chordal* if it does not contain a hole as an induced subgraph. Farber [6] defined a graph to be *strongly chordal* if it is chordal and every cycle in the graph on $2k$ vertices, $k \geq 3$, has a chord uv such that each segment of the cycle from u to v has an odd number of edges. We denote by *k-sun* the graph obtained from a cycle of length $2k$ ($k \geq 3$) by adding edges to make the even-indexed vertices pairwise adjacent. Figure 1 shows a 5-sun. A *sun* is simply a k-sun for some $k \geq 3$. Farber showed [6] that a graph is strongly chordal if and only if it is chordal and does not contain a sun as induced subgraph. Farber's motivation was a polynomial-time algorithm for the minimum weighted dominating set problem for strongly chordal graphs. The problem is NP-hard for chordal graphs [1]. In this paper, we prove that it is NP-hard to find a sun in an arbitrary graph. This result is motivated by the following discussion on chordal and strongly chordal graphs. For more information on this topic, see [3, 7].

We use $N(x)$ to denote the set of vertices adjacent to vertex x in a graph G. Define $N[x] = N(x) \cup \{x\}$. A vertex x in a graph is *simplicial* if $N(x)$ induces a complete graph. It is well known [4] that graph G is chordal if and only if every induced subgraph H of G contains a simplicial vertex of H. Farber proved [6] an analogous characterization for strongly chordal graphs. A vertex x in a graph is *simple* if the vertices in $N(x)$ can be ordered as $x_1, x_2, \ldots, x_k$ such that $N[x_1] \subseteq N[x_2] \subseteq \cdots \subseteq N[x_k]$. Thus, every simple vertex is simplicial. For a graph G, let $\mathcal{R} = v_1, v_2, \ldots, v_n$ be an ordering of vertices of G. Let $G(i) = G[\{v_i, v_{i+1}, \ldots, v_n\}]$, i.e., the subgraph induced in G by the set v_i through v_n of vertices. $\mathcal{R}$ is a *simple elimination ordering* for G if v_i is simple in $G(i)$, $1 \leq i \leq n$. The following is due to Farber [6].

THEOREM 1 (see [6]). *The following are equivalent for any graph G:*
- *G is strongly chordal.*
- *G is chordal and does not contain a sun.*
- *Vertices of G admit a simple elimination ordering.*

*Received by the editors July 3, 2008; accepted for publication (in revised form) September 2, 2009; published electronically January 22, 2010. This author's research was supported by NSERC.
http://www.siam.org/journals/sidma/23-4/72928.html

†Department of Physics and Computer Science, Wilfrid Laurier University, Waterloo, Ontario, Canada (choang@wlu.ca).

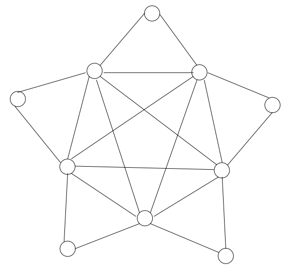

FIG. 1. *The 5-sun.*

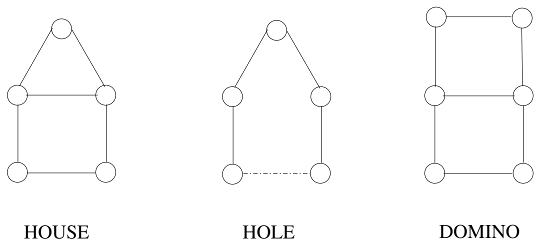

FIG. 2. *The house, the hole, and the domino.*

Thus, suns play an important role in the studies of chordal and strongly chordal graphs. There are well known algorithms [16, 12] to test whether a chordal graph is strongly chordal and thus whether it contains a sun. It is natural to investigate the problem of sun testing for larger classes of graphs. A graph is HHD-free if it does not contain a house, a hole, or a domino (see Figure 2). Every chordal graph is an HHD-free graph. HHD-free graphs [10] have several properties analogous to those of chordal graphs. Brandstädt [2] proposed the problem of finding a sun in an HHD-free graph. This problem was proved to be polynomial-time solvable in [13] and [5]. The absence of a sun in a graph seems to suggest that the graph has a certain structure. The author has thought, but has not been able to prove, that a sun-free HHD-free graph contains a homogeneous set or a simple vertex (a set H of vertices of a graph $G = (V, E)$ is homogeneous if $2 \leq |H| < |V|$ and every vertex outside H is adjacent to all, or to no vertices of H; homogeneous sets are also known as nontrivial modules). One may wonder whether the existence of the algorithms in [16, 12, 13, 5] is due to the property of being sun-free or of being chordal (or HHD-free). This has led several researchers to ask for the complexity of finding a sun in a graph. In this paper, we will prove the following.

THEOREM 2. *It is NP-complete to decide whether a graph contains a sun.*

The above theorem suggests that it is the property of being chordal (or HHD-free) that allows us to test for a sun efficiently and it is unlikely there is a polynomial-time algorithm for finding a sun in an arbitrary graph. Denote by *k-hole* the hole on k vertices. A *k-antihole* is the complement of a k-hole. A graph is *weakly chordal* [8] if it does not contain a k-hole or k-antihole with $k \geq 5$. Weakly chordal graphs generalize chordal graphs in a natural way, and they are known to be perfect and have many interesting algorithmic properties (see [9]). In spite of Theorem 2, it is conceivable there are polynomial-time algorithms to solve the sun recognition problem for weakly chordal graphs or even perfect graphs [15]. In this spirit, we will refine Theorem 2 to obtain a stronger result.

THEOREM 3. *It is NP-complete to decide whether a graph G contains a sun, even when G does not contain a k-antihole with $k \geq 7$.*

Let k-CLIQUE (respectively, k-SUN) be the problem whose instance is a graph G and an integer k, for which the question to be answered is whether G contains a clique on k vertices (respectively, k-SUN). It is well known [11] that k-CLIQUE is NP-complete. It is not difficult to prove but perhaps interesting to note that k-SUN is also NP-complete. Observe that if k is a constant (not part of the input), then the two problems can obviously be solved in polynomial time.

THEOREM 4. *k-SUN is NP-complete.*

Note that Theorem 2 implies Theorem 4: To decide whether a graph contains a sun, we need only to solve $O(n)$ instances of k-SUN with k running from 3 to $n/2$, where n is the number of vertices of the graph. However, we have a short and direct proof of Theorem 4. We will give the proofs of Theorems 2, 3, and 4 in the remainder of the paper.

2. The proofs. First, we need to introduce some definitions. For simplicity, we will say a vertex x *sees* a vertex y if x is adjacent to y; otherwise, we will say x *misses* y. Let G, F be two vertex-disjoint graphs, and let x be a vertex of G. We say that a graph H is obtained from G by *substituting* F for x if H is obtained by replacing x by F in G and adding the edge ab for any $a \in V(G) - \{x\}$ and any $b \in F$ whenever ax is an edge of G. In the proofs, we will often use the observation that every vertex in $H - F$ either sees all, or misses all, vertices of F.

By $(c_1, c_2, \ldots, c_k,\ r_1, r_2, \ldots, r_k)$ we denote the k-sun with vertices $c_1, c_2, \ldots, c_k$, $r_1, r_2, \ldots, r_k$ such that $c_1, c_2, \ldots, c_k$ induce a clique and $r_1, r_2, \ldots, r_k$ induce a stable set; each r_i has degree two and sees c_i, c_{i+1} with the subscripts taken modulo k. The vertices r_i will be called the *rays* of the k-sun. A *triangle* is a clique on three vertices.

We will rely on the following NP-complete problem due to Poljak [14].

STABLE SET IN TRIANGLE-FREE GRAPHS.
Instance: A triangle-free graph G, an integer k.
Question: Does G contain a stable set with k vertices?

Proof of Theorem 2. We will reduce *stable set in triangle-free graphs* to the problem of finding a sun in a graph.

Let $G = (V, E)$ be a triangle-free graph with $V = \{v_1, v_2, \ldots, v_n\}$, and without loss of generality assume $k \geq 4$. Define a graph $f(G, k)$ from G as follows. Substitute for each vertex v_i a clique $V_i = \{v_i^1, v_i^2, \ldots, v_i^k\}$; add a clique W with vertices $u_1, w_1, \ldots, u_k, w_k$; add a stable set X with vertices $x_1, \ldots, x_k$; for $i = 1, 2, \ldots, k$, add edges $x_i w_i$ and $x_i u_{i+1}$ (the subscripts are taken module k); for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots k$, add edges $v_i^j u_j, v_i^j w_j$. Figure 3 shows a graph G whose graph $f(G, 4)$ is shown in Figure 4 (for clarity, we do not show all edges of $f(G, 4)$; all adjacency between V_1 and W and between V_2 and W are shown, and adjacency between V_3 and W are not shown; the thick line between V_1 and V_2 (and between V_2 and V_3) represents all possible edges between the two sets; there are no edges between V_1 and V_3; each of the sets V_i, W induces a clique; the set X induces a stable set). We will often rely on the following observations.

Observation 1. Suppose G is triangle-free. Then $f(G, k)$ does not contain a triangle each of whose vertices belongs to a distinct V_i. □

Observation 2. Let x be a vertex in V_i, and y be a vertex in V_j with $i \neq j$. If x and y have a common neighbor z in W, then $N(x) \cap W = N(y) \cap W$. □

The theorem follows from the following claim.

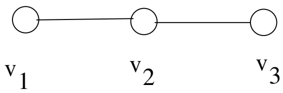

FIG. 3. *The graph G.*

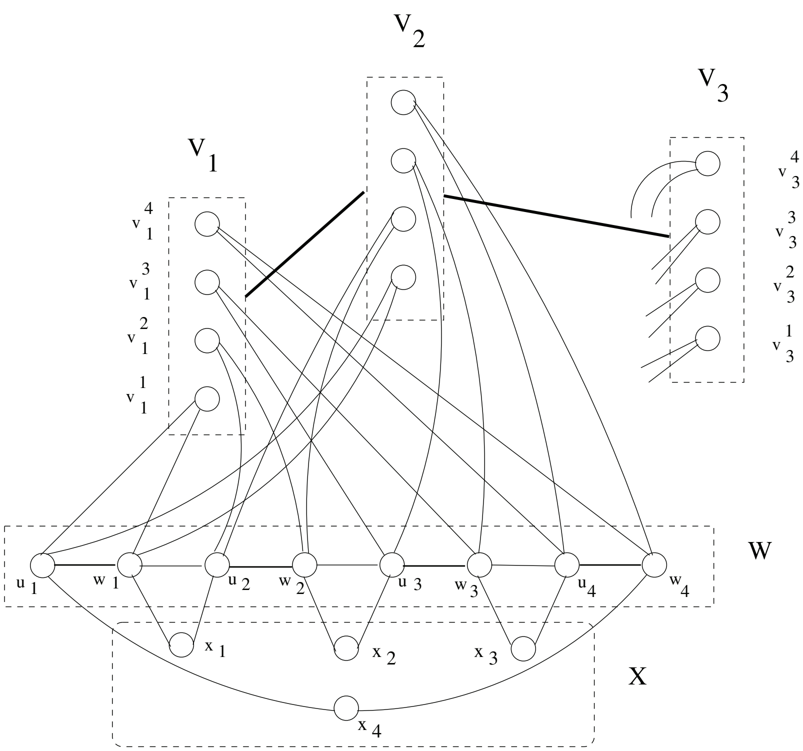

FIG. 4. *The graph f(G,4).*

CLAIM 1. *G has a stable set with k vertices if and only if $f(G,k)$ contains a sun.*

Proof of Claim 1. Suppose G has a stable set with vertices $v_1, v_2, \ldots, v_k$. Then $f(G,k)$ has a $2k$-sun $(c_1, c_2, \ldots, c_{2k}, r_1, r_2, \ldots, r_{2k})$ with $r_{2i-1} = v_i^i$, $r_{2i} = x_i$, $c_{2i-1} = u_i$, and $c_{2i} = w_i$ for $i = 1, 2, \ldots, k$.

Now, suppose $f(G,k)$ contains a sun. Write $T = V_1 \cup V_2 \cup \cdots \cup V_n$. We will establish that

(1) any sun S of $f(G,k)$ is a $2k$-sun with k rays in T.

Consider a sun $S = (c_1, c_2, \ldots, c_t, r_1, r_2, \ldots, r_t)$ of $f(G,k)$. First, we claim that (with the subscript taken modulo k)

(2) if a ray r_j lies in X, then r_{j-1}, r_{j+1} lie in T.

Let x_i be a vertex in X that is a ray r_j of S. We may assume that $c_j = w_i$ and $c_{j+1} = u_{i+1}$. Since r_{j-1} sees w_i and misses u_{i+1}, we have $r_{j-1} \in V_s$ for some s. Similarly, we have $r_{j+1} \in V_r$ for some r. Note that $r \neq s$. So, (2) holds.

Since W is a clique, S must have a ray in $T \cup X$. (2) implies that

(3) $$T \text{ contains a ray of } S.$$

Next, we will prove

(4) $$\text{if } r_i \in V_j, \text{ then } c_i, c_{i+1} \in W.$$

Suppose (4) is false. For simplicity, we may assume $i = 1$ and $j = 1$ (we can always rename the vertices of $f(G, k)$ and S so that this is the case). We will often implicitly use the fact that a vertex in V_a either sees all, or misses all, vertices of V_b whenever $a \neq b$. Note that c_1, c_2 cannot be in X. We will distinguish among several cases.

Case 1. $c_1, c_2 \in V_1$. Since c_3 sees c_1, c_2 and misses r_1, c_3 cannot be in T. Thus, c_3 is in W. But no vertex in W can see two vertices in V_1, a contradiction.

Case 2. $c_1 \in V_1, c_2 \in V_j$ *for some* $j \neq 1$. We may write $j = 2$. Since c_3 (respectively, r_t) sees c_1 and misses r_1, c_3 (respectively, r_t) cannot be in T. Thus, c_3 and r_t are in W. Observation 2, with $z = c_3, x = c_1$, and $y = c_2$, implies r_t sees c_2, a contradiction to the definition of S.

Case 3. $c_1 \in V_1, c_2 \in W$. This case is not possible since a vertex in W can have at most one neighbor in any V_j.

Case 4. $c_1, c_2 \in V_j$ *for some* $j \neq 1$. We may write $j = 2$. Since r_2 sees c_2 and misses c_1, r_2 is in W. Since r_t sees c_1 and misses c_2, r_t is in W. But then r_2 sees r_t, a contradiction.

Case 5. $c_1 \in V_j, c_2 \in V_r$ *with* $j \neq r, j \neq 1$, *and* $r \neq 1$. In this case, r_1, c_1, and c_2 contradict Observation 1.

Case 6. $c_1 \in V_j$ *for some* $j \neq 1$ *and* $c_2 \in W$. We may let $j = 2$. If $c_3 \in W$, then Observation 2, with $z = c_2, x = r_1$, and $y = c_1$, implies c_3 sees r_1, a contradiction to the definition of S. So, we have $c_3 \in T$. Since c_3 misses r_1, we have $c_3 \notin V_1 \cup V_2$. So, we may assume $c_3 \in V_3$. We have $r_2 \notin W$; otherwise Observation 2, with $z = c_2, x = c_1$, and $y = c_3$, implies r_2 sees c_1, a contradiction to the definition of S. We have $r_2 \notin V_1 \cup V_2 \cup V_3$ since r_2 misses r_1 and c_1. So, we may assume $r_2 \in V_4$. Since r_3 (respectively, c_4 if it exists) sees c_3 and misses r_1, Observation 2, with $z = c_2, x = r_1$, and $y = c_3$, implies $r_3 \notin W$ (respectively, $c_4 \notin W$). Since r_3 (respectively, c_4 if it exists) misses r_1 and r_2, we have $r_3 \notin V_1 \cup V_2 \cup V_3 \cup V_4$ (respectively, $c_4 \notin V_1 \cup V_2 \cup V_3 \cup V_4$). Now, if $t = 3$, then the three vertices r_3, c_1, and c_3 contradict Observation 1. But if $t > 3$, then the three vertices c_4, c_1, and c_3 contradict Observation 1.

So, (4) holds. Next, we will establish two more assertions (where the subscripts are taken modulo k) below.

(5) $$\text{If a ray } r_j \text{ lies in } T, \text{ then } r_{j-1}, r_{j+1} \text{ lie in } X.$$

By (4) and the definition of $f(G, k)$, we may assume $c_j = u_i, c_{j+1} = w_i$. Since x_i is the only vertex of $f(G, k)$ that sees w_i and misses u_i, we have $x_i = r_{j+1}$. Similarly, we have $x_{i-1} = r_{j-1}$. So, (5) holds.

(6) $$\text{If some vertex } x_i \in X \text{ is a ray of } S, \text{ then } x_{i+1} \text{ is also a ray of } S.$$

Let x_i be a vertex in X that is a ray r_j of S. We may assume that $c_j = w_i$ and $c_{j+1} = u_{i+1}$. By (2), we have $r_{j+1} \in V_a$ for some a. By (4), we have $c_{j+2} = w_{i+1}$. By (5), r_{j+2} lies in X, and so we have $r_{j+2} = x_{i+1}$. Thus, (6) holds.

We are now in a position to prove (1). From (3), we may assume r_1 lies in T. By (5), we have $r_2 \in X$. By (6), all x_j's are rays of S for $j = 1, 2, \ldots, k$. It follows from (2) that S has exactly k rays in T. Therefore, S is a $2k$-sun. We have proved (1).

We continue with the proof of Claim 1 (and Theorem 2). Consider the k rays of S that belong to T. Since each V_i is a clique, it contains at most one ray. So, there are k sets V_i containing a ray of S. Let these sets be $V_1, V_2, \ldots, V_k$. Clearly, in G, the vertices $v_1, v_2, \ldots, v_k$ form a stable set. $\square$

Proof of Theorem 3. We will use the notation defined in the proof of Theorem 2 with G being a triangle-free graph. We need only to prove the graph $f(G, k)$ does not contain a t-antihole with $t \geq 7$. We will prove by contradiction. Suppose $f(G, k)$ contains a t-antihole A with vertices $a_1, a_2, \ldots, a_t$ with $t \geq 7$ such that a_i misses a_{i+1} with the subscripts taken modulo k. Since the vertices in X have degree two, none of them can belong to A. Since each V_i is a clique,

(7) no two consecutive vertices of A can belong to the same V_i.

Similarly,

(8) no two consecutive vertices of A can belong to W.

Now, we claim that

(9) one of a_i, a_{i+1} must lie in W for all i.

Suppose (9) is false for a_i. For simplicity, we may assume $i = 1$, and so we have $a_1, a_2 \in T$. By (7), we may assume $a_1 \in V_1, a_2 \in V_2$. Clearly, we have $a_t \notin V_1$.

Suppose $a_t \in V_2$. Then a_3 has to be in W; otherwise a_3 lies in some V_j and so it misses a_t (since it misses a_2) implying $t = 4$, a contradiction. By symmetry, we have $a_{t-1} \in W$. Since a_1 sees a_3, and a_{t-1} is a common neighbor of a_1 and a_2, Observation 2 implies that a_2 sees a_3, a contradiction to the definition of A. So, we have $a_t \notin V_2$.

Suppose $a_t \in W$. By (8), we have $a_{t-1} \in V_j$. If $j = 2$, then a_1 misses a_{t-1}, a contradiction to the definition of A. If $j = 1$, then a_2 misses a_{t-1} implying $t = 4$, a contradiction. So, we may assume $a_{t-1} \in V_3$. Let $j \in \{t-2, t-3\}$. If $a_j \in W$, then since a_2 sees a_t, Observation 2 with $z = a_j, x = a_2$, and $y = a_1$ implies a_1 sees a_t, a contradiction to the definition of A. So, we have $a_{t-2} \in V_m$ for some m, and $a_{t-3} \in V_p$ for some p. Since a_{t-2} misses a_{t-1}, we have $a_{t-2} \notin V_1 \cup V_2 \cup V_3$. So, we may assume $m = 4$. We have $a_{t-3} \in V_2 \cup V_3$; otherwise the three vertices a_{t-3}, a_{t-1}, and a_2 contradict Observation 1. Since a_{t-2} sees a_2, a_{t-2} sees all of V_2. Thus, we have $a_{t-3} \notin V_2$, and so $a_{t-3} \in V_3$. Since $t \geq 7$, the vertex a_{t-4} exists. Since a_{t-4} misses a_{t-3} but sees a_{t-1}, a_{t-4} is not in T; so we have $a_{t-4} \in W$. Observation 2 with $z = a_{t-4}, x = a_{t-1}$, and $y = a_{t-2}$ implies a_{t-1} sees a_t, a contradiction to the definition of A.

Thus, a_t belongs to some V_j which is distinct from V_1, V_2. It follows from symmetry and the definition of $f(G, k)$ that a_3, a_{t-1} also belong to distinct V_i. Now, the three vertices a_{t-1}, a_1, and a_3 contradict Observation 1. So, (9) holds.

From (8) and (9), we may assume without loss of generality that $a_i \in T$ whenever i is odd, and $a_i \in W$ whenever i is even. In particular, t is even and at least eight. The definition of A implies that a_1 sees a_4, a_6. Thus, we have $\{a_4, a_6\} = \{u_i, w_i\}$ for some i. The definition of $f(G, k)$ means that every vertex of T either sees both a_4, a_6 or misses both of them. But a_3 misses a_4 and sees a_6, a contradiction. $\square$

Proof of Theorem 4. We will reduce k-CLIQUE to k-SUN. Let G, k be an instance of k-CLIQUE. We may assume $k \geq 4$. Construct a graph $h(G)$ from G by adding a vertex $v(a,b)$ for each edge ab of G and joining $v(a,b)$ to a and b by an edge of $h(G)$. Let Y be the set of vertices $v(a,b)$. It is easy to see that if G has a clique K on k vertices, then $h(G)$ has a k-sun induced by K and some k vertices in Y. If $h(G)$ has a k-sun $(c_1,\dots,c_k,r_1,\dots,r_k)$, then since the vertices in Y have degree two, none of them can be a vertex c_i; thus, the vertices $c_1,\dots,c_k$ induce a clique on k vertices in G. $\square$

REFERENCES

[1] K. S. Booth and J. H. Johnson, *Dominating sets in chordal graphs*, SIAM J. Comput., 11 (1982), pp. 191–199.

[2] A. Brandstädt, *Problem session*, in Dagstuhl Seminar on Robust and Approximative Algorithms for Particular Graph Classes, Seminar 04221, Wadern, Germany, 2004, Dagstuhl Seminar Proceedings, Internationales Begegnungs- und Forschungszentrum für Informatik (IBFI), Schloss, Dagstuhl, Germany, 2005.

[3] A. Brandstädt, V. B. Le, and J. P. Spinrad, *Graph Classes: A Survey*, SIAM Monogr. Discrete Math. Appl. 3, SIAM, Philadelphia, 1999.

[4] G. A. Dirac, *On rigid circuit graphs*, Abh. Math. Sem. Univ. Hamburg, 25 (1961), pp. 71–76.

[5] E. M. Eschen, C. T. Hoàng, and R. Sritharan, *An $O(n^3)$ recognition algorithm for hhds-free graphs*, Graphs Combin., 23 (2007), pp. 209–231.

[6] M. Farber, *Characterizations of strongly chordal graphs*, Discrete Math., 43 (1983), pp. 173–189.

[7] M. C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs*, Academic Press, New York, 1980.

[8] R. B. Hayward, *Weakly triangulated graphs*, J. Combin. Theory Ser. B, 39 (1985), pp. 200–209.

[9] R. B. Hayward, J. Spinrad, and R. Sritharan, *Improved algorithms for weakly chordal graphs*, ACM Trans. Algorithms, 3 (2007), article 14.

[10] C. T. Hoàng and N. Khouzam, *On brittle graphs*, J. Graph Theory, 12 (1988), pp. 391–404.

[11] R. Karp, *Reducibility among combinatorial problems*, in Complexity of Computer Computations, R. E. Miller and J. W. Thatcher, eds., Plenum Press, New York, 1972, pp. 85–103.

[12] A. Lubiw, *Doubly lexical orderings of matrices*, SIAM J. Comput., 16 (1987), pp. 854–879.

[13] S. D. Nikolopoulos and L. Palios, *Recognizing hhds-free graphs*, in Proceedings of the 31st International Workshop on Graph Theoretic Concepts in Computer Science (WG 2005), Metz, France, Lecture Notes in Comput. Sci. 3787, Springer, Berlin, 2005, pp. 456–467.

[14] S. Poljak, *A note on stable sets and coloring of graphs*, Comment. Math. Univ. Carolin., 15 (1974), pp. 307–309.

[15] J. L. Ramírez-Alfonsín and B. A. Reed, eds., *Perfect Graphs*, Wiley, New York, 2001.

[16] R. Paige and R. E. Tarjan, *Three partition refinement algorithms*, SIAM J. Comput., 16 (1987), pp. 973–989.