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A linear array of superconducting weak links with a common bias current can exhibit synchronization provided there is nearest-neighbor coupling between the links. A perturbation method is applied to the equations governing this system, and the results indicate how the domain of locking depends on various parameters. Numerical simulations confirm the predicted behavior.

I. INTRODUCTION

It is now generally understood that a series pair of current-biased superconducting weak links will synchronize their Josephson oscillations (frequency or voltage locking) provided a coupling mechanism exists between the two-phase slip sites.[1] From a theoretical point of view, a variety of publications have investigated inductive coupling,[2] capacitive coupling,[3] resistive shunt coupling,[4] wave-function mixing,[5] and quasiparticle nonequilibrium coupling.[6] The latter is now generally regarded as operative in situations where the two microbridges are situated in close proximity to one another. This point has been made by Jillie *et al.*,[7] Lindelof and Hansen,[8] Palmer and Mercereau,[9] and Meyer and Tidecks.[10] In two recent papers, Jillie *et al.*[11] and Nerenberg *et al.*[12] examined this synchronization phenomenon in detail with particular emphasis on the range of locking.

A logical extension of the system of two links would be a series chain of N microbridges with or without individual bias currents. Sandell *et al.*[13] studied the radiated power and linewidth from resistively shunted *arrays* containing up to eight microbridges. Palmer and Mercereau[9] measured the Josephson rf potential as a function of interbridge spacing for a series chain consisting of 100 weak links. They found that for spacings of less than 1.5 μm the potential was proportional to N, the number of elements, while for spacings greater than 5 μm the proportionality was to $\sqrt{N}$. This provided direct evidence of complete synchronization in the chain. Between 1.5 and 5.0 μm the synchronization is presumably incomplete. Another important conclusion of this study, based on the failure of ground plane or strip line coupling to alter the results, was that the coupling mechanism was internal and not electromagnetic—that is nonequilibrium quasiparticle distributions are mediating the interaction. Jillie[14] also noted the apparent partial coherence in series arrays.

In this paper we examine the behavior of a linear chain of weak links in which nearest-neighbor coupling is operative and show that complete synchronization will occur if the system parameters are located within certain subregions of the entire parameter space.

It is interesting to note that such a mutually synchronized state may be possible in other extended nonlinear systems[15]; however, little theoretical work has been done. In particular, analytical calculations are very rare, computer simulation being virtually the exclusive tool. In biology the alpha rhythm of the brain[15] and the heart pacemaker[16] are two examples of phenomena which may be due to mutual synchronization of electrical or chemical rhythms in large clusters of cells. An advantage of such a state biologically is that cells may, without being exactly identical, produce a signal whose strength is reinforced by coherence, as in the case of microwave emission from voltage-locked weak link arrays.

II. THEORY

The equations for a pair of closely spaced links in series are[11]

$$\dot{\phi}_1 = \frac{2er_1}{\hbar}[i_1 - i_{c_1}\sin\phi_1 + \alpha_1(i_2 - i_{c_2}\sin\phi_2)] \ , \quad (1)$$

$$\dot{\phi}_2 = \frac{2er_2}{\hbar}[i_2 - i_{c_2}\sin\phi_2 + \alpha_2(i_1 - i_{c_1}\sin\phi_1)] \ , \quad (2)$$

where ϕ_1 and ϕ_2 are the phase differences across junctions 1 and 2, respectively, r_1 and r_2 are their respective shunt resistances, i_{c_1} and i_{c_2} their critical currents, i_1 and i_2 the bias currents, and finally α_1 and α_2 are coupling parameters measuring the strength of the mutual interactions.

A *linear array* of N links connected in series may:

(i) have each link independently biased; or (ii) have a common bias current for the entire chain. The generalization of Eqs. (1) and (2) become for the former case

$$\dot{\phi}_n = \frac{2er_n}{\hbar}[i_n - i_{c_n}\sin\phi_n + \alpha_{n,n+1}(i_{n+1} - i_{c_{n+1}}\sin\phi_{n+1}) + \alpha_{n,n-1}(i_{n-1} - i_{c_{n-1}}\sin\phi_{n-1})] \quad , \tag{3}$$

$$n = 1, 2, \ldots, N \quad .$$

The symbols have the same meaning as in Eqs. (1) and (2) except that the subscript n refers to the nth link in the chain, and $\alpha_{n,n+1}$ and $\alpha_{n,n-1}$ are the coupling parameters between nearest-neighbor links n and $n+1$, and n and $n-1$, respectively. The "end" links can be subsumed in Eq. (3) by artificially defining both $\alpha_{1,0}$ and $\alpha_{N,N+1}$ as zero.

It is easy to see that if the links are commonly biased, Eq. (3) is still valid except that all i_n must be set at a common biasing current, i.

We now convert to dimensionless quantities by measuring time in units of $\hbar/(2e\bar{r}\bar{i}_c)$, currents in units of the average critical current $\bar{i}_c$, and resistance in units of the average resistance $\bar{r}$. Thus $t^* = 2e\bar{r}\bar{i}_c t/\hbar$, $I_n = i_n/\bar{i}_c$, $I_{c_n} = i_{c_n}/\bar{i}_c$, and $R_n = r_n/\bar{r}$. In addition, assuming that coupling is small and approximately equal between nearest-neighbor pairs we write

$$\alpha_{n,n+1} = \alpha\gamma_{n,n+1}, \quad \alpha_{n,n-1} = \alpha\gamma_{n,n-1}, \quad n = 1, 2, \ldots, N \quad ,$$

(where to include the end junctions in the chain we define $\gamma_{1,0}$ and $\gamma_{N,N+1}$ to be zero). The parameter α will be a measure of the smallness of the coupling generally, while $\gamma_{n,n+1} \sim 1$ and $\gamma_{n,n-1} \sim 1$.

This allows Eq. (3) to be written as

$$\dot{\phi}_n^* = A_n - C_n\sin\phi_n - \alpha(B_{n,n-1}\sin\phi_{n-1} - D_{n,n-1}) - \alpha(B_{n,n+1}\sin\phi_{n+1} - D_{n,n+1}) \quad , \tag{4}$$

$$n = 1, 2, \ldots, N \quad ,$$

where $(*)$ refers to differentiation with respect to t^*,

$$A_n = R_n I_n, C_n = R_n I_{c_n} \quad , \quad B_{n,n-1} = R_n I_{c_{n-1}}\gamma_{n,n-1} \quad ,$$

$$B_{n,n+1} = R_n I_{c_{n+1}}\gamma_{n,n+1} \quad , \quad D_{n,n-1} = R_n I_{n-1}\gamma_{n,n-1} \quad ,$$

and

$$D_{n,n+1} = R_n I_{n+1}\gamma_{n,n+1} \quad .$$

The cases $n = 1$ and $n = N$ are subsumed in Eq. (4) by taking $B_{1,0}$, $D_{1,0}$, $B_{N,N+1}$, and $D_{N,N+1}$ all zero. Writing the equations in this form permits us to proceed to a perturbation solution of them.

III. PERTURBATION CALCULATION

We solve the system (4) by a general perturbation procedure based on the assumption of α being small. For reasons which will become apparent below it is convenient to write $\phi_n(t^*)$ as a general asymptotic expansion in the parameter α:

$$\phi_n(t^*) \sim \sum_{m=0}^{\infty}[\phi_n^{(m)}(\alpha, t^*)]\alpha^m, \quad \alpha \to 0 \quad . \tag{5}$$

Equation (5) will be a truly asymptotic expansion providing

$$[\phi_n^{(m+1)}(\alpha, t^*)]\alpha^{m+1} = o([\phi_n^{(m)}(\alpha, t^*)]\alpha^m), \quad \alpha \to 0 \quad .$$

This order relation will be satisfied in particular if all $\phi_n^{(m)}(\alpha, t^*)$ are analytic functions of α [with $\phi_n^{(m)}(0, t^*) \not\equiv 0$].

Then substituting Eq. (5) into Eq. (4) leads to

$$\dot{\phi}_n^{*(0)} = A_n - C_n\sin\phi_n^{(0)}, \quad n = 1, 2, \ldots, N \tag{6}$$

as the zeroth order system, while the first-order system is

$$\dot{\phi}_n^{*(1)} = (C_n\cos\phi_n^{(0)})\phi_n^{(1)} + D_{n,n+1} - B_{n,n+1}\sin\phi_{n+1}^{(0)} + D_{n,n-1} - B_{n,n-1}\sin\phi_{n-1}^{(0)}, \tag{7}$$

$$n = 1, 2, \ldots, N \quad .$$

The systems (6) and (7) can be solved analytically, however a "secularity" arises in the solving of Eq. (7). $\phi_n^{(1)}$ will be found to oscillate with an amplitude which tends to infinity linearly with time (see Appendix). This means in particular that there is no average value for $d\phi_n^{(1)}/dt^*$ in the limit as time tends to infinity—the first-order term is meaningless. This problem can be remedied[12] by a "renormalization" process at the cost of $\phi_n^{(0)}$ becoming an (analytic) function of α, instead of being independent of this parameter while $\phi_n^{(1)}$ becomes $O(1)$ as $t^* \to \infty$.

To see this most clearly we rewrite Eq. (4) adding and subtracting a constant k_n in each equation

$$\dot{\phi}_n^* = \omega_n - C_n\sin\phi_n - \alpha(B_{n,n-1}\sin\phi_{n-1} - D_{n,n-1} + B_{n,n+1}\sin\phi_{n+1} - D_{n,n+1} + k_n) \quad , \tag{8}$$

where $\omega_n = A_n + \alpha k_n$.

The new zeroth order and new first-order equations are, respectively,

$$\dot{\phi}_n^{*(0)} = \omega_n - C_n\sin\phi_n^{(0)}, \quad n = 1, 2, \ldots, N \tag{9}$$

and

$$\dot{\phi}_n^{*(1)} = -(C_n\cos\phi_n^{(0)})\phi_n^{(1)} - k_n + B_{n,n-1}\sin\phi_{n-1}^{(0)} - D_{n,n-1} + B_{n,n+1}\sin\phi_{n+1}^{(0)} - D_{n,n+1} \quad . \tag{10}$$

The k_n are constants to be chosen so that $\phi_n^{(1)} = O(1)$ as $t^* \to \infty$. If this is possible, then clearly the secularity problem will have been solved (to first order). In addition, since there is no voltage locking region in zeroth order of the original perturbation theory this procedure becomes essential to the analysis, and crucial to establishing that a region of voltage locking does indeed exist in parameter space.

Solving the new zeroth-order Eq. (9) yields

$$\phi_n^{(0)}(t^*) = 2\tan^{-1}\left\{\frac{1}{\omega_n}\left[C_n + \Omega_n \tan\left[\frac{\Omega_n t^*}{2} + \psi_n\right]\right]\right\} , \tag{11}$$

where $\Omega_n = (\omega_n^2 - C_n^2)^{1/2}$ and

$$\psi_n = \tan^{-1}\left[\frac{\omega_n \tan[\phi_n^{(0)}(0)/2] - C_n}{\Omega_n}\right] .$$

From Eq. (11) it is clear that

$$\phi_n^{(0)}(t^*) = \Omega_n t^* + O(1) \tag{12}$$

as $t^* \to \infty$, for all n. Now the dc voltages V_n are given by

$$V_n \equiv \lim_{T\to\infty} \frac{\hbar}{2eT}\int_0^T \frac{d\phi_n}{dt}dt = \lim_{T^*\to\infty} \frac{\bar{i}_c\bar{r}}{T^*}\int_0^{T^*} \frac{d\phi_n}{dt^*}dt^* ,$$

where $T^* = 2e\bar{r}\bar{i}_c T/\hbar$. Assuming that the k_n can be found so that $\phi_n^{(1)}(t^*) = O(1)$ as $t^* \to \infty$, and using Eq. (12), we find, accurate to *first* order of perturbation theory, that

$$V_n = \bar{i}_c\bar{r}\,\Omega_n, \quad \forall n . \tag{13}$$

We therefore see that voltage locking at equal voltages of the entire system is given by

$$\Omega_1 = \Omega_2 = \Omega_3 = \cdots = \Omega_N . \tag{14}$$

We next determine the appropriate k_n, which in turn will give us the final expression for the Ω_n.

From Eq. (11) it can be shown that

$$\sin\phi_n^{(0)}(t^*) = \frac{1 + Z_n \sin(\Omega_n t^* + \theta_n)}{Z_n + \sin(\Omega_n t^* + \theta_n)} , \tag{15}$$

where $Z_n = \omega_n/C_n$, and $\theta_n = \tan^{-1}(C_n/\Omega_n) + 2\psi_n$. It will be useful below that from Eq. (15), $\sin\phi_n^0(t^*)$, aside from a zero-frequency part, contains only the frequency Ω_n and its harmonics.

Solving Eq. (10) in the same fashion that Eq. (7) is solved in the Appendix, results in the condition that k_n be chosen so that

$$\int_0^{t^*} \frac{-k_n - D_{n,n+1} - D_{n,n-1} + B_{n,n+1}\sin\phi_{n+1}^0(\tau) + B_{n,n-1}\sin\phi_{n-1}^{(0)}(\tau)}{1 - Z_n \sin\phi_n^{(0)}(\tau)} d\tau = O(1) \quad \text{as } t^* \to \infty , \quad n = 1, 2, \ldots, N . \tag{16}$$

[$B_{1,0}$ and $B_{N,N+1}$ are defined as zero in order to handle the end-link cases in Eq. (16).] Using Eq. (15) we obtain for one part of the integral

$$\int_0^{t^*} \frac{(k_n - D_{n,n+1} - D_{n,n-1})}{1 - Z_n \sin\phi_n^{(0)}(\tau)} d\tau = \frac{\omega_n^2}{\Omega_n^2}(k_n - D_{n,n+1} - D_{n,n-1})t^* + O(1) \quad \text{as } t^* \to \infty . \tag{17}$$

One of the two remaining parts of the integral on the left-hand side of Eq. (16) [again using Eq. (15)] is

$$\int_0^{t^*} \frac{\sin\phi_{n+1}^{(0)}(\tau)}{1 - Z_n \sin\phi_n^{(0)}(\tau)} d\tau = \left(\frac{\omega_n}{\Omega_n}\right)^2 \int_0^{t^*} \frac{1 + Z_{n+1}\sin(\Omega_{n+1}\tau + \theta_{n+1})}{Z_{n+1} + \sin(\Omega_{n+1} + \theta_{n+1})} d\tau$$
$$+ \left(\frac{\omega_n}{\Omega_n}\right)^2 Z_n^{-1} \int_0^{t^*} \frac{1 + Z_{n+1}\sin(\Omega_{n+1}\tau + \theta_{n+1})}{Z_{n+1} + \sin(\Omega_{n+1}\tau + \theta_{n+1})} \sin(\Omega_n\tau + \theta_n) d\tau . \tag{18}$$

The second integral on the right-hand side of Eq. (18) is $O(1)$ as $t^* \to \infty$ unless $\Omega_n = M\Omega_{n+1}$, $M = 1, 2, \ldots$. Since we are interested in the totally locked state at equal voltages given by Eqs. (14) we consider Eq. (18) only when $\Omega_n = \Omega_{n+1}$. Evaluating both integrals on the right-hand side of Eq. (18) under this condition we get

$$\int_0^{t^*} \frac{\sin\phi_{n+1}^{(0)}(\tau)}{1 - Z_n \sin\phi_n^{(0)}(\tau)} d\tau = \frac{\omega_n^2}{\Omega_n^2}(Y_{n+1} - Z_{n+1})\left(1 + \frac{Z_{n+1}}{Z_n}\frac{\Omega_{n+1}}{\omega_{n+1}}\cos(\theta_n - \theta_{n+1})\right)t^* + O(1) \quad \text{as } t^* \to \infty , \tag{19}$$

where $Y_n = \Omega_n/C_n$ and the second term in large brackets is present only if $\Omega_n = \Omega_{n+1}$.

The final part of the left-hand side of Eq. (16),

$$\int_0^{t^*} \frac{\sin\phi_{n-1}^{(0)}(\tau)}{1 - Z_n \sin\phi_n^{(0)}(\tau)} d\tau ,$$

is evaluated in the same way so that Eq. (16) will be satisfied if the constants k_n are given by

$$k_n = D_{n,n+1} + D_{n,n-1} + B_{n,n+1}(Y_{n+1} - Z_{n+1})\left(1 + \frac{Z_{n+1}}{Z_n}\frac{\Omega_{n+1}}{\omega_{n+1}}\cos(\theta_n - \theta_{n+1})\right)$$
$$+ B_{n,n-1}(Y_{n-1} - Z_{n-1})\left(1 + \frac{Z_{n-1}}{Z_n}\frac{\Omega_{n-1}}{\omega_{n-1}}\cos(\theta_n - \theta_{n-1})\right), \quad n = 1, 2, \ldots, N \ , \tag{20}$$

where the terms involving $\cos(\theta_n - \theta_{n-1})$ and $\cos(\theta_n - \theta_{n+1})$ are present only if $\Omega_{n-1} = \Omega_n$ and $\Omega_{n+1} = \Omega_n$, respectively. (The end cases are again contained within the formalism by choosing $B_{1,0} = 0$ and $B_{N,N+1} = 0$.)

The condition for voltage locking (accurate to first order) therefore consists of the self-consistent combination of the expressions (20) for the k_n, and [from Eq. (11)] those for

$$\Omega_n = [(A_n + \alpha k_n)^2 - C_n^2]^{1/2}, \quad n = 1, 2, \ldots, N \ , \tag{21}$$

together with the equal-voltage locking equations (14). The circumstances under which the system will satisfy this condition are discussed below.

IV. VOLTAGE LOCKING (SYNCHRONIZATION)

As mentioned above, the condition that all the dc voltages be equal is given by Eqs. (14) thereby justifying our interest in including in Eq. (20) terms present only when $\Omega_n = \Omega_{n-1}$ and when $\Omega_n = \Omega_{n+1}$. Voltages may also lock in a harmonic relation,[12] and this formalism can deal with such a situation. However the number of possible different combinations for an extended system is staggering, even if we would restrict ourselves to the first few harmonics in each link. Therefore only the case of equal-voltage locking is considered here.

The parameters of the differential system (4) are: I, $\{I_{c_n}\}_{n=1}^{N}$, $\{R_n\}_{n=1}^{N}$, $\{\gamma_{n,n+1}\}_{n=1}^{N-1}$, $\{\gamma_{n+1,n}\}_{n=1}^{N-1}$, and α. There are not in fact $1 + N + N + (N-1) + (N-1) + 1 = 4N$ independent parameters, but rather $4N - 2$, because the normalization of the currents and resistances implies $\sum_{n=1}^{N} I_{c_n} = 1$ and $\sum_{n=1}^{N} R_n = 1$. We have for definiteness restricted ourselves to the case of common biasing.

Equations (14) comprise $(N-1)$ independent conditions and hence would apparently serve to determine $(N-1)$ parameters in terms of the remaining $(3N-1)$ parameters in order to have voltage locking. It would seem that once $(3N-1)$ parameters were fixed, one, or possibly a few, set(s) of parameter values for the remaining $(N-1)$ would result in voltage locking. One implication might be that this state would be impossible to achieve in reality since experimentally one cannot fix *precisely* the values of parameters. Stated more mathematically, the voltage locking condition would *seem* to restrict the parameters in the $(4N-2)$-dimensional space to a $(3N-1)$-dimensional subspace, a region of measure zero in the entire space.

However as can be seen from Eq. (20) *in the synchronized state* there are additional $(N-1)$ free variables which are related to the initial relative phases of the links, viz., $\{\cos(\theta_n - \theta_{n-1})\}_{n=2}^{N}$. Therefore, in fact, the constraint that Eqs. (14) places on the parameters is that they be such as to preserve the *inequalities* $|\cos(\theta_n - \theta_{n-1})| \leq 1$, $n = 2, 3, \ldots, N$. This then restricts $(N-1)$ of the parameters to a bounded region, but one of positive measure in their subspace of $(N-1)$ dimensions, while the remaining $3N-1$ variables are essentially free. Synchronization is seen to be (in principle) experimentally realizable in that one can have random fluctuations in the parameters, yet still remain in the region of voltage locking. It should be noted that this region will shrink to zero as α goes to zero. That is, the finite interaction between junctions is essential for synchronization. The spread in the $(N-1)$ "nonfree" parameters is a result of the intervention of the relative phases in the k_n. Now, although the k_n do not vanish in the limit as α tends to zero, as we shall see, the equal voltage condition will determine a region in parameter space which *scales with* α.

This is easiest seen in detail in the case where $I_2, I_3, \ldots, I_N$ are those chosen as the $(N-1)$ "nonfree" variables. Equations (14) become with the use of Eq. (21)

$$[(A_n + \alpha k_n)^2 - C_n^2]^{1/2} = [(A_1 + \alpha k_1)^2 - C_1^2]^{1/2} \ , \tag{22}$$
$$n = 2, 3, \ldots, N \ .$$

Using $A_n = I_n R_n$ and retaining terms to first order in α yields

$$I_n = \frac{1}{R_n}(C_n^2 - C_1^2 + A_1^2)^{1/2}\left(1 + \alpha\frac{(k_1 A_1 - k_n R_n I_n)}{(C_n^2 - C_1^2 + A_1^2)}\right) \ , \tag{23}$$
$$n = 2, 3, \ldots, N \ .$$

For $\alpha \neq 0$ the system of Eqs. (23) must be solved iteratively for the $\{I_n\}_{n=2}^{N}$ since the right-hand side of this system contains them implicitly (in the k_n) as well as explicitly; however it is clear that there will be a *spread* in the I_n values satisfying Eq. (23), due to the variation of $\{\cos(\theta_{n+1} - \theta_n)\}_{n=1}^{N-1}$ (the latter set of

variables appearing in the set $\{k_n\}_{n=1}^{N}$), and that to first order this will be linear in α. The region thus disappears in the limit as α tends to zero.

In the ensuing discussion we therefore take α to be one of the $(3N-1)$ free parameters. We also assume that $\gamma_{n,n+1}, \gamma_{n+1,n}$; $n=1, 2, \ldots, N-1$, belong to this set of free parameters and that none of them are zero. That is we ensure a non-null interaction between *all* nearest neighbors. If this were violated an examination of Eq. (20) would reveal that the region of synchronization for the entire system would revert to zero measure in parameter space.

On the basis of this analysis one would expect the following. First assume $(3N-1)$ parameters are fixed. Then there corresponds to each set of values for the $(N-1)$ relative initial phases a single point in the $(N-1)$-dimensional space of the remaining parameters which permits locking. This mapping is mediated by the voltage locking condition (14), via the dependence of the k_n on the relative phases through Eq. (20). Therefore if the system has such a set of parameter values it will synchronize provided the relative initial phases are the corresponding correct ones. However if the initial phases are *not* correct, the frequencies will consequently start out being unequal, thereby causing the relative phases to slip. This will lead to the correct alignment of phases at some later time, and from then on synchronization will ensue. Certain qualifications need to be added to this last scenario, however we postpone this to the next section.

It is of practical significance therefore to map the region of synchronization and this is discussed below. We have also performed a numerical simulation of the exact equations to verify the predictions of the perturbation calculation and to determine the sensitivity of voltage locking to the initial relative phases.

V. NUMERICAL SIMULATION AND COMPARISON WITH PERTURBATION THEORY

A numerical simulation of an arbitrarily large system of differential equations cannot be carried out. In our case, finding and mapping the region of synchronization for a system even of moderate size $(10 > N \geq 3)$ would be virtually impossible by numerical methods alone. The perturbation approach does have the merit of solving the problem for any N, albeit approximately.

Various tests were performed. As a guide and for comparison the region of synchronization for the case $N=3$ was determined easily with the perturbation results by applying the condition as described in the final paragraph of Sec. III. One has various possible choices of the $(N-1)$ parameters to be linked with the $(N-1)$ relative phases. In the case of the common bias current case, it is appropriate to determine the locking region in critical-current space. By our choice of units $I_{c_1}+I_{c_2}+\cdots+I_{c_N}=1$. Fixing all other parameters of the system we can find the region in this hyperplane which produces synchronization.

On the other hand, in the case of individually biased links it is more suitable to fix all other parameters as well as one bias current (say I_1), and then determine the locking region in the $(N-1)$-dimensional space $\{I_2, I_3, \ldots, I_N\}$.

In order that computer runs be moderate, N had to be chosen to be rather small. We selected a common bias triple junction for the main calculation to be presented here. The region of synchronization in the plane $I_{c_1}+I_{c_2}+I_{c_3}=1$ was first determined using the perturbation results.

Next the numerical simulation was employed. This was performed using Adam's predictor-corrector scheme. The critical currents and initial relative phases were selected using the perturbation results as a guide. As Fig. 1 shows the projected region as determined numerically matches almost exactly the perturbation results. The average relative phases at different locking points tested were also virtually exactly as predicted by perturbation theory. (Under synchronization the short-time-average relative phases remain constant for all time.)

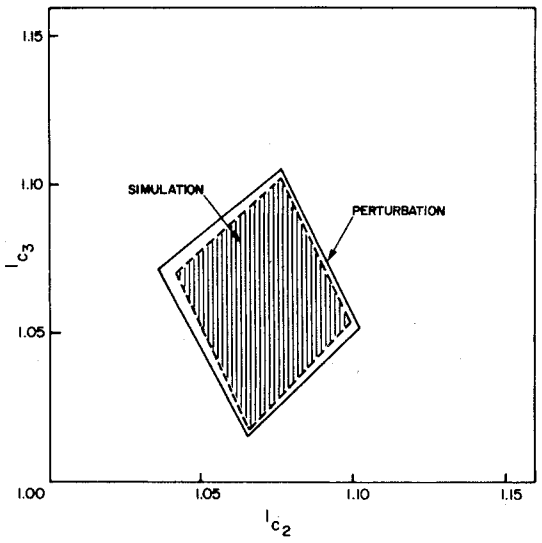

FIG. 1. Projection of region of synchronization from the plane defined by $I_{c_1}+I_{c_2}+I_{c_3}=1$ to a plane perpendicular to the I_{c_1} axis. $I=2$, $R_1=1.0$, $R_2=0.95$, $R_3=1.05$, $\gamma_{1,2}=\gamma_{2,1}=\gamma_{2,3}=\gamma_{3,2}=1$, and $\alpha=0.1$. The solid line represents the border of this region determined by perturbation theory, while the dashed line represents the border as determined by the numerical simulation of the exact equations.

It was found in some cases that if the starting values of the relative phases were too different from those predicted for locking by perturbation theory, the system would not fall into the totally synchronized state. This is explained as follows. The region of total synchronization for the entire system of N links may intersect regions which permit subsets of junctions to be voltage locked (still at equal voltage), but with each subset at different voltage. If we are at a point within this intersection, with the initial relative phases more appropriate for this situation rather than those "correct" for the totally locked state, then the oscillations in internal relative phases within each subset may not be of sufficient amplitude to permit the eventual requisite alignment of phases for total locking, despite the slipping of relative phases between subsets. Nonetheless, it was dramatic to restart these cases with only roughly correct relative phases (according to perturbation theory), and find total locking taking place almost immediately with a dc voltage close to that predicted by perturbation theory yet different from each of those voltages prevailing in the partially locked state; this with the parameters of the system remaining the same.

Stated more mathematically, the totally locked state was always found to be stable, but its domain of attraction, though large, was not always the entire space of possible phases. The consequences for experiment of this property will be discussed below.

A further test of the perturbation calculation was carried out for seven links in series, again with $\alpha = 0.1$ and other parameters similar in value to the three link chain. A few points in the six-dimensional region of locking (on the plane $I_{c_1} + I_{c_2} + \cdots + I_{c_7} = 1$) were determined easily from the perturbation results. The numerical simulation again verified the prediction of total locking; the relative phases and the voltage at locking matching extremely closely the analytic calculation.

In all calculations the dc voltages were in agreement usually to about 0.2% and always better than 0.5%. These discrepancies are in fact of the same order of magnitude as the fluctuations in $\phi_n(t^*)/t^*$ (at large t^*) as determined numerically. For stronger coupling (larger α values) undoubtedly greater differences between perturbation theory and numerical simulation would result. However the simulation indicates (as might be expected in a nearest-neighbor situation) that the magnitudes of the discrepancies are independent of the number of links.

VI. DISCUSSION

The perturbation calculation shows that the linear chain of N weak links can be totally synchronized over a region in parameter space. The numerical simulation verified this behavior in particular cases using $N = 3$ and $N = 7$. However, the simulation also pointed to a problem of starting the system correctly. On occasion, if the initial phase differences were far from those required for synchronization, the system would not synchronize totally despite the parameters being in the domain of synchronization. This happens only when subsets of the units lock in at different frequencies, and the internal oscillations within these subsets are insufficient to allow the relative phases to align themselves as required for total synchronization.

This latter observation may have consequences in attempting to produce synchronization in large-scale systems. Since initial phases cannot be adjusted experimentally it could happen that total synchronization would not be achieved, despite the parameters being in the correct domain. However since the synchronized state is stable the natural fluctuations (noise) in the parameter values might serve to produce sufficient phase slippage for the correct alignment to take place. Or, if this is insufficient, a means of disturbing the system until locking occurs would be needed. It might be noted that thermal noise has otherwise been neglected in this discussion. We have not, for example, considered the possibility that it could in some circumstances break the locking.

The foregoing suggests that a linear chain of microbridges under some circumstances may achieve a kind of "domain" structure in which synchronization exists only within subgroups of the weak links. If m domains contain $N_1, N_2, \ldots, N_m$ links, respectively, (where $N_1 + N_2 + \cdots + N_m = N$), then the radiated power from the linear array would fall between the values for a totally coherent single domain of N junctions and for N decoupled single junctions, i.e., $N^2 > N_1^2 + N_2^2 + \cdots + N_m^2 > N$. This could be the origin of the experimental results apparent in Fig. 1(a) of Ref. 9, in which there is a region of transition from total coherence to total incoherence as a function of interbridge spacing.

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APPENDIX

We show here the existence of a secularity in the original first-order equations. We solve Eqs. (6) and (7), that is the perturbation equations prior to the "renormalization."

Solving Eq. (6) yields

$$\phi_n^0(t^*) = 2\tan^{-1}\left\{\frac{1}{A_n}\left[C_n + \Omega_n \tan\left(\frac{\Omega_n t^*}{2} + \psi_n\right)\right]\right\},$$

where here $\Omega_n = (A_n^2 - C_n^2)^{1/2}$ and

$$\psi_n = \tan^{-1}\frac{[A_n \tan(\phi_n^{(0)}(0)/2) - C_n]}{\Omega_n} , \tag{A1}$$

which translates into

$$\sin\phi_n(t^*) = \frac{1 + Z_n \sin(\Omega_n t^* + \theta_n)}{Z_n + \sin(\Omega_n t^* + \theta_n)} ,$$
$$Z_n = \frac{A_n}{C_n}, \quad \theta_n = \tan^{-1}\left[\frac{C_n}{\Omega_n}\right] + 2\psi_n . \tag{A2}$$

Now the integrating factor for the nth of the equations in Eq. (7) is $\exp(\int C_n \cos\phi_n^{(0)} dt^*)$.

Since $dt^* = (\dot{\phi}_n^{(0)})^{-1} d\phi_n^{(0)}$ and in view of Eq. (6) we get this factor to be

$$\exp\left[\int \frac{C_n \cos\phi_n^0}{A_n - C_n \sin\phi_n^{(0)}} d\phi_n^0\right] = \frac{1}{A_n - C_n \sin\phi_n^{(0)}} . \tag{A3}$$

Solving the nth equation in Eq. (7) we obtain

$$\phi_n^{(1)} = (A_n - C_n \sin\phi_n^{(0)}) \int_0^{t^*} \left[D_{n,n+1} - B_{n,n+1} \sin\phi_{n+1}^{(0)} + \frac{D_{n,n-1} - B_{n,n-1} \sin\phi_{n-1}^{(0)}}{A_n - C_n \sin\phi_n^{(0)}} \right] d\tau . \tag{A4}$$

From Eq. (A2) it follows that

$$A_n - C_n \sin\phi_n^{(0)} = \frac{A_n}{Z_n}(Z_n^2 - 1)[Z_n + \sin(\Omega_n t^* + \theta_n)]^{-1} . \tag{A5}$$

Using Eqs. (A5) and (A2) in Eq. (A4) we see that the integral in Eq. (A4) in general will have a term proportional to t^*. As can be seen from Eq. (A5), the prefactor of the integral in Eq. (A4) has an oscillating term and therefore we have a "secularity." In particular, $\phi_n^{(1)}(t^*)/t^*$ will have no limit as $t^* \to \infty$.

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