1998

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(Received 29 September 1997; revised manuscript received 13 March 1998)

Numerical simulations have been carried out for a pair of unidirectionally coupled identical pendula under the action of a common external ac torque. Both the master pendulum and the slave pendulum were in chaotic states. The only form of persistent locking appeared to be a computational artifact; otherwise the synchronization of slave to master was found to be intermittent. [S0031-9007(98)06630-7]

PACS numbers: 05.45.+b

The possibility that coupling between chaotic systems or subsystems could result in synchronization of the dynamics was introduced by Pecora and Carroll [1]. A considerable body of theoretical work has now emerged on this general topic, the investigations focusing on a number of prototype systems including Rössler [2,3], Lorenz [4,5], and the so-called double scroll circuit [6]. Experimental work, often conceived with respect to possible communication applications, has mainly been done on coupled Lorenz-based circuits [7], Rössler systems [8], Chua circuits [9], and lasers [10].

The impact of the choice of coupling scheme on the synchronizing outcome is a matter of current interest [11]. A fundamental factor is whether bidirectional (mutual) or unidirectional coupling is assumed. Many examples of the latter are to be found: Duffing's equations [12], Lorenz model [13], Rössler systems [14], mixed Lorenz-Rössler [15], and unidirectionally coupled analog circuits [16,17].

The behavior of a unidirectionally coupled pair of chaotic pendula is the subject of this Letter. Harmonically driven pendula have often served as prototypical chaotic systems [18]. Beyond the rewards to be found in exploring the rich dynamics of the pendulum within the context of contemporary nonlinear dynamics, this system possesses the additional significance of being an analog of a capacitive Josephson junction—a superconducting device of considerable practical importance [19].

In this work we find that permanent synchronization of the two pendula does not occur except as a numerical artifact arising from finite computational precision. Instead, intermittent locking is seen to be an essential property of this system. These results constitute a warning that care must be exercised in assessing the results of numerical studies which apparently lead to synchronized chaos.

The master pendulum is described in dimensionless form by the usual nonautonomous expression in the angular coordinate θ_m

$$\ddot{\theta}_m + Q^{-1}\dot{\theta}_m + \sin\theta_m = \Gamma_0 \cos(\Omega t)\,, \qquad (1)$$

where time t has been normalized in units of ω_0^{-1}, ω_0 being the small-angle resonant frequency of the pendulum, Q^{-1} is a damping coefficient, Γ_0 is the drive torque amplitude normalized by the pendulum critical torque ($mg\ell$), and the drive frequency Ω is expressed in units of ω_0. Overdots denote derivatives in normalized time.

The slave pendulum is identical to the master, but is coupled to it via the instantaneous angular coordinates as follows:

$$\ddot{\theta}_s + Q^{-1}\dot{\theta}_s + \sin\theta_s = \Gamma_0 \cos(\Omega t) + c[\sin\theta_s - \sin\theta_m]\,. \qquad (2)$$

Here c is treated as a parameter which sets the coupling strength. The additional torque acting on the slave is in this particular model taken to be proportional to the difference in the gravitational restoring torques.

We note in passing that in Josephson devices, $\sin\theta$ represents a normalized junction supercurrent, so the form of coupling assumed here would in that situation be proportional to the supercurrent difference. Another coupling for current-biased Josephson junctions, based on the phase difference $\theta_s - \theta_m$, was treated by Doedel *et al.* [20].

The nonautonomous second order equations (1) and (2) can be written as equivalent sets of autonomous coupled first order equations.

$$\begin{aligned} \dot{\theta}_m &= \omega_m\,, \\ \dot{\omega}_m &= -Q^{-1}\omega_m - \sin\theta_m + \Gamma_0\cos\varphi_m\,, \\ \dot{\varphi}_m &= \Omega\,, \end{aligned} \qquad (3)$$

and

$$\begin{aligned} \dot{\theta}_s &= \omega_s + c(\omega_m - \omega_s)\,, \\ \dot{\omega}_s &= -Q^{-1}\omega_s - \sin\theta_s + \Gamma_0\cos\varphi_s\,, \\ \dot{\varphi}_s &= \Omega\,. \end{aligned} \qquad (4)$$

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The overall structure here is reminiscent of other unidirectional coupling schemes, such as [14].

All of the results to be presented were computed with the (not exceptional) values $Q = 5.0$, $\Gamma_0 = 1.2$, and $\Omega = 0.5$. This places the master pendulum in a chaotic state. Numerical solutions to Eqs. (1) and (2) were obtained with a fourth order Runge-Kutta routine which employed continuous half-step error monitoring as described in [21]. Generally, time grids of $0.01(2\pi/\Omega)$ were used, although in some cases the grid was $0.001(2\pi/\Omega)$.

Our interest is in the synchronization which the slave may achieve with respect to the free-running chaotic master. The master pendulum exhibits a familiar strange attractor when a Poincaré section is plotted. If ideal synchronization is achieved, the strange attractor for the slave would be precisely registered, point-for-point, with the strange attractor of the master. In the absence of synchronization, Poincaré points move on the two attractors in an uncorrelated fashion. An intuitive measure of the quality of synchronization is thus

$$\eta = \sqrt{(\theta_s - \theta_m)^2 + (\dot{\theta}_s - \dot{\theta}_m)^2}, \qquad (5)$$

which is the distance between a point on the master attractor and its corresponding point on the slave attractor.

A typical result is depicted in Fig. 1. Two key properties are illustrated in this figure. First, after approximately 5000 forcing cycles (as shown in the figure inset) the synchronization of slave to master becomes permanent (a close inspection of the data reveals that $\eta = 0$). We will show that this perfect locking is an artifact of the computational process. Second, the slave achieves near synchronization ($\eta \approx 0$) for significant intervals during the earlier phase of the dynamics, but these intervals always end with a loss of synchronization, which inevitably is followed by relocking, and so on. This form of synchronization is *intermittent.* We now deal separately with these two features.

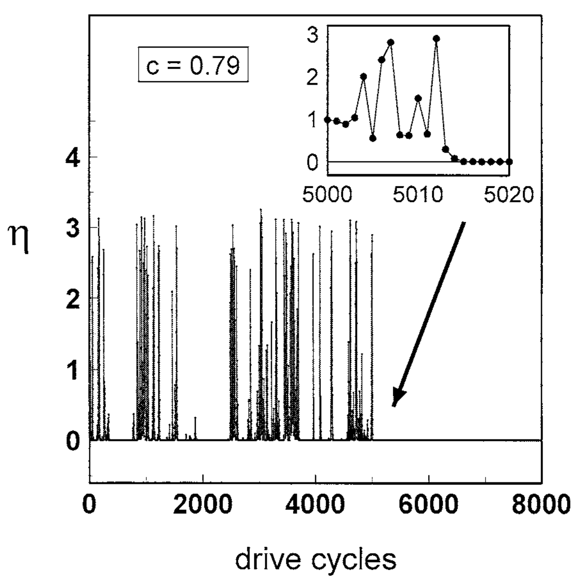

FIG. 1. Synchronization error η versus time measured in units of drive cycles. The simulation was done in double precision with a coupling coefficient $c = 0.79$. The inset is a magnified portion of the moment when hard locking sets in.

Perfect synchronization ($\eta = 0$) is possible for this coupling scheme, but it is a computational artifact in the following sense. Models based upon ODE's [as are Eqs. (3) and (4)] assume that variables such as θ_m and θ_s are real numbers and can therefore never be perfectly matched. (The inability to match variables is also typically true of real world applications, although the underlying causes may be various.) However, in computers, the use of finite-bit representations of numbers reduces any uncountable interval of real numbers to a finite set, and therefore the precise equality of two computed variables *is* possible. If the condition $\theta_m = \theta_s$ ever occurs in a numerical simulation, then according to Eqs. (1) and (2) it will remain exactly so from then on because the coupling term has vanished. This kind of perfect synchronization was observed in both single precision and double precision calculations, with the moment of onset seemingly unpredictable.

It was found that this hard locking could be suppressed by adding a feature to the program which randomly scrambled the nth decimal digit in the variables (θ_m, $\dot{\theta}_m$, θ_s, $\dot{\theta}_s$), where $n = 7$ for single precision and $n = 16$ for double precision. Effectively, this intervention added a bounded rectangular distribution of noise thereby limiting the randomization to the smallest level detectable by either single or double precision arithmetic. When this process was invoked, the intermittency persisted indefinitely. It should also be noted that while η is not exactly zero in the intervals of softer locking, nevertheless, it is very small and the locking is genuine. For example, in double precision simulations it is typically found that $(\theta_m - \theta_s) \leq 10^{-10}$ during synchronization.

In this connection we make two related cautionary remarks. First, the onset of hard locking ($\eta = 0$) could be misinterpreted as the end of a chaotic transient, which clearly it is not. Second, any simulation run which happens to finish at some point within an extended synchronizing window (say, several thousand cycles in duration) where $\eta \approx 0$ could be misinterpreted as having become permanently synchronized. This also would be an illusion.

We turn now to intermittent synchronization. The interleaved intervals of synchronized and unsynchronized dynamics during any simulation run can be saved and then sorted according to the duration of the individual segments. The results of one such simulation—which extended to a total of 10^6 drive cycles—are shown in Fig. 2. The linearity of the log version of the plot suggests that $P(\tau) = A\exp(-\frac{\tau}{T})$, where P is the probability of observing a locking interval of duration τ and T is the mean locking time. The slope of the log plot implies a mean synchronizing time of $(0.036 \times \ln 10)^{-1} = 12.1$ drive cycles.

The distribution of synchronizing intervals shown in Fig. 2 partially characterizes the intermittency which

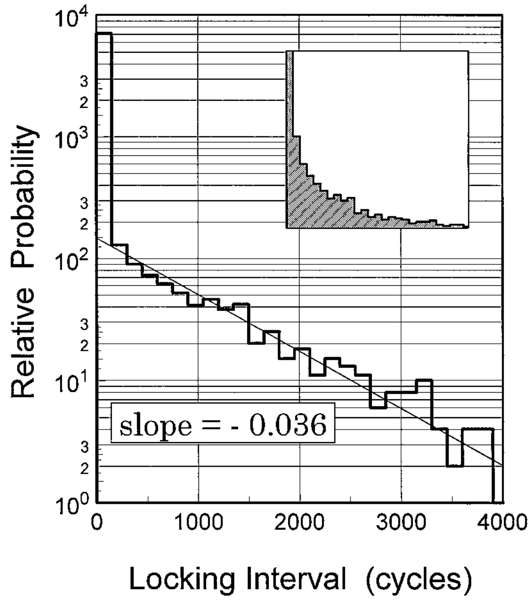

FIG. 2. Results of simulating the coupled pendula with $c = 0.80$ for a total time of 10^6 cycles. Sorting the data yields the histogram which represents the probability of occurrence of locking times. Inset: Original data before log scaling.

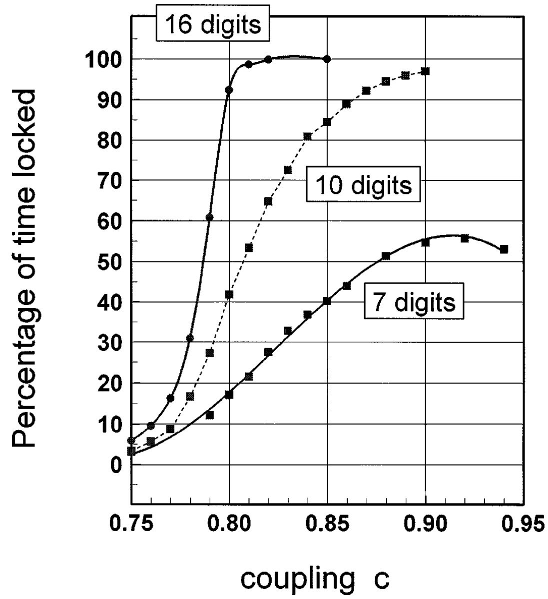

FIG. 3. Effect of the coupling coefficient c on the fraction of a total observation interval during which the slave is synchronized to the master. Clearly evident is the strong influence of the number of significant digits used in a simulation.

occurs for a selected coupling parameter c. But the interleaved intervals of broken synchronization are also significant. In a simulation run of specified length, the ratio of accumulated locked time to the total represents the fraction of time which the system generally spends in a synchronized condition. Figure 3 displays this type of data as a function of coupling parameter c. While intermittency was observed at all c, stronger coupling resulted in an increasingly prevalent synchronized state and longer intervals of locking.

However, as is evident in Fig. 3, the *precision* of the calculations was observed to significantly affect the properties of the intermittency. A possible interpretation of this effect can be found in the fact that a computation to n_1 significant digits has a larger *equivalent noise* (arising from round off uncertainties) than a computation to n_2 digits, if $n_1 < n_2$. In this connection, we mention two recent reports of experimentally observed intermittent loss of synchronization in coupled chaotic electronic oscillators, attributed in [22] to attractor bubbling, and in [23] to on-off intermittency. Numerical simulations of unidirectionally coupled Rössler oscillators with additive noise [24] also exhibited on-off intermittency.

Further support for intermittency comes from the application of two analytic tests of synchronization. First, we examine the system using the approach of Fujisaka and Yamada [25,26] with a linear approximation of the system just off the synchronization manifold (for which $\eta \simeq 0$). In compact notation the pendula modeled by Eqs. (3) and (4) can be expressed as

$$\begin{aligned} \dot{\mathbf{x}}_m &= \mathbf{F}(\mathbf{x}_m)\,, \\ \dot{\mathbf{x}}_s &= \mathbf{F}(\mathbf{x}_s) - \mathbf{c}(\mathbf{x}_m - \mathbf{x}_s)\,, \end{aligned} \tag{6}$$

where c is now a 3 by 3 coupling matrix (in the present case it has only one nonzero component). These equations are linearized to give the approximate dynamics transverse to the synchronization manifold,

$$\delta \dot{\mathbf{x}}_\perp = \{D\mathbf{F}[\mathbf{x}_{\rm sm}(t)] - \mathbf{c}\}\delta \mathbf{x}_\perp\,, \tag{7}$$

where $D\mathbf{F}[\mathbf{x}_{\rm sm}(t)]$ is the Jacobian of the flow evaluated on the synchronization manifold. The vector $\mathbf{x}_{\rm sm}(t)$ is the identical orbit for both transmitter and receiver on the manifold. For the two-pendulum system, Eq. (7) becomes

$$\begin{vmatrix} \dot{\omega}_m - \dot{\omega}_s \\ \dot{\theta}_m - \dot{\theta}_s \\ \dot{\varphi}_m - \dot{\varphi}_s \end{vmatrix} = \begin{vmatrix} -Q^{-1} & -\cos\theta_{\rm sm} & -\Gamma_0 \sin\varphi_{\rm sm} \\ 1 - c & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \times \begin{vmatrix} \omega_m - \omega_s \\ \theta_m - \theta_s \\ \varphi_m - \varphi_s \end{vmatrix}. \tag{8}$$

The corresponding eigenvalue problem yields

$$\begin{aligned} \lambda_{1,2} &= -\frac{1}{2Q}[1 \pm \sqrt{1 - (2Q)^2(1 - c)\cos\theta_{\rm sm}}\,]\,, \\ \lambda_3 &= 0 \quad \text{because } \varphi_m = \varphi_s\,. \end{aligned} \tag{9}$$

Fujisaka and Yamada [25,26] suggest that high-quality synchronization occurs when the largest eigenvalue is negative. However, if the term in the radical is greater than one, then one of the eigenvalues will be negative. Numerical simulation of this term shows that it varies chaotically about unity with an average value that is typically slightly less than one. The chaotic variation is due to the chaotic angle coordinate $\theta_{\rm sm}$. Therefore the largest eigenvalue is not always less than zero, and this particular criterion for synchronization is not obeyed.

As a further test, we apply the more stringent criterion for strong synchronization suggested by Gauthier and Bienfang [22], namely, that a sufficient condition that all perturbations decay to the manifold without transient growth is that

$$\frac{dL}{dt} = 2\delta\mathbf{x}_\perp(t)\{(D\mathbf{F}[\mathbf{x}_{\rm sm}(t)] - \mathbf{c})\delta\mathbf{x}_\perp\} < 0 \qquad (10)$$

for all times. The Lyapunov function $L = |\delta\mathbf{x}_\perp(t)|^2$ is equal to the square of the magnitude of the vector describing the distance from the synchronization manifold as defined above. Straightforward calculation for the pendula leads to condition (10) becoming

$$-\frac{1}{Q}(\omega_m - \omega_s)^2 + [1 - c - \cos\theta_{\rm sm}] \times (\theta_m - \theta_s)(\omega_m - \omega_s) < 0 \qquad (11)$$

for synchronization. Again numerical simulation shows that this condition is not uniformly obeyed. The time series for this expression is intermittently positive or negative with a typical average value very slightly less than zero in the region of synchronization, and randomly slightly positive or negative elsewhere. Therefore intermittent synchronization of the coupled pendulums remains a plausible behavior.

In conclusion, we have used numerical and analytic tests to show that the synchronization between unidirectionally coupled pendula is intermittent. This work indicates the need for care in distinguishing between intermittency, truly persistent synchronization, and the numerical artifact of hard locking. As a final note, we can report that we have also observed evidence of intermittent synchronization in coupled Duffing oscillators (see Fig. 4).

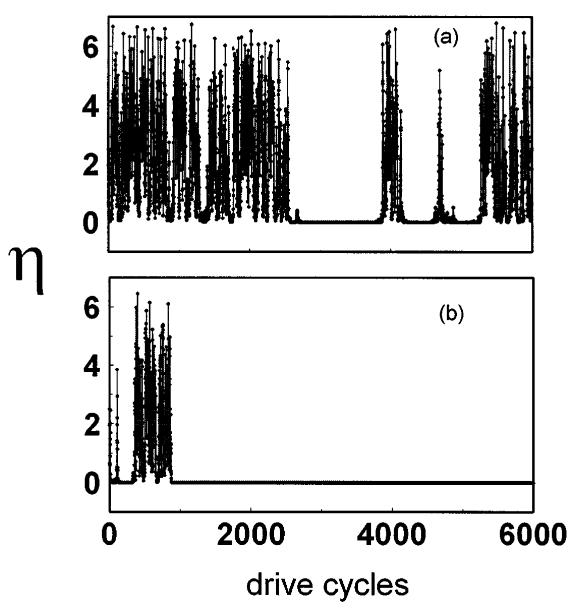

FIG. 4. Synchronized chaos in a pair of unidirectionally coupled Duffing oscillators governed by $\ddot{y} + 0.1\dot{y} + y^3 = 10\cos t$ and $\ddot{x} + 0.1\dot{x} + x^3 = 10\cos t + 1.5(y \quad x)$. (a) Intermittent synchronization resulting from the addition of noise in the 8th digit; (b) occurrence of hard locking in the noiseless case.

Financial support was provided by the Natural Sciences and Engineering Research Council of Canada and by the Research Committee of the Academy of the New Church.

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